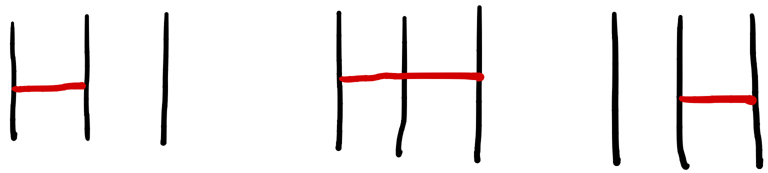
3 strands

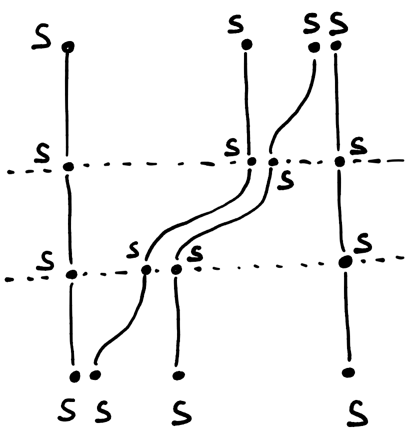
2023/10/20

$$\Phi_{(ss)s} = \exp\left(a\,t^{12} + b\,t^{13} + c\,t^{23}\right) = 1 + a\,t^{12} + b\,t^{13} + c\,t^{23}$$



t^{12} t^{13} t^{23}

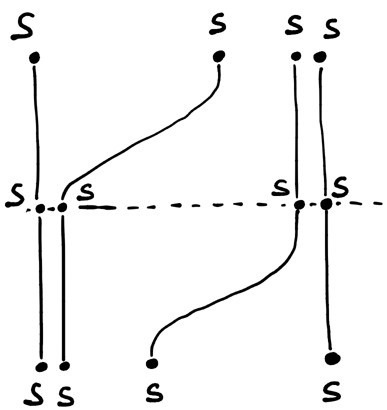
pentagon eq.



$$1 + a\,t^{23} + b\,t^{24} + c\,t^{34}$$

$$1 + a(t^{12} + t^{13}) + b\,t^{14} + c(t^{24} + t^{34})$$

$$1 + a\,t^{12} + b\,t^{13} + c\,t^{23}$$



$$1 + a\,t^{12} + b(t^{13} + t^{14}) + c(t^{23} + t^{24})$$

$$1 + a(t^{13} + t^{23}) + b(t^{14} + t^{24}) + c\,t^{34}$$

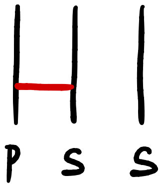
$$2a\,t^{12} + (a+b)t^{13} + b\,t^{14} + (a+c)t^{23} + (b+c)t^{24} + 2c\,t^{34}$$

$$a\,t^{12} + (a+b)t^{13} + 2b\,t^{14} + (a+c)t^{23} + (b+c)t^{24} + c\,t^{34}$$

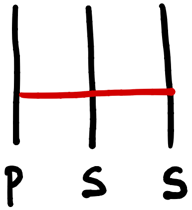
$\rightsquigarrow$ $a = b = c = 0$

$$\boxed{\Phi_{(ss)s} = 1}$$

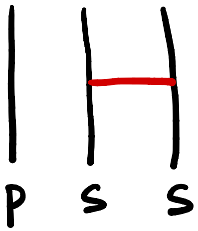
1 pole & 2 strands



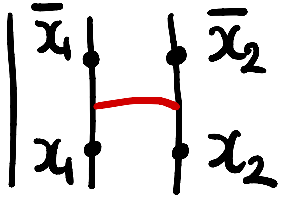
x_1



x_2



t^{12}

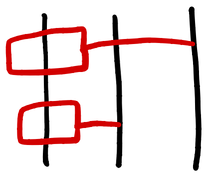


4 term relation

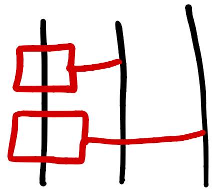
$$x_1 + x_2 = \bar{x}_1 + \bar{x}_2$$

normal form :

$\mathbb{O}_{12}$:



$\mathbb{O}_{21}$:



<u>Lem 1</u> : $F(x_1, x_2),\ G(x_1, x_2) \in \mathbb{Q}[[x_1, x_2]]$

$$\mathbb{O}_{12}\big(F(x_1,x_2)\big) \cdot \mathbb{O}_{12}\big(G(x_1,x_2)\big)$$

$$= \mathbb{O}_{12}\left(F(x_1,x_2)\,G(x_1,x_2) + t^{12}\,\frac{\big(F(x_1,x_2) - F(x_1,\bar{x}_2)\big)\big(G(x_1,\bar{x}_2) - G(\bar{x}_1,\bar{x}_2)\big)}{x_2 - \bar{x}_2}\right)$$

<u>proof</u>

- Check that

$$\mathbb{D}_{21}(x_1^m x_2^n) = \mathbb{D}_{12}\left(x_1^m x_2^n + t^{12} \frac{x_1^m - \bar{x}_1^m}{x_1 - \bar{x}_1} \frac{x_2^n - \bar{x}_2^n}{x_2 - \bar{x}_2} \cdot (x_1 - \bar{x}_1) \right)$$

- For monomials, we compute

$$\mathbb{D}_{12}(x_1^a x_2^b) \cdot \mathbb{D}_{12}(x_1^c x_2^d)$$

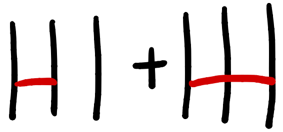
$$= \mathbb{D}_{12}(x_1^a) \cdot \mathbb{D}_{21}(x_2^b x_1^c) \cdot \mathbb{D}_{12}(x_2^d)$$

$$= \mathbb{D}_{12}(x_1^a) \cdot \mathbb{D}_{12}\left(x_1^c x_2^b + t^{12} \frac{x_1^c - \bar{x}_1^c}{x_1 - \bar{x}_1} \frac{x_2^b - \bar{x}_2^b}{x_2 - \bar{x}_2} (x_1 - \bar{x}_1) \right) \cdot \mathbb{D}_{12}(x_2^d)$$

$$= \mathbb{D}_{12}\left(x_1^a x_2^b \cdot x_1^c x_2^d + t^{12} \frac{(x_1^a x_2^b - x_1^a \bar{x}_2^b)(x_1^c \bar{x}_2^d - \bar{x}_1^c \bar{x}_2^d)}{x_2 - \bar{x}_2} \right)$$

//

$$\left[\begin{array}{l}\underline{\text{Lem2}}\ \mathbb{D}_{12}\left(e^{x_1+x_2}\right) = \mathbb{D}_{12}\left(e^{x_1}e^{x_2} + t^{12}\, e^{x_1+x_2}\left(\frac{e^{\bar{x}_2-x_2}-1}{\bar{x}_2-x_2}-1\right)\right)\\ \qquad\qquad \Vert \\ \qquad \exp\Big(\qquad\qquad\qquad\qquad\Big)\end{array}\right]$$



$\underline{\text{proof}}$ Let $s \in \mathbb{Q}$ and write $\mathbb{D}_{12}\left(e^{s(x_1+x_2)}\right) = \mathbb{D}_{12}\left(F_s(x_1,x_2) + t^{12} G_s(x_1,\bar{x}_1,x_2,\bar{x}_2)\right)$

$$\frac{d}{ds}\mathbb{D}_{12}\left(e^{s(x_1+x_2)}\right) = \mathbb{D}_{12}(x_1+x_2)\cdot \mathbb{D}_{12}\left(e^{s(x_1+x_2)}\right)$$

$$\Vert$$

$$= \mathbb{D}_{12}(x_1+x_2)\cdot \mathbb{D}_{12}\left(F_s(x_1,x_2) + t^{12} G_s(x_1,\bar{x}_1,x_2,\bar{x}_2)\right)$$

$$\mathbb{D}_{12}\left(\frac{dF_s}{ds} + t^{12}\frac{dG_s}{ds}\right) \underset{\text{Lem1}}{=} \mathbb{D}_{12}\Big((x_1+x_2)F_s(x_1,x_2) + {} + t^{12}\left(F_s(x_1,\bar{x}_2) - F_s(\bar{x}_1,\bar{x}_2) + (x_1+x_2)G_s(x_1,\bar{x}_1,x_2,\bar{x}_2)\right)\Big)$$

$\rightsquigarrow$ ODE

$$\begin{cases} \frac{d}{ds}F_s(x_1+x_2) = (x_1+x_2)\,F_s(x_1+x_2) & \text{— ①} \\ \frac{d}{ds}G_s(x_1,\bar{x}_1,x_2,\bar{x}_2) = F_s(x_1,\bar{x}_2) - F_s(\bar{x}_1,\bar{x}_2) + (x_1+x_2)\,G_s(x_1,\bar{x}_1,x_2,\bar{x}_2) & \text{②} \end{cases}$$

with the intitial condition $F_0(x_1,x_2)=1$ & $G_0(x_1,\bar{x}_1,x_2,\bar{x}_2)=0$.

By ①, $F_s(x_1,x_2)=e^{s(x_1+x_2)}$. Then ② becomes

$$\frac{d}{ds}G_s = \underbrace{(x_1+x_2)}_{A}G_s + \underbrace{e^{s(x_1+\bar{x}_2)} - e^{s(\bar{x}_1+\bar{x}_2)}}_{B(s)} \qquad x_1+x_2 \quad 4T$$

By generalities on ODE,

$$G_s = e^{sA}\int e^{-sA}B(s)\,ds = e^{s(x_1+x_2)}\int\left(e^{s(\bar{x}_2-x_2)}-1\right)ds$$

$$= e^{s(x_1+x_2)}\left(\frac{e^{s(\bar{x}_2-x_2)}}{\bar{x}_2-x_2} - s + C\right)$$

Since $G_0=0$, the constant C is equal to $-(\bar{x}_2-x_2)^{-1}$.

$$\rightsquigarrow G_s = e^{s(x_1+x_2)}\left(\frac{e^{s(\bar{x}_2-x_2)}-1}{\bar{x}_2-x_2} - 1\right).$$ Putting $s=1$ yields the result.

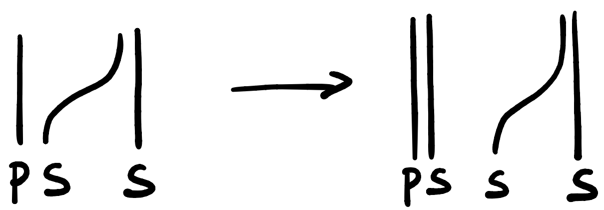
// Lem2

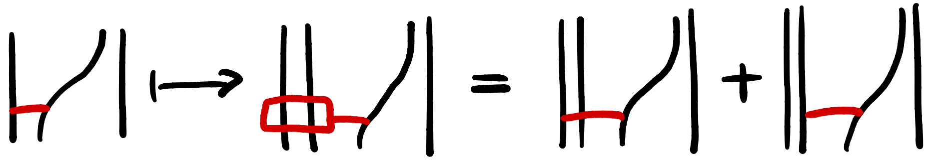
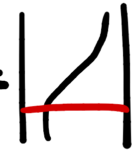
$\underline{\Phi_{(PS)S}}$ Write $\Phi = \Phi_{(PS)S} = \mathbb{O}_{12}\left(F(x_1, x_2) + t^{12} G(x_1, \bar{x}_1, x_2, \bar{x}_2)\right)$

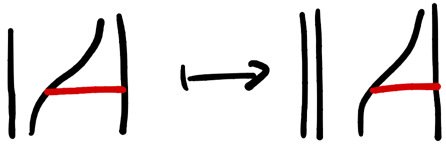
Assumption

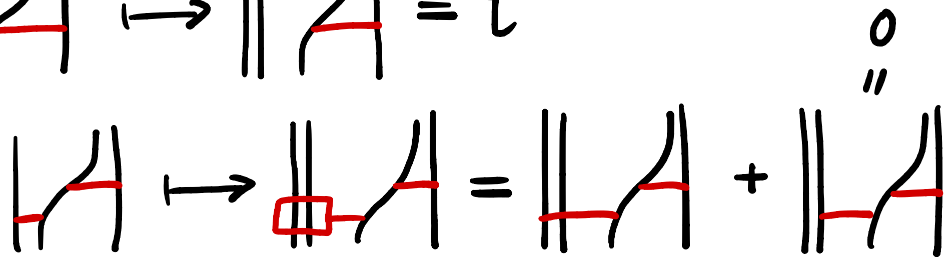
Φ is group-like $\rightsquigarrow$ $F = \exp(f)$, where $f \in \mathrm{Prim}(\mathbb{Q}[[x_1, x_2]]) = \mathbb{Q}x_1 \oplus \mathbb{Q}x_2$

Let us consider pentagon & hexagon eq. for Φ.

Lem 3  induces the following map:

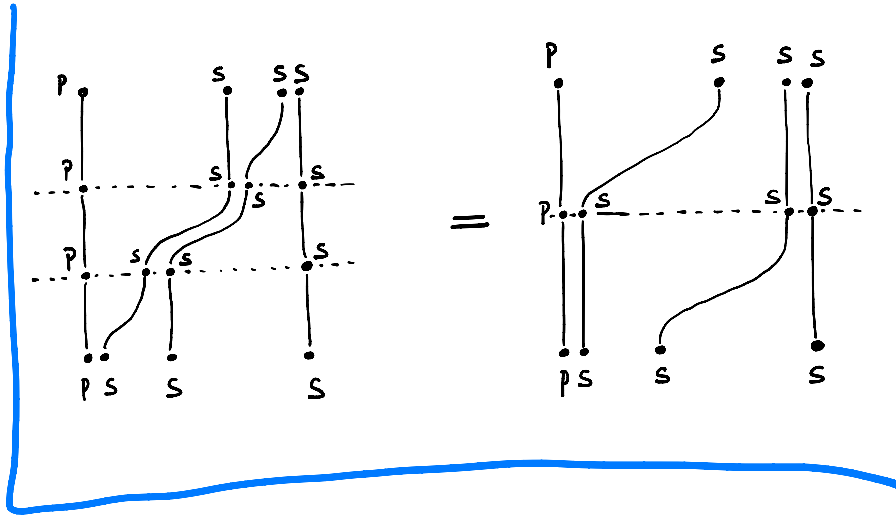
$x_1 =$  $= x_2 + t^{12}$, $x_2 =$  $\longmapsto x_3 + t^{13}$

$t^{12} =$  $= t^{23}$

$t^{12} x_1 =$  $= t^{23} x_2$

Similarly, $t^{12} x_2 \mapsto t^{23} x_3$, $t^{12} \bar{x}_1 \mapsto t^{23} \bar{x}_2$, $t^{12} \bar{x}_2 \mapsto t^{23} \bar{x}_3$

Pentagon eq.

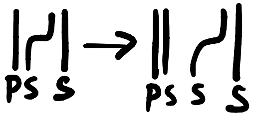


$$\mathrm{LHS} = \left(\Phi_{(p1)2} \;\big|_3\right) \times \left(\Phi_{(p(12))3}\right) \times \overbrace{\left(\big|_p \;\Phi_{(ss)s}\right)}^{1 \,=}$$

$$= \mathbb{D}_{123}\left(F(x_1, x_2) + t^{12} G(x_1, \bar{x}_1, x_2, \bar{x}_2)\right)$$

$$\times\, \mathbb{D}_{123}\left(F(x_1+x_2, x_3) + t^{13} G(x_1+x_2, \bar{x}_1+x_2, x_3, \bar{x}_3) + t^{23} G(x_1+x_2, x_1+\bar{x}_2, x_3, \bar{x}_3)\right)$$

$$\mathrm{RHS} = \left(\Phi_{(p1)2}\right) \times \Phi_{(p1)(23)}$$



$$= \mathbb{D}_{123}\left(F(x_2+t^{12}, x_3+t^{13}) + t^{23} G(x_2, \bar{x}_2, x_3, \bar{x}_3)\right)$$

$$\times\, \mathbb{D}_{123}\left(F(x_1, x_2+x_3) + t^{12} G(x_1, \bar{x}_1, x_2+x_3, \bar{x}_2+x_3) + t^{13} G(x_1, \bar{x}_1, x_2+x_3, x_2+\bar{x}_3)\right)$$

Taking the first projection (the t-degree $= 0$ part),

$$F(x_1,x_2)\,F(x_1+x_2,x_3) = F(x_2,x_3)\,F(x_1,x_2+x_3)$$

So, $f := \log F$ satisfies the (additive) 2-cocycle condition

$$\boxed{f(x_1,x_2) + f(x_1+x_2,x_3) = f(x_2,x_3) + f(x_1,x_2+x_3)}$$

Since we assume that F is group-like, $f(x,y) = ax+by$ $(a,b\in\mathbb{Q})$

$$2a x_1 + (a+b)x_2 + bx_3 = ax_1 + (a+b)x_2 + 2b x_3$$

We obtain $a=b=0$, which implies $f=0$ & $F=1$.

Rem W/O primitive assumption, $f(x,y) = x^m + y^m - (x+y)^m$ sat the 2-cocycle eq.

Conclusion so far: If $\Phi = \Phi_{(ps)s}$ satisfies the pentagon eq., it must be of the form $\Phi = \mathbb{O}_{12}\left(1 + t^{12}\, G(x_1,\bar{x}_1,x_2,\bar{x}_2)\right)$. Furthermore, the pentagon for $\Phi_{(ps)s}$ is equivalent to the following set of three eqs:

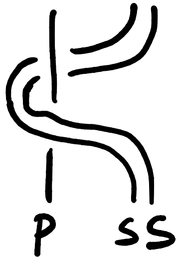
$\underline{t^{12}}$: $G(x_1, \bar{x}_1, x_2, \bar{x}_2) = G(x_1, \bar{x}_1, x_2 + x_3, \bar{x}_2 + x_3)$

$\underline{t^{13}}$: $G(x_1 + x_2, \bar{x}_1 + x_2, x_3, \bar{x}_3) = G(x_1, \bar{x}_1, x_2 + x_3, x_2 + \bar{x}_3)$

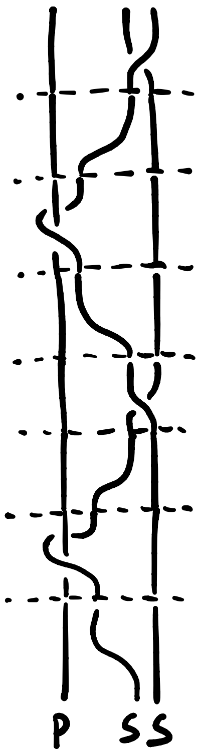
$\underline{t^{23}}$: $G(x_1 + x_2, x_1 + \bar{x}_2, x_3 + \bar{x}_3) = G(x_2, \bar{x}_2, x_3, \bar{x}_3)$

(negative) hexagon eq.

positive ?



$=$



$1-\frac{1}{2}(1|X)$

Φ

e^{x_1}

Φ^{-1}

$1+\frac{1}{2}(1|X)$

Φ

e^{x_1}

Φ^{-1}

$\parallel$

$\mathbb{D}_{12}(e^{x_1+x_2})$

Assumption

$\Phi = \Phi_{(ps)s}$ is of the form

$\Phi = \mathbb{D}_{12}\left(1+t^{12}G(x_1,\bar{x}_1,x_2,\bar{x}_2)\right)$

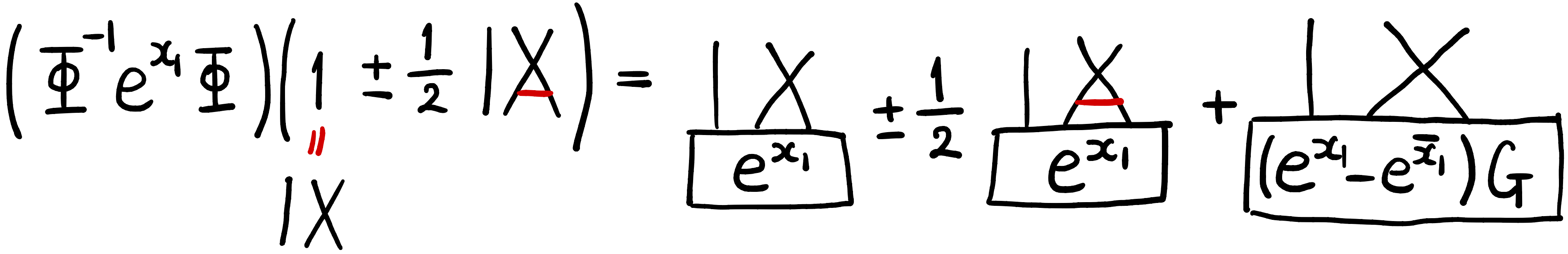
(true if Φ sat pentagon)

$\rightsquigarrow \Phi^{-1} = \mathbb{D}_{12}(1-t^{12}G)$

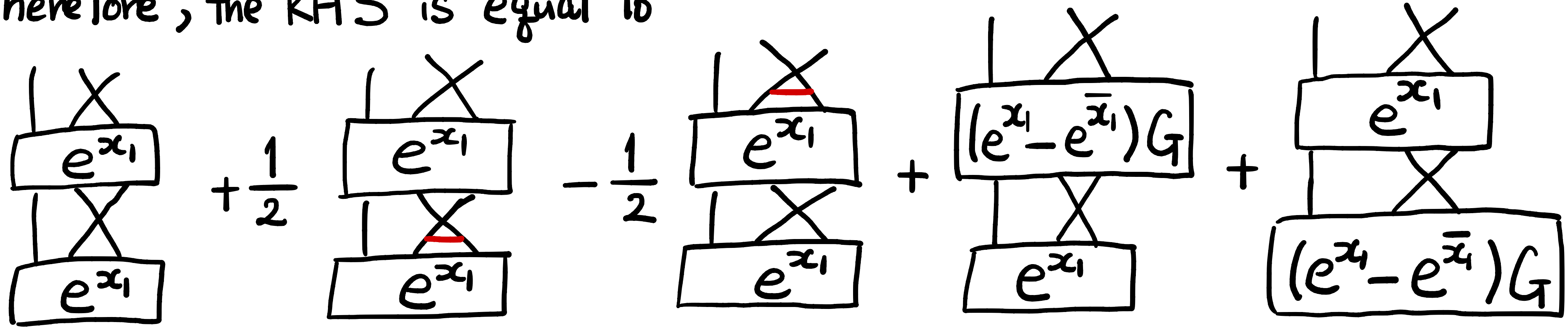
$$\Phi^{-1}e^{x_1}\Phi = \mathbb{D}_{12}(1-t^{12}G)\cdot\mathbb{D}_{12}(e^{x_1})\cdot\mathbb{D}_{12}(1+t^{12}G)$$

$$= \mathbb{D}_{12}(e^{x_1}-t^{12}Ge^{\bar{x}_1})\cdot\mathbb{D}_{12}(1+t^{12}G)$$

$$= \mathbb{D}_{12}\left(e^{x_1}+t^{12}(e^{x_1}-e^{\bar{x}_1})G\right)$$



Therefore, the RHS is equal to



$$= \mathbb{D}_{12}\left(e^{x_1}e^{x_2} + t^{12}\left(\frac{1}{2}\, e^{x_1}\left(e^{\bar{x}_2} - e^{x_2}\right) + e^{x_1}\left(e^{x_2} - e^{\bar{x}_2}\right)G^{(21)} + \underbrace{\left(e^{x_1} - e^{\bar{x}_1}\right)e^{\bar{x}_2}G}_{\overset{4T}{=}\; e^{x_1}\left(e^{\bar{x}_2} - e^{x_2}\right)}\right)\right)$$

$$= \mathbb{D}_{12}\left(e^{x_1}e^{x_2} + t^{12}\, e^{x_1}\left(e^{\bar{x}_2} - e^{x_2}\right)\left(\frac{1}{2} + G - G^{(21)}\right)\right)$$

Here, $G^{(21)} = G(x_2, \bar{x}_2, x_1, \bar{x}_1)$

On the other hand, the LHS is computed by Lem 2. The hexagon for $\Phi = \Phi_{(PS)S}$ is equivalent to

$$e^{x_1+x_2}\left(\frac{e^{\bar{x}_2-x_2}-1}{\bar{x}_2-x_2}-1\right) = e^{x_1}(e^{\bar{x}_2}-e^{x_2})\left(\frac{1}{2}+G-G^{(21)}\right) \quad \cdots ☆$$

$$☆ \iff \frac{e^{\bar{x}_2}-e^{x_2}}{\bar{x}_2-x_2}-e^{x_2} = (e^{\bar{x}_2}-e^{x_2})\left(\frac{1}{2}+G-G^{(21)}\right)$$

$$\iff G-G^{(21)} = \frac{1}{\bar{x}_2-x_2}-\frac{1}{e^{(\bar{x}_2-x_2)}-1}-\frac{1}{2} \quad \cdots ☆☆$$

<u>Rem</u> $\eta(z) := \frac{1}{z}-\frac{1}{e^z-1}-\frac{1}{2} = -\sum_{k\geq 2}^{\infty}\frac{B_k}{k!}z^{k-1} = -\frac{z}{12}+\frac{z^3}{720}-\frac{z^5}{30240}+\cdots$

Since η is odd, $\eta(\bar{x}_2-x_2)^{(21)} = \eta(\bar{x}_1-x_1) \underset{4T}{=} \eta(-(\bar{x}_2-x_2)) = -\eta(\bar{x}_2-x_2)$.

A speial solution to ☆☆ is given by $G = \frac{1}{2}\eta(\bar{x}_2-x_2)$, and

[**Prop 1** $\{\text{Solutions to } \star\star\} = \frac{1}{2}\eta(\bar{x}_2 - x_2) + \{H \mid H = H^{(21)}\}$]

Notice that $G := \frac{1}{2}\eta(\bar{x}_2 - x_2)$ satisfies the three eqs. $\underline{t}^{12}, \underline{t}^{13}$ & $\underline{t}^{23}$.

$G = \frac{1}{2}\eta(\bar{x}_2 - x_2) + H$ satisfies the pentagon & hexagon

$\iff$ $\boxed{H = H^{(21)} \ \& \ \underline{t}^{12}, \underline{t}^{13}, \underline{t}^{23} \text{ for } H}$ $\cdots \star\star\star$

[**Prop 2** $\star\star\star \iff H = \beta(x_1 - \bar{x}_1)$ for some even fct $\beta \in \mathbb{Q}[[z]]$]

$(= \beta(\bar{x}_2 - x_2))$

proof

($\Leftarrow$) Use $\beta(x_1 - \bar{x}_1)^{(21)} = \beta(x_2 - \bar{x}_2) \underset{4T}{=} \beta(-(x_1 - \bar{x}_1)) \underset{\text{even}}{=} \beta(x_1 - \bar{x}_1)$

$(\Longrightarrow)$ By 4T, eliminate $\bar{x}_2$ and regard H as a fct in variables $x_1, \bar{x}_1, x_2$

$H(x_1, \bar{x}_1, x_2) = H(x_1, \bar{x}_1, x_2, \bar{x}_2)$. Let us assume

$$H = H^{(21)} : \quad H(x_1, \bar{x}_1, x_2) = H(x_2, x_1 + x_2 - \bar{x}_1, x_1) \quad \text{— (1)}$$

$$\underline{t^{12}} : \quad H(x_1, \bar{x}_1, x_2) = H(x_1, \bar{x}_1, x_2 + x_3) \quad \text{— (2)}$$

$$\underline{t^{13}} : \quad H(x_1 + x_2, \bar{x}_1 + x_2, x_3) = H(x_1, \bar{x}_1, x_2 + x_3) \quad \text{— (3)}$$

$$\underline{t^{23}} : \quad H(x_1 + x_2, x_1 + \bar{x}_2, x_3) = H(x_2, \bar{x}_2, x_3) \quad \text{— (4)}$$

(2) $\rightsquigarrow \dfrac{\partial H}{\partial x_2} = 0 \rightsquigarrow H(x_1, \bar{x}_1, x_2) = \bar{H}(x_1, \bar{x}_1)$

$$(1)' : \quad \bar{H}(x_1, \bar{x}_1) = \bar{H}(x_2, x_1 + x_2 - \bar{x}_1)$$

$$(3)' : \quad \bar{H}(x_1 + x_2, \bar{x}_1 + x_2) = \bar{H}(x_1, \bar{x}_1)$$

$$(4)' : \quad \bar{H}(x_1 + x_2, x_1 + \bar{x}_2) = \bar{H}(x_2, \bar{x}_2)$$

equivalent by $1 \leftrightarrow 2$

$$\bar{H}(x_1,\bar{x}_1) \overset{③' \text{ w/ } x_2=-x_1}{=} \bar{H}(x_1-\bar{x}_1, 0)$$

$$\Big\| \ ①' \text{ w/ } x_2 = 0$$

$$\bar{H}(0, x_1-\bar{x}_1) \underset{③' \text{ w/}}{=\!=} \bar{H}(\bar{x}_1-x_1, 0)$$

$\beta(z) := \bar{H}(z,0)$.

Since the map

$$\mathbb{Q}[[z]] \to \mathbb{Q}[[x,y]],\ f(z) \mapsto f(x-y)$$

is injective, $\bar{H}(x_1-\bar{x}_1,0) = \bar{H}(\bar{x}_1-x_1,0)$ implies that $\beta(z) = \beta(-z)$, i.e., β is even. Putting things together, we conclude

$$H(x_1,\bar{x}_1,x_2,\bar{x}_2) = \bar{H}(x_1,\bar{x}_1)$$
$$= \bar{H}(x_1-\bar{x}_1, 0)$$
$$= \beta(x_1-\bar{x}_1)$$

// Prop 2

Conclusion:

$$\left\{ \begin{array}{l} \text{Solutions to pentagon} \\ \text{\& hexagon for } \Phi_{(PS)S} \end{array} \right\}$$

$$= \left\{ \Phi_{12}\left(1+t^{12}\left(\tfrac{1}{2}\eta(\bar{x}_2-x_2)+\beta(\bar{x}_2-x_2)\right)\right) \,\middle|\, \beta \in \mathbb{Q}[[z]]_{\text{even}} \right\}$$