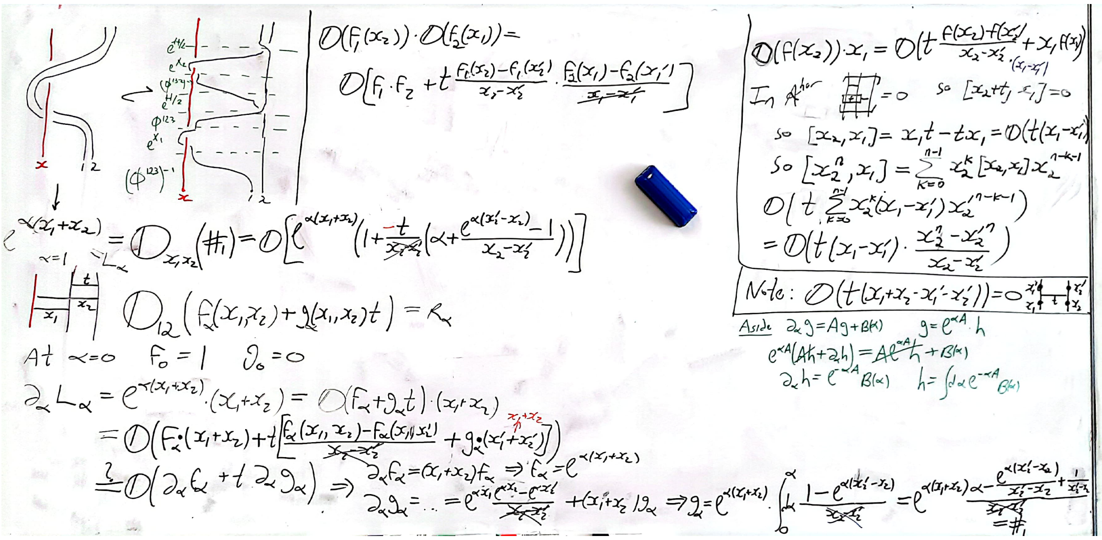
Pensieve header: Computations in 1-pole chord diagrams mod 1ss, Continues pensieve://People/Hogan. A detailed writeup with conclusions a full solution set for Φ in PSS is in PhiInPSS.pdf.



$\mathbb{O}[f, g]$ stands for $\mathbb{O}_{1,2}[f + t\, g]$.

```
In[•]:= CF[𝕆[f_, g_]] := 𝕆[Simplify[f], Simplify[g /. x̄1 → x1 + x2 − x̄2]]
```

```
In[•]:= 𝕆[f1_, g1_] ≡ 𝕆[f2_, g2_] := Simplify[(f1 == f2) ∧ (g1 == g2)]
```

```
In[•]:= 𝕆 /: 𝕆[f1_, g1_] ** 𝕆[f2_, g2_] :=
  CF@𝕆[f1 f2, ((f1 − (f1 /. x2 → x̄2)) ((f2 /. x2 → x̄2) − (f2 /. {x1 → x̄1, x2 → x̄2}))) / (x2 − x̄2) +
     f1 g2 + g1 (f2 /. {x1 → x̄1, x2 → x̄2})]
```

```
In[•]:= 𝕆[f[x2], 0] ** 𝕆[x1, 0]
Out[•]=
  𝕆[f[x2] x1, −f[x2] + f[x̄2]]
```

```
In[•]:= {h1, h2, h3} = {𝕆[f1[x1, x2], g1[x1, x2, x̄1, x̄2]],
   𝕆[f2[x1, x2], g2[x1, x2, x̄1, x̄2]], 𝕆[f3[x1, x2], g3[x1, x2, x̄1, x̄2]]}
Out[•]=
  {𝕆[f1[x1, x2], g1[x1, x2, x̄1, x̄2]],
   𝕆[f2[x1, x2], g2[x1, x2, x̄1, x̄2]], 𝕆[f3[x1, x2], g3[x1, x2, x̄1, x̄2]]}
```

```
In[•]:= h1 ** h2
Out[•]=
  𝕆[f1[x1, x2] f2[x1, x2],
   f2[x1 + x2 − x̄2, x̄2] g1[x1, x2, x1 + x2 − x̄2, x̄2] + f1[x1, x2] g2[x1, x2, x1 + x2 − x̄2, x̄2] +
    ((f1[x1, x2] − f1[x1, x̄2]) (f2[x1, x̄2] − f2[x1 + x2 − x̄2, x̄2])) / (x2 − x̄2)]
```

In[∘]:= **lhs = (h1 ** h2) ** h3**

Out[∘]=

$$\mathbb{O}\Big[f1[x_1,x_2]\, f2[x_1,x_2]\, f3[x_1,x_2],\ f1[x_1,x_2]\, f2[x_1,x_2]\, g3[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2] + f3[x_1+x_2-\overline{x}_2,\overline{x}_2]\Big(f2[x_1+x_2-\overline{x}_2,\overline{x}_2]\, g1[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2] + f1[x_1,x_2]\, g2[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2] + \frac{(f1[x_1,x_2]-f1[x_1,\overline{x}_2])\,(f2[x_1,\overline{x}_2]-f2[x_1+x_2-\overline{x}_2,\overline{x}_2])}{x_2-\overline{x}_2}\Big) + \frac{(f1[x_1,x_2]\, f2[x_1,x_2]-f1[x_1,\overline{x}_2]\, f2[x_1,\overline{x}_2])\,(f3[x_1,\overline{x}_2]-f3[x_1+x_2-\overline{x}_2,\overline{x}_2])}{x_2-\overline{x}_2}\Big]$$

In[∘]:= **rhs = h1 ** (h2 ** h3)**

Out[∘]=

$$\mathbb{O}\Big[f1[x_1,x_2]\, f2[x_1,x_2]\, f3[x_1,x_2],\ f2[x_1+x_2-\overline{x}_2,\overline{x}_2]\, f3[x_1+x_2-\overline{x}_2,\overline{x}_2]\, g1[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2] + f1[x_1,x_2]\Big(f3[x_1+x_2-\overline{x}_2,\overline{x}_2]\, g2[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2] + f2[x_1,x_2]\, g3[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2] + \frac{(f2[x_1,x_2]-f2[x_1,\overline{x}_2])\,(f3[x_1,\overline{x}_2]-f3[x_1+x_2-\overline{x}_2,\overline{x}_2])}{x_2-\overline{x}_2}\Big) + \frac{(f1[x_1,x_2]-f1[x_1,\overline{x}_2])\,(f2[x_1,\overline{x}_2]\, f3[x_1,\overline{x}_2]-f2[x_1+x_2-\overline{x}_2,\overline{x}_2]\, f3[x_1+x_2-\overline{x}_2,\overline{x}_2])}{x_2-\overline{x}_2}\Big]$$

In[∘]:= **lhs ≡ rhs**

Out[∘]=

True

In[∘]:= $\mathbf{e12[\alpha\_] := \mathbb{O}\left[e^{\alpha\,(x_1+x_2)},\ e^{\alpha\,(x_1+x_2)}\, g[\alpha]\right]}$

In[∘]:= $\mathbf{lhs = (\partial_\alpha \#)\ \&\ /@\ e12[\alpha]}$

Out[∘]=

$$\mathbb{O}\left[e^{\alpha\,(x_1+x_2)}\,(x_1+x_2),\ e^{\alpha\,(x_1+x_2)}\, g[\alpha]\,(x_1+x_2) + e^{\alpha\,(x_1+x_2)}\, g'[\alpha]\right]$$

In[∘]:= $\mathbf{rhs = e12[\alpha] ** \mathbb{O}[x_1+x_2, 0]}$

Out[∘]=

$$\mathbb{O}\left[e^{\alpha\,(x_1+x_2)}\,(x_1+x_2),\ e^{\alpha\,x_1}\left(-e^{\alpha\,x_2}+e^{\alpha\,\overline{x}_2}+e^{\alpha\,x_2}\, g[\alpha]\,(x_1+x_2)\right)\right]$$

In[∘]:= $\mathbf{FullSimplify\left[e^{\alpha\,(x_1+x_2)}\, g[\alpha]\ /.\ DSolve[lhs \equiv rhs \land g[0] == 0,\ g[\alpha],\ \alpha][[1]]\right]}$

Out[∘]=

$$-\frac{e^{\alpha\,(x_1+x_2)}\left(-1+e^{\alpha\,(-x_2+\overline{x}_2)}+\alpha\, x_2-\alpha\,\overline{x}_2\right)}{x_2-\overline{x}_2}$$

In[∘]:= $\mathbf{e12[\alpha\_] := \mathbb{O}\left[e^{\alpha\,(x_1+x_2)},\ \frac{e^{\alpha\,(x_1+\overline{x}_2)}-e^{\alpha\,(x_1+x_2)}}{\overline{x}_2-x_2}-\alpha\, e^{\alpha\,(x_1+x_2)}\right]}$

In[∘]:= `lhs = CF[(∂_α #) & /@ e12[α]]`

Out[∘]=

$$\mathbb{O}\left[e^{\alpha\,(x_1+x_2)}\,(x_1+x_2),\ -e^{\alpha\,(x_1+x_2)} - e^{\alpha\,(x_1+x_2)}\,\alpha\,(x_1+x_2) + \frac{-e^{\alpha\,(x_1+x_2)}\,(x_1+x_2) + e^{\alpha\,(x_1+\overline{x}_2)}\,(x_1+\overline{x}_2)}{-x_2+\overline{x}_2}\right]$$

In[∘]:= `rhs = e12[α] ** 𝕆[x₁ + x₂, 0]`

Out[∘]=

$$\mathbb{O}\left[e^{\alpha\,(x_1+x_2)}\,(x_1+x_2),\ -e^{\alpha\,(x_1+x_2)} + e^{\alpha\,(x_1+\overline{x}_2)} + \frac{(x_1+x_2)\left(e^{\alpha\,(x_1+x_2)} - e^{\alpha\,(x_1+\overline{x}_2)} - e^{\alpha\,(x_1+x_2)}\,\alpha\,(x_2-\overline{x}_2)\right)}{x_2-\overline{x}_2}\right]$$

In[∘]:= `lhs ≡ rhs`

Out[∘]=

True

In[∘]:=

$$\text{FullSimplify}\Big[\text{g2}[x_1, x_2, \overline{x}_2] \ /.\ \text{Solve}\Big[\mathbb{O}[f[x_1,x_2], g[x_1,x_2,\overline{x}_1,\overline{x}_2]] ** \mathbb{O}\big[f[x_1,x_2]^{-1}, \text{g2}[x_1,x_2,\overline{x}_2]\big] \equiv \mathbb{O}[1,0],\ \text{g2}[x_1,x_2,\overline{x}_2]\Big][[1]]\Big]$$

Out[∘]=

$$-\frac{\frac{g[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2]}{f[x_1+x_2-\overline{x}_2,\overline{x}_2]} + \frac{(f[x_1,x_2]-f[x_1,\overline{x}_2])\left(\frac{1}{f[x_1,\overline{x}_2]} - \frac{1}{f[x_1+x_2-\overline{x}_2,\overline{x}_2]}\right)}{x_2-\overline{x}_2}}{f[x_1,x_2]}$$

In[∘]:=

$$\mathbb{O}\ /:\ \mathbb{O}[f\_, g\_]^{-1} := \mathbb{O}\left[f^{-1},\ -\frac{\frac{g\ /.\ \overline{x}_1\to x_1+x_2-\overline{x}_2}{f\ /.\ \{x_1\to x_1+x_2-\overline{x}_2,\ x_2\to\overline{x}_2\}} + \frac{(f-(f\ /.\ x_2\to\overline{x}_2))\left(\frac{1}{f\ /.\ x_2\to\overline{x}_2} - \frac{1}{f\ /.\ \{x_1\to x_1+x_2-\overline{x}_2,\ x_2\to\overline{x}_2\}}\right)}{x_2-\overline{x}_2}}{f}\right]$$

In[∘]:= $\mathbb{O}[1, g]^{-1}$

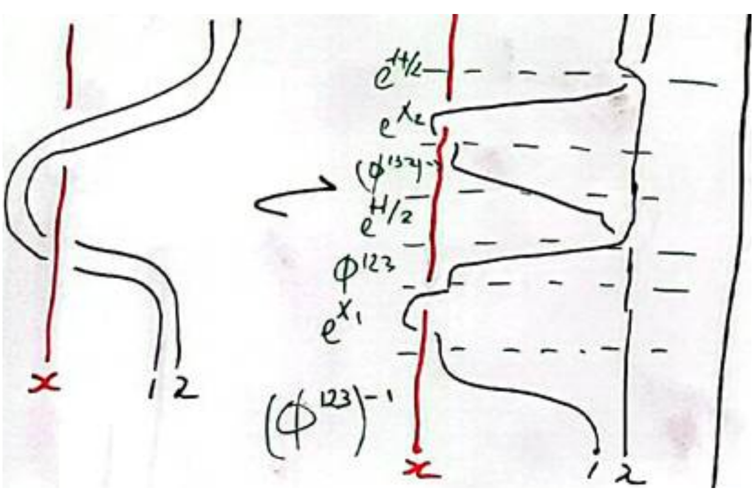
Out[∘]=

$\mathbb{O}[1, -g]$

In[∘]:= $\text{h3} ** \text{h3}^{-1}$

Out[∘]=

$\mathbb{O}[1, 0]$



In[∘]:= $\Phi = \mathbb{O}[1, \phi[x_1, x_2, \overline{x}_2]]$

Out[∘]=

$\mathbb{O}[1, \phi[x_1, x_2, \overline{x}_2]]$

In[∘]:= $\text{rhslist} = \{\Phi^{-1}, \mathbb{O}[e^{x_1}, 0], \Phi, \mathbb{O}[1, 1/2], \text{CF}[\Phi^{-1} /. \{x_1 \to x_2, x_2 \to x_1, \overline{x}_1 \to \overline{x}_2, \overline{x}_2 \to \overline{x}_1\}], \mathbb{O}[e^{x_2}, 0], \text{CF}[\Phi /. \{x_1 \to x_2, x_2 \to x_1, \overline{x}_1 \to \overline{x}_2, \overline{x}_2 \to \overline{x}_1\}], \mathbb{O}[1, -1/2]\}$

Out[∘]=

$\{\mathbb{O}[1, -\phi[x_1, x_2, \overline{x}_2]], \mathbb{O}[e^{x_1}, 0], \mathbb{O}[1, \phi[x_1, x_2, \overline{x}_2]], \mathbb{O}[1, \frac{1}{2}], \mathbb{O}[1, -\phi[x_2, x_1, x_1 + x_2 - \overline{x}_2]], \mathbb{O}[e^{x_2}, 0], \mathbb{O}[1, \phi[x_2, x_1, x_1 + x_2 - \overline{x}_2]], \mathbb{O}[1, -\frac{1}{2}]\}$

In[∘]:= rhs = NonCommutativeMultiply @@ rhslist

Out[∘]=

$\mathbb{O}\left[e^{x_1+x_2}, -\frac{1}{2} e^{x_1} \left(e^{x_2} - e^{\overline{x}_2}\right) \left(1 + 2\phi[x_1, x_2, \overline{x}_2] - 2\phi[x_2, x_1, x_1 + x_2 - \overline{x}_2]\right)\right]$

In[∘]:= lhs = e12[1]

Out[∘]=

$\mathbb{O}\left[e^{x_1+x_2}, -e^{x_1+x_2} + \frac{-e^{x_1+x_2} + e^{x_1+\overline{x}_2}}{-x_2 + \overline{x}_2}\right]$

In[∘]:= lhs ≡ rhs

Out[∘]=

$\frac{1}{2} e^{x_1} \left(-2 e^{x_2} + \frac{2\left(e^{x_2} - e^{\overline{x}_2}\right)}{x_2 - \overline{x}_2} + \left(e^{x_2} - e^{\overline{x}_2}\right) \left(1 + 2\phi[x_1, x_2, \overline{x}_2] - 2\phi[x_2, x_1, x_1 + x_2 - \overline{x}_2]\right)\right) = 0$

In[∘]:= $\text{Apart}\left[g /. \text{First@Solve}\left[-2 e^{x_2} + \frac{2\left(e^{x_2} - e^{\overline{x}_2}\right)}{x_2 - \overline{x}_2} + \left(e^{x_2} - e^{\overline{x}_2}\right)(1 + 2g) == 0, g\right]\right]$

Out[∘]=

$-\frac{e^{x_2}}{-e^{x_2} + e^{\overline{x}_2}} - \frac{2 + x_2 - \overline{x}_2}{2(x_2 - \overline{x}_2)}$

In[∘]:= $\text{Apart}\left[-\frac{e^{x_2}}{-e^{x_2} + e^{\overline{x}_2}} - \frac{2 + x_2 - \overline{x}_2}{2(x_2 - \overline{x}_2)} /. \{x_1 \to x_2, x_2 \to x_1, \overline{x}_1 \to \overline{x}_2, \overline{x}_2 \to \overline{x}_1\} /. \overline{x}_1 \to x_1 + x_2 - \overline{x}_2\right]$

Out[∘]=

$\frac{e^{x_2}}{-e^{x_2} + e^{\overline{x}_2}} + \frac{2 + x_2 - \overline{x}_2}{2(x_2 - \overline{x}_2)}$

In[∘]:= $\text{Simplify}\left[(\text{lhs} \equiv \text{rhs}) /. \phi[\_, x2\_, x2b\_] :\to \frac{-1}{2}\left(\frac{e^{x2}}{-e^{x2} + e^{x2b}} + \frac{2 + x2 - x2b}{2(x2 - x2b)}\right)\right]$

Out[∘]=

True

In[∘]:=

$$\Phi = \mathbb{O}\left[1,\ \phi = -\frac{e^{x_2}/2}{-e^{x_2}+e^{\overline{x}_2}} - \frac{2+x_2-\overline{x}_2}{4\,(x_2-\overline{x}_2)} + \varphi[x_1, x_2, \overline{x}_2] + (\varphi[x_1, x_2, \overline{x}_2]\ /.\ \{x_1\to x_2, x_2\to x_1, \overline{x}_1\to\overline{x}_2, \overline{x}_2\to\overline{x}_1\}\ /.\ \overline{x}_1\to x_1+x_2-\overline{x}_2)\right];$$

$$\text{lhs} = \text{e12}[1]$$

$$\text{rhs} = \Phi^{-1} ** \mathbb{O}\left[e^{x_1}, 0\right] ** \Phi ** \mathbb{O}[1, 1/2] ** \left(\Phi^{-1}\ /.\ \{x_1\to x_2, x_2\to x_1, \overline{x}_1\to\overline{x}_2, \overline{x}_2\to\overline{x}_1\}\right) ** \mathbb{O}\left[e^{x_2}, 0\right] ** (\Phi\ /.\ \{x_1\to x_2, x_2\to x_1, \overline{x}_1\to\overline{x}_2, \overline{x}_2\to\overline{x}_1\}) ** \mathbb{O}[1, -1/2]$$

$$\text{lhs} \equiv \text{rhs}$$

Out[∘]=

$$\mathbb{O}\left[e^{x_1+x_2},\ -e^{x_1+x_2} + \frac{-e^{x_1+x_2}+e^{x_1+\overline{x}_2}}{-x_2+\overline{x}_2}\right]$$

Out[∘]=

$$\mathbb{O}\left[e^{x_1+x_2},\ \frac{e^{x_1}\left(e^{x_2}-e^{\overline{x}_2}-e^{x_2}\,x_2+e^{x_2}\,\overline{x}_2\right)}{x_2-\overline{x}_2}\right]$$

Out[∘]=

True

In[∘]:=

$$\text{lhs} = \text{e12}[-1]$$

$$\text{rhs} = \Phi^{-1} ** \mathbb{O}\left[e^{-x_1}, 0\right] ** \Phi ** \mathbb{O}[1, -1/2] ** \left(\Phi^{-1}\ /.\ \{x_1\to x_2, x_2\to x_1, \overline{x}_1\to\overline{x}_2, \overline{x}_2\to\overline{x}_1\}\right) ** \mathbb{O}\left[e^{-x_2}, 0\right] ** (\Phi\ /.\ \{x_1\to x_2, x_2\to x_1, \overline{x}_1\to\overline{x}_2, \overline{x}_2\to\overline{x}_1\}) ** \mathbb{O}[1, 1/2]$$

$$\text{lhs} \equiv \text{rhs}$$

Out[∘]=

$$\mathbb{O}\left[e^{-x_1-x_2},\ e^{-x_1-x_2} + \frac{-e^{-x_1-x_2}+e^{-x_1-\overline{x}_2}}{-x_2+\overline{x}_2}\right]$$

Out[∘]=

$$\mathbb{O}\left[e^{-x_1-x_2},\ \frac{e^{-x_1-x_2-\overline{x}_2}\left(-e^{x_2}+e^{\overline{x}_2}+e^{\overline{x}_2}\,x_2-e^{\overline{x}_2}\,\overline{x}_2\right)}{x_2-\overline{x}_2}\right]$$

Out[∘]=

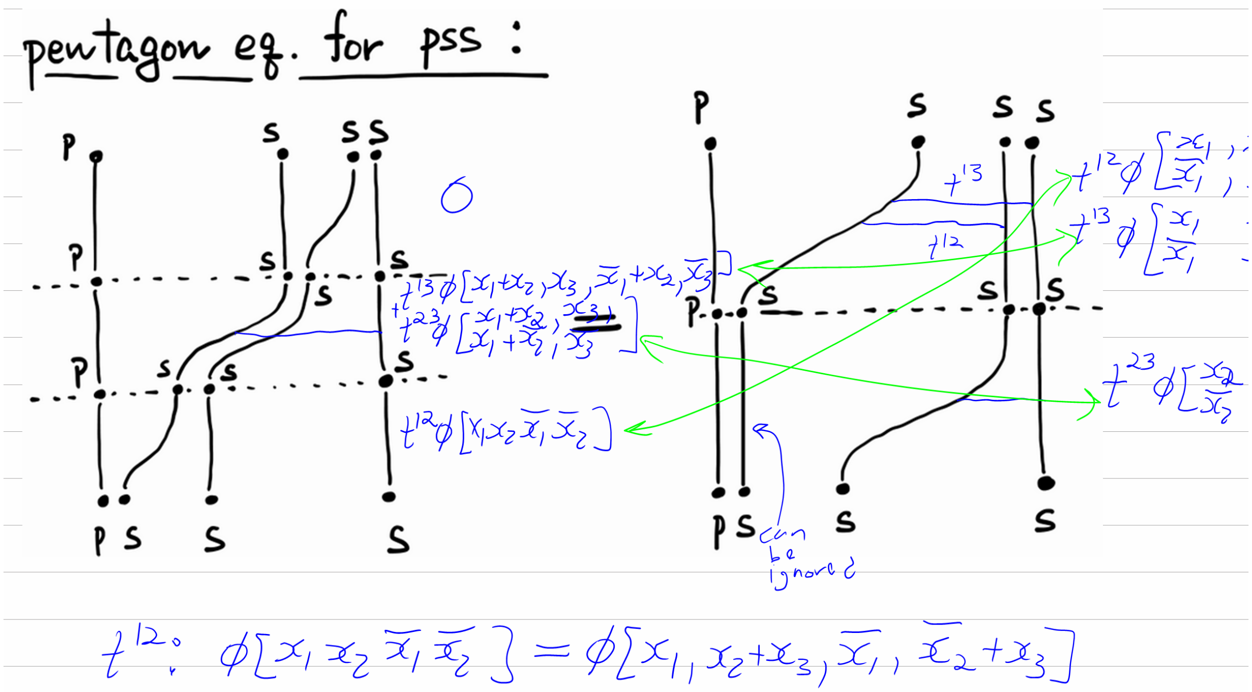
True

In[∘]:=

$$\Phi ** (\text{MapAt}[-\#\ \&, \Phi, 2]\ /.\ \{x_1\to -x_1, x_2\to -x_2, \overline{x}_1\to -\overline{x}_1, \overline{x}_2\to -\overline{x}_2\})$$

Out[∘]=

$$\mathbb{O}\left[1,\ \frac{e^{x_2}}{2\,e^{x_2}-2\,e^{\overline{x}_2}} + \frac{e^{\overline{x}_2}}{2\,e^{x_2}-2\,e^{\overline{x}_2}} + \frac{1}{-x_2+\overline{x}_2} - \varphi[-x_1, -x_2, -\overline{x}_2] + \varphi[x_1, x_2, \overline{x}_2] - \varphi[-x_2, -x_1, -x_1-x_2+\overline{x}_2] + \varphi[x_2, x_1, x_1+x_2-\overline{x}_2]\right]$$



In[∘]:= $\phi = -\frac{e^{x_2}}{2\left(-e^{x_2}+e^{\bar{x}_2}\right)} - \frac{2+x_2-\bar{x}_2}{4\,(x_2-\bar{x}_2)} + \varphi[x_1, x_2, \bar{x}_2] + \varphi[x_2, x_1, x_1+x_2-\bar{x}_2] + 7 + a_1 x_1 + a_2 x_2 + \bar{a}_2 \bar{x}_2$

Out[∘]=

$$7 - \frac{e^{x_2}}{2\left(-e^{x_2}+e^{\bar{x}_2}\right)} + a_1 x_1 + a_2 x_2 - \frac{2+x_2-\bar{x}_2}{4\,(x_2-\bar{x}_2)} + \bar{a}_2 \bar{x}_2 + \varphi[x_1, x_2, \bar{x}_2] + \varphi[x_2, x_1, x_1+x_2-\bar{x}_2]$$

In[∘]:= (* t^{12} *) Simplify[ϕ == (ϕ /. {$x_1 \to x_1$, $x_2 \to x_2+x_3$, $\bar{x}_1 \to \bar{x}_1$, $\bar{x}_2 \to \bar{x}_2+x_3$}) /. $\bar{x}_1 \to x_1+x_2-\bar{x}_2$]

Out[∘]=

$$a_2 x_3 + x_3 \bar{a}_2 + \varphi[x_1, x_2+x_3, x_3+\bar{x}_2] + \varphi[x_2+x_3, x_1, x_1+x_2-\bar{x}_2] == \varphi[x_1, x_2, \bar{x}_2] + \varphi[x_2, x_1, x_1+x_2-\bar{x}_2]$$

In[∘]:= (* t^{23} *) Simplify[(ϕ /. {$x_1 \to x_1+x_2$, $x_2 \to x_3$, $\bar{x}_1 \to x_1+\bar{x}_2$, $\bar{x}_2 \to \bar{x}_3$}) ==
(ϕ /. {$x_1 \to x_2$, $x_2 \to x_3$, $\bar{x}_1 \to \bar{x}_2$, $\bar{x}_2 \to \bar{x}_3$}) /. $\bar{x}_2 \to x_2+x_3-\bar{x}_3$]

Out[∘]=

$$a_1 x_1 + \varphi[x_1+x_2, x_3, \bar{x}_3] + \varphi[x_3, x_1+x_2, x_1+x_2+x_3-\bar{x}_3] == \varphi[x_2, x_3, \bar{x}_3] + \varphi[x_3, x_2, x_2+x_3-\bar{x}_3]$$

In[∘]:= (* t^{13} *) Simplify[(ϕ /. {$x_1 \to x_1+x_2$, $x_2 \to x_3$, $\bar{x}_1 \to \bar{x}_1+x_2$, $\bar{x}_2 \to \bar{x}_3$}) ==
(ϕ /. {$x_1 \to x_1$, $x_2 \to x_2+x_3$, $\bar{x}_1 \to \bar{x}_1$, $\bar{x}_2 \to x_2+\bar{x}_3$}) /. $\bar{x}_1 \to x_1+x_3-\bar{x}_3$]

Out[∘]=

$$a_1 x_2 + \varphi[x_1+x_2, x_3, \bar{x}_3] + \varphi[x_3, x_1+x_2, x_1+x_2+x_3-\bar{x}_3] == a_2 x_2 + x_2 \bar{a}_2 + \varphi[x_1, x_2+x_3, x_2+\bar{x}_3] + \varphi[x_2+x_3, x_1, x_1+x_3-\bar{x}_3]$$