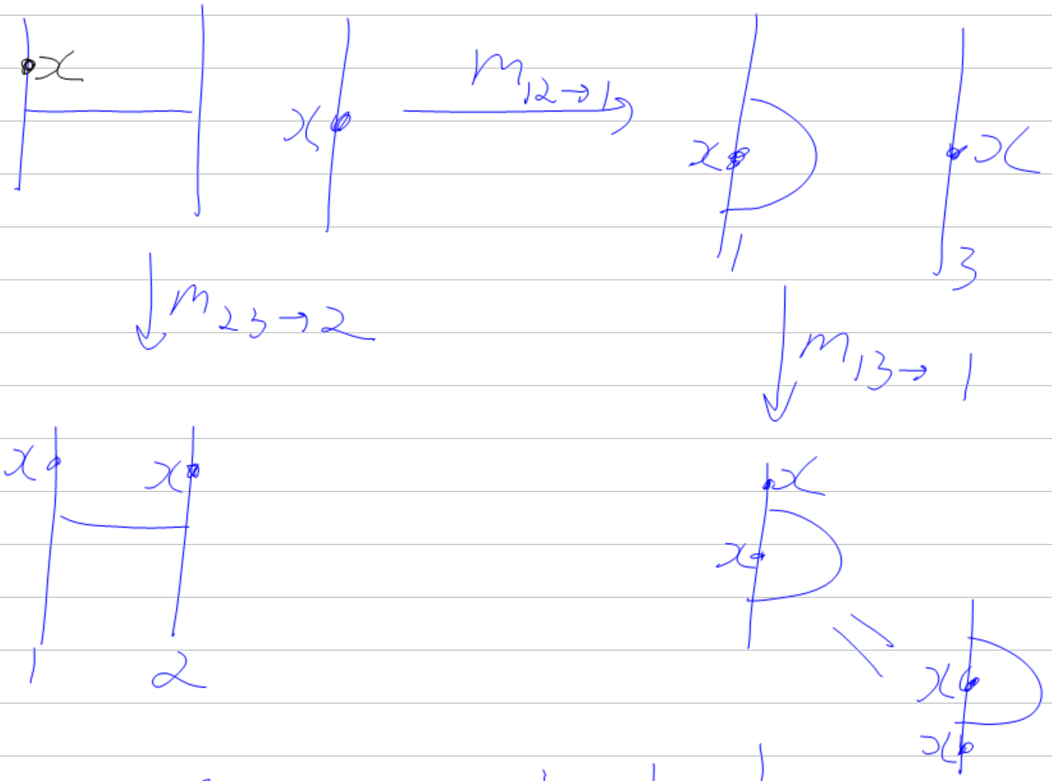
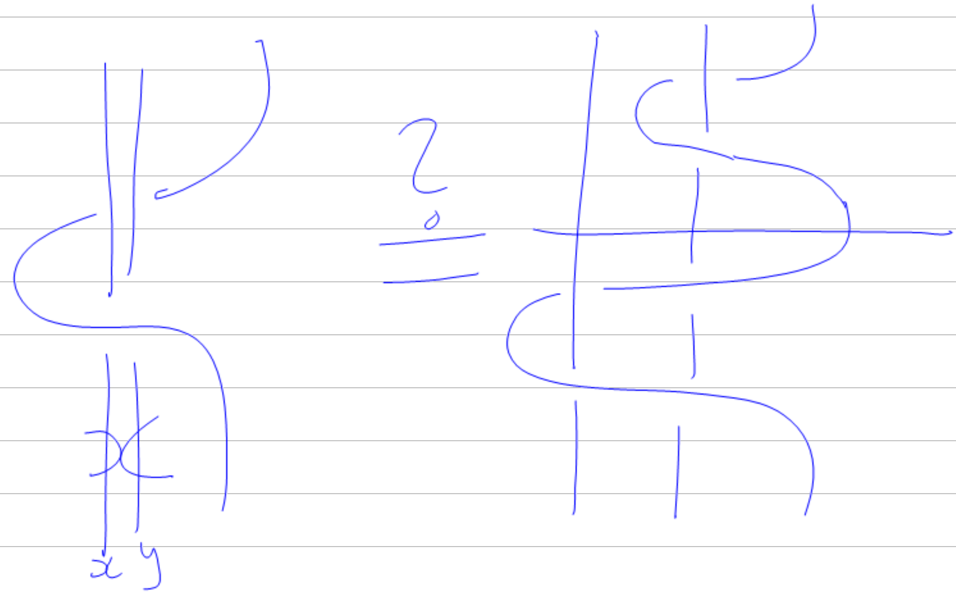
$O_{AR,\{x,y\},\{1,2,3\}}[A_{C[1,2]}[AW_1[]\ AW_2[]\ AW_3[x]\ AW_1[x]\ AW_2[]]]$




$$FA[x_1 \ldots x_n] \xrightarrow[\Phi_{\lambda_1 \ldots \lambda_n}]{x_i \mapsto \lambda_i} FA[y_1 \ldots y_m]$$

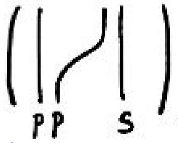
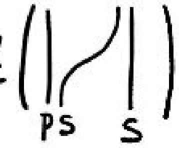
$FA : Vect \longrightarrow FreeAlg$

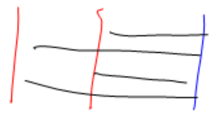
$FA[x_1 \to \sum a_{1j} y_j,\ x_2 \longrightarrow \sum a_{2j} y_j \ldots]\}$

$\Phi^{1,23,4}$

See 231013_Journal.pdf

23/10/4

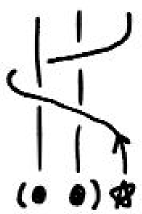
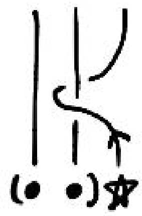
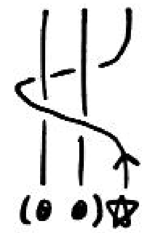
$\Phi_{\bullet\bullet\star} = Z\Big($  $\Big)$, $\quad \Phi_{\bullet\star\star} = Z\Big($  $\Big)$, $\quad \Phi_{\star\star\star} = Z\Big($  $\Big)$



solved $\qquad$ solved $(=1)$

[ Q. Does $\Phi^{psp}_{\bullet\star\bullet}$ make sense?
If so, is it related to $\Phi_{\bullet\bullet\star}$? ]

$\Phi^{213}$

$\alpha =$  , $\quad \beta =$  , $\quad \alpha\beta =$  $= \Delta_\bullet \Big($  $\Big)$

$\gamma$

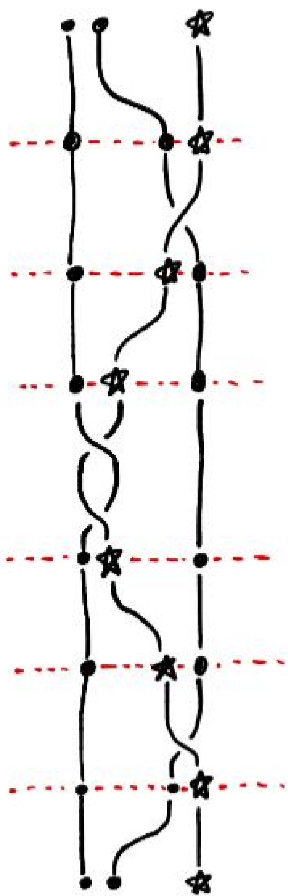
$\gamma$

$$f^{-1}\exp(x) f \cdot g^{-1}\exp(y) g = \exp(x+y) \longleftarrow Z$$

$\{\text{solutions}\} \to \exp(L)$
$(f, g) \longmapsto g$

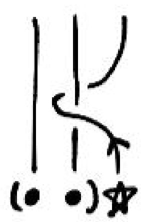
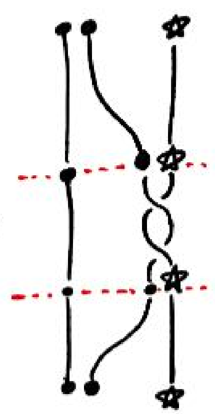
[ Q. How $f$ and $g$ are related? ]

$\alpha = \gamma \cdot \beta^{-1}$

$\alpha =$  $=$ 

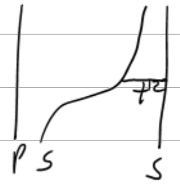
$\exp(x)$

$\left.\begin{array}{l} \Phi^{-1}_{(\bullet\star)\bullet} \\ \exp(\frac{1}{2}y) \\ \Phi_{(\bullet\bullet)\star} \end{array}\right\} \xrightarrow{Z} f^{-1}$

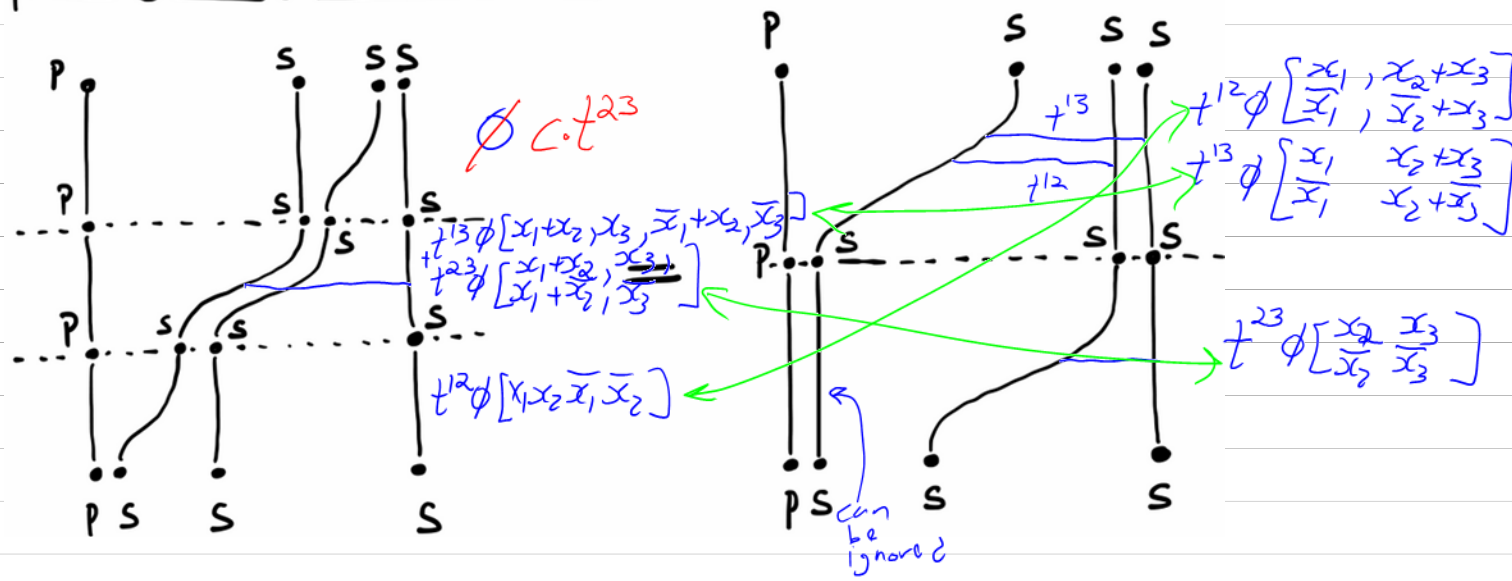
$\beta =$  $=$  $\exp(y)$

$\Phi_{(\bullet\bullet)\star} \xrightarrow{Z} g^{-1}$

$\Phi = \Phi[1, t^{12}\phi]$



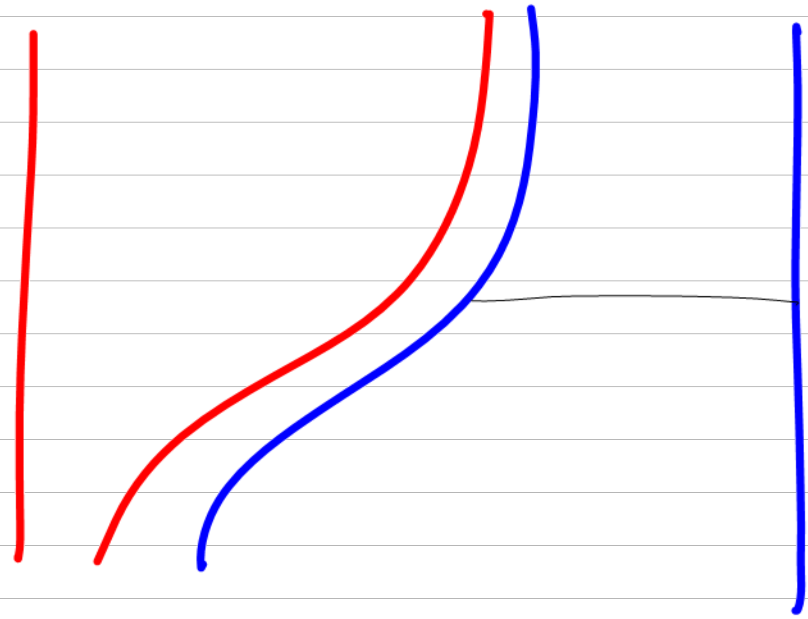

pentagon eq. for pss :

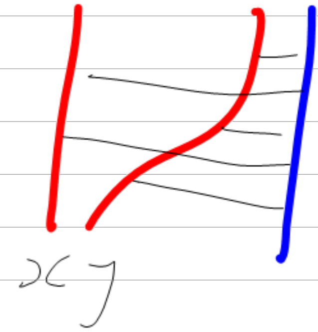


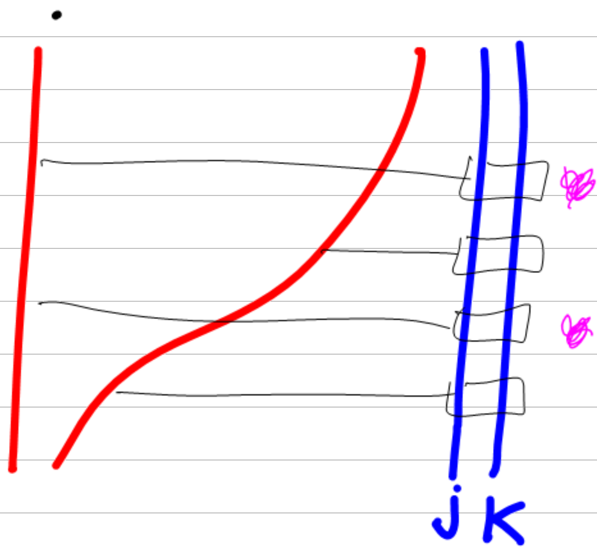

$t^{12}$: $\phi[x_1 x_2 \bar{x}_1 \bar{x}_2] = \phi[x_1, x_2+x_3, \bar{x}_1, \bar{x}_2+x_3]$

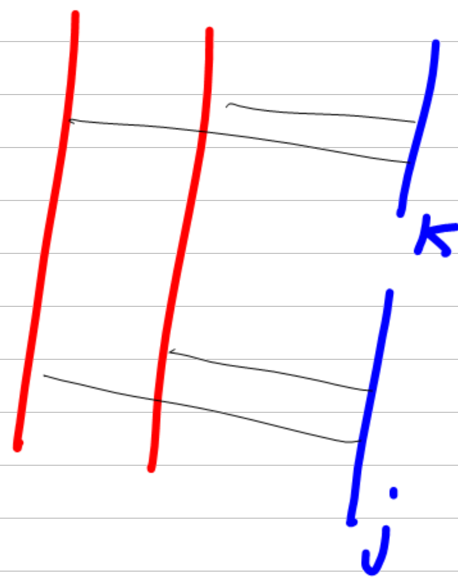
---

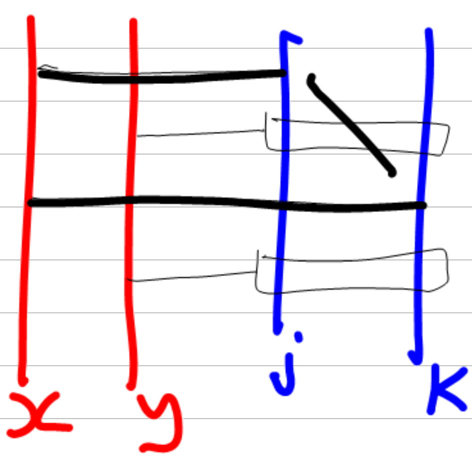
$$\psi = \frac{1}{2}\eta(\bar{x}_2 - x_2) + \beta(\bar{x}_2 - x_2)$$





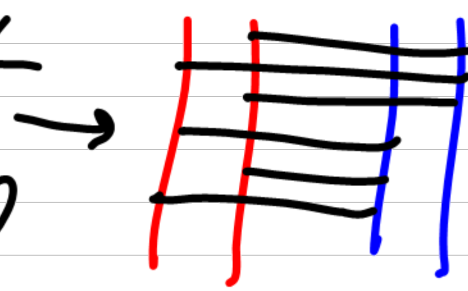



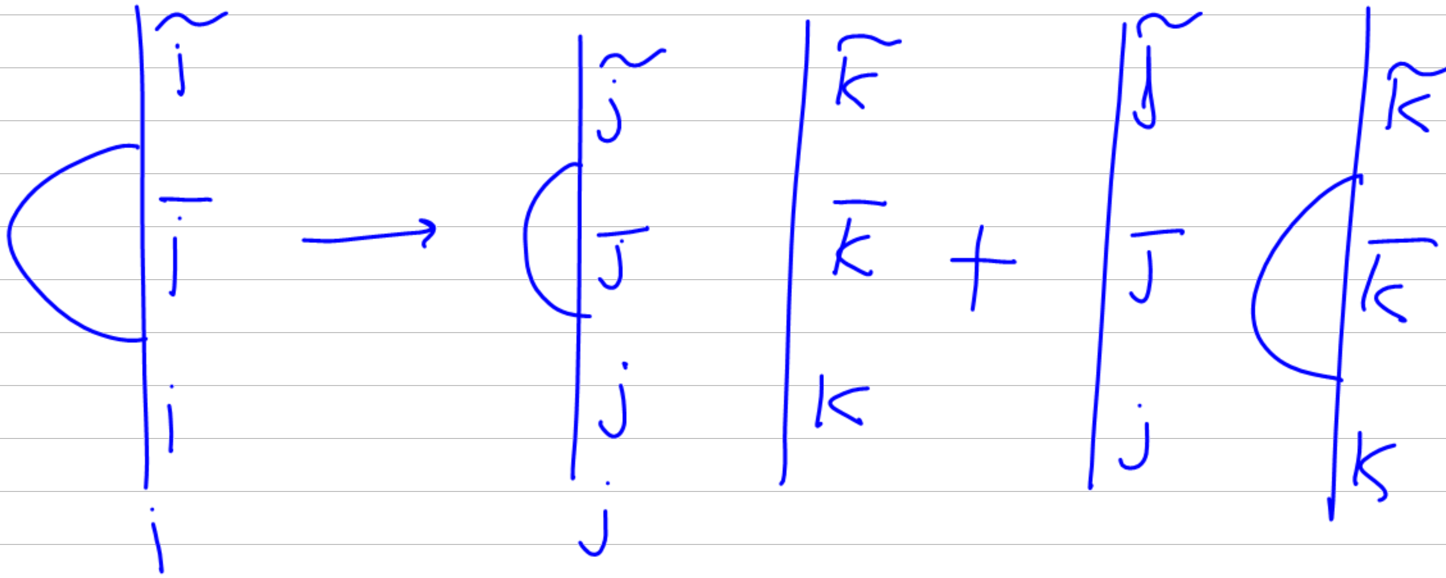



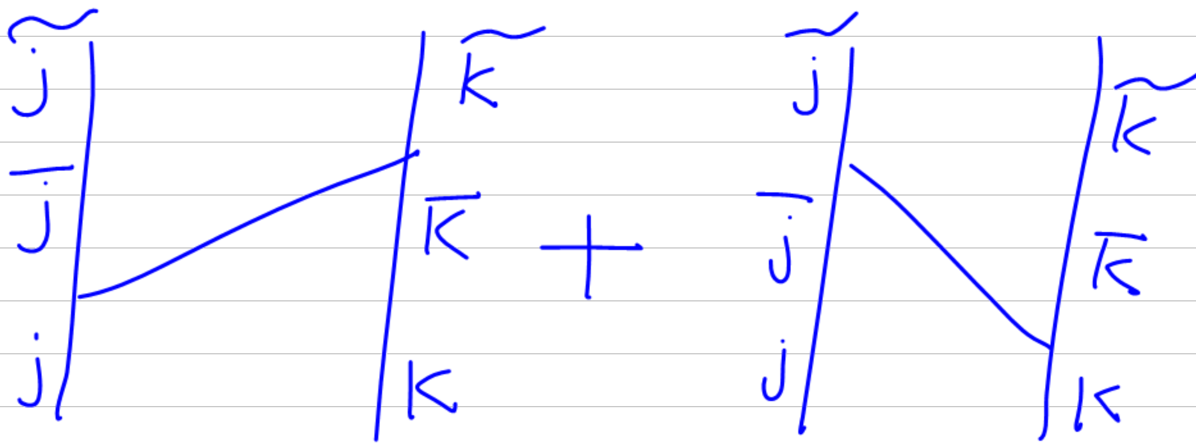

j: xyx

K: yxy

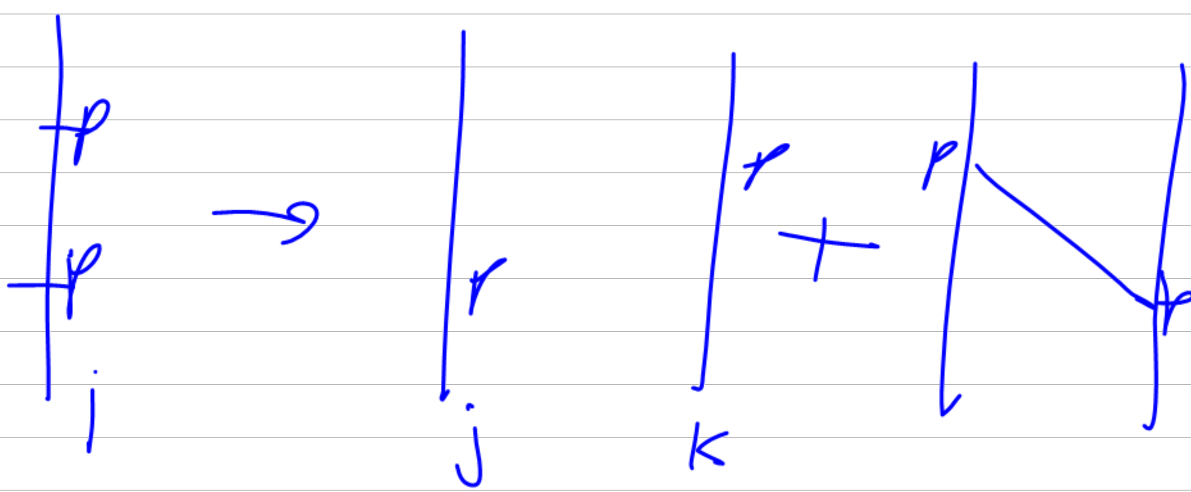




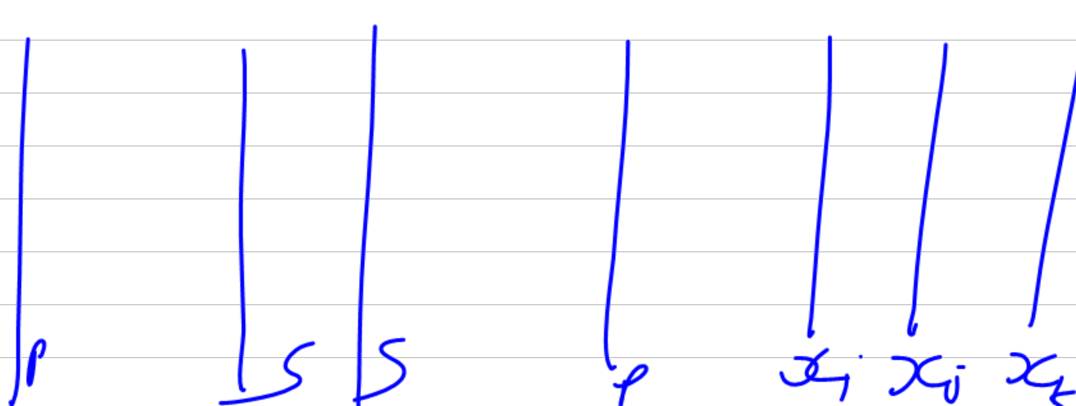
j̃ j̄ j i
j̃ j̄ j j k̃ k̄ k + j̃ j̄ j k̃ k̄ k

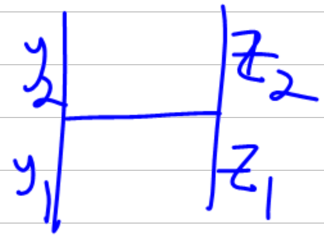
j̃ j̄ j k̃ k̄ k + j̃ j̄ j k̃ k̄ k

p p i
r j r k + p p

p s s p x_i x_j x_k




$$y_1+z_1=y_2+z_2$$

$$z_2=z_1+y_1-y_2$$


---

Pent($\Phi$)

$\cancel{\Phi^5 1}$  $\Phi_{m-1}$ s.t. $Pent(\Phi_{m-1}) = 0 + \underset{\deg m}{E_m} + \dots$

$$\Phi_m = \Phi_{m-1} + \overset{\deg m}{\varphi_m} \qquad Pent(\Phi_m) = Pent(\Phi_{m-1}+\varphi_m)$$

$$(\Phi_{m-1}+\varphi_m)^{123}(\Phi_{m-1}+\varphi_m)^{1(23)4}(\ +\ )(\ +\ )(\ +\ ) - 1 \quad \text{at deg } m$$

$$= E_m + \varphi^{234} - \varphi^{(12)34} + \varphi^{1(23)4} - \varphi^{12(34)} + \varphi^{123} = E_m + d^3\varphi = 0$$

$$d^3\varphi = \sum_{i=0}^{3+1} (-1)^i d_i^3 \varphi$$

$d_i^3\varphi$ : add a strand/pole in position $i$

Need * $d^4 E_m = 0$ . . . Follow from "associahedra" see my paper on Non-Associative Tangles

* $H^4(\mathcal{G}) = 0$ sometimes works, sometimes doesn't.

---

$e(x) = \underline{\underline{1}} + x + \dots$ (hom) $e(x+y) = e(x)e(y)$ should not exist!

Suppose $e_{m-1}$ satisfies $\text{hom}_{m-1}$ meaning $e_{m-1}(x+y) - e_{m-1}(x)e_{m-1}(y) = \underset{\substack{\text{to deg}\\ m-1}}{0} + \underset{\substack{\uparrow\\ \text{mistake in}\\ \text{deg } m.}}{M_m} + \dots$

$$e_m = e_{m-1} + \underset{\text{a polynomial of deg } m}{C_m}$$

$$\text{hom}(e_m) = (e_{m-1} + C_m)(x+y) - (e_{m-1}+C_m)(x) \cdot (e_{m-1}+C_m)(y)$$

$$= M_m + C_m(x+y) - C_m(x) - C_m(y) = M_m - d^1 C_m$$

Need $*\ \mathcal{C}$ $\quad * \ dM_m = 0$ $\quad * \ H(\mathcal{C})$

Need an eq'n that $M$ satisfies?

$$\begin{tikzcd}
& e_{m-1}(x+y)e_{m-1}(z) \arrow[dr, dash, "M_m(x,y)"] & \\
e_{m-1}(x+y+z) \arrow[ur, dash, "M_m(x+y,z)"] \arrow[dr, dash, "M_m(x,y+z)"'] & & e_{m-1}(x)e_{m-1}(y)e_{m-1}(z) \\
& e_{m-1}(x)e_{m-1}(y+z) \arrow[ur, dash, "M_m(y,z)"'] &
\end{tikzcd}$$

in deg $m$

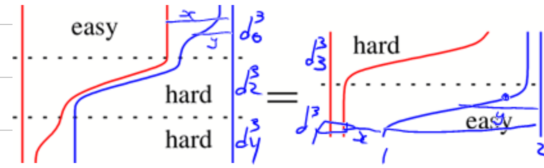
$$d^2 M_m = M_m(y,z) - M_m(x+y,z) + M_m(x,y+z) - M_m(x,y) = 0$$

$C^k = \mathbb{Q}[x_1 \dots x_k]$ $\qquad d^k : C^k \to C^{k+1}$

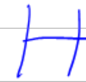
$d^k = \sum_{j=0}^{k+1} (-1)^j d^k_j$ $\qquad d^k_j$: "insert variable in position $j$"

claim $d^k // d^{k+1} = 0$

$$\text{Pentagon}_d[\Phi\_] := \mathrm{IM}_d\Big[\Phi /\!/ s\eta_2,\ \Phi /\!/ s\sigma_{1\to 2} /\!/ p\Delta_{y\to y,z} /\!/ p2s_{z\to 1},\ \underbrace{\Phi /\!/ s\sigma_{1\to 2} /\!/ p2s_{y\to 1} /\!/ p\sigma_{x\to y} /\!/ p\eta_x}_{d_0^3}\Big] - \mathrm{IM}_d\Big[\underbrace{\Phi /\!/ s\sigma_{1\to 2} /\!/ p2s_{y\to 1} /\!/ p\Delta_{x\to x,y}}_{d_1^3},\ \Phi /\!/ s\Delta_{1\to 1,2}\Big]$$




$d^3 : \mathcal{A}_{pps} \to \mathcal{A}_{ppss}$ $\quad d^3 = \sum_{j=0}^{4} (-1)^j d_j^3$ $\quad \mathcal{A}_{pps} = FA\langle x,y\rangle = \{\varphi\}$



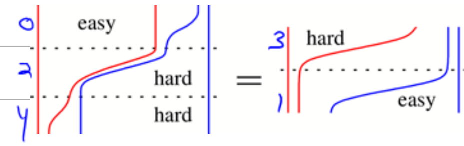
$d_0^3 \varphi = 1\otimes \varphi(y,0) + t\,(\partial_y''\varphi)(y,0)\otimes(\partial_y'\varphi)(y,0)$ $\quad \mathcal{A}_{ppss} = FA\langle x,y\rangle^{\otimes 2} \oplus FA\langle x,y\rangle^{\otimes 2}[1]$

$d_0^3 x^k = 1\otimes y^k \quad d_0^3 x^k y x^l = t\, y^l \otimes y^k \quad d_0^3(\text{otherwise}) = 0$

$d_1^3 \varphi = 1\otimes \varphi(x+y,0) + t\,(\partial_y''\varphi)(x+y,0)\otimes(\partial_y'\varphi)(x+y,0)$

$d_1^3 x^k = 1\otimes (x+y)^k \quad d_1^3 x^k y x^l = t\,(x+y)^l \otimes (x+y)^k \quad d_1^3(\text{otherwise}) = 0$

$d_2^3 \varphi = 1\otimes\varphi + t\,(\partial_y''\varphi)\otimes(\partial_y'\varphi)$




$d_3^3 \varphi = \Delta\varphi /\!/ \mathcal{O}$

$d_4^3 \varphi = \varphi \otimes 1$

---

$d^n : \mathcal{A}_{pps^{n-2}} \to \mathcal{A}_{pps^{n-1}}$ $\quad d^n = \sum_{j=0}^{n+1} (-1)^j d_j^n$

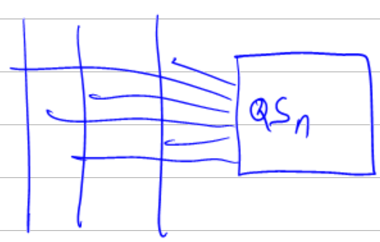
$d_0^n$: "Add pole on left" & convert

$d_1^n$: Double pole #1 & convert

$d_2^n$: Double pole #2 & convert

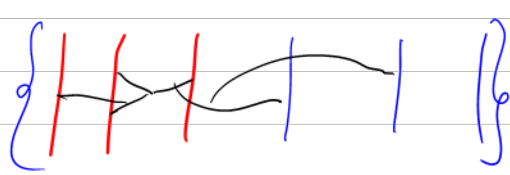
$d_j^n$ for $3\le j<n$ double strand $j-2$

$d_{n+1}^n$: add strand on right

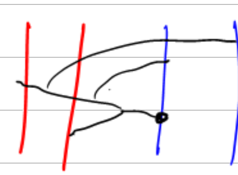



---

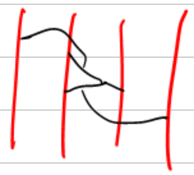
$d_{\{x,y,z\},\{1,2,3\}} =$



can touch blue at most twice.

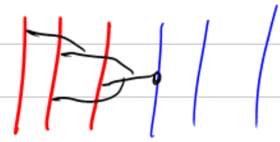


Doesn't touch blue

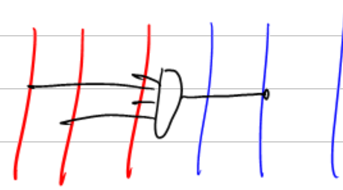


$sder_4$

Touches blue once

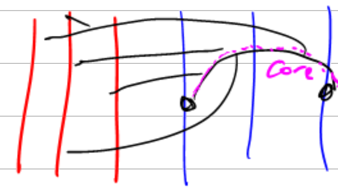


$\bigoplus_{ss} FL\langle Ps\rangle$



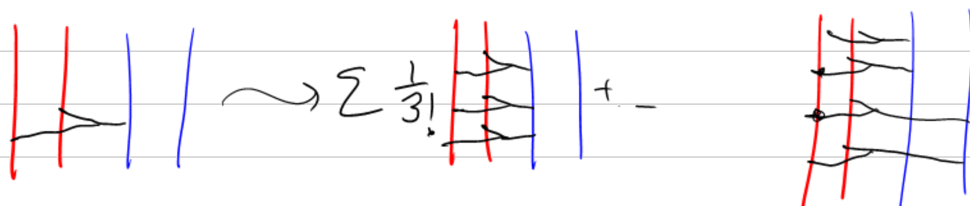
$\bigoplus_{ss} FA\langle Ps\rangle[1]$

Touches blue twice



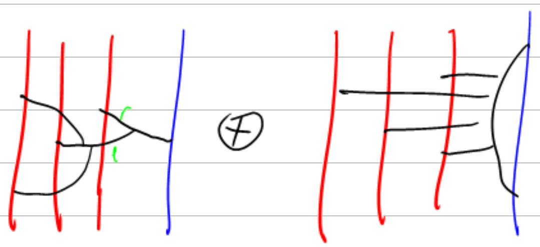

$\bigoplus_{p_1<p_2} FA\langle P_s\rangle[1]$

Div div




Claim There is a funny Lie algebra structure on $\overset{\text{Funny}:=}{FL\langle PS\rangle \oplus FA\langle PS\rangle[1]}$:

not canonical.

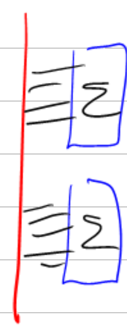


$$0 \to \underset{\text{Abelian}}{FA[1]} \to \text{Funny} \to FL \to 0$$

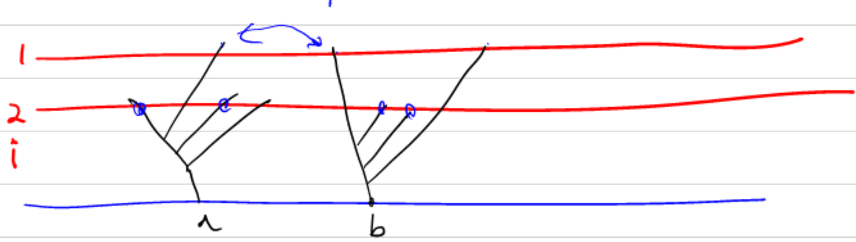
$\partial_i : FL(x_1 \ldots x_n) \xrightarrow[-1]{\deg} FA(x_1 \ldots x_n)$

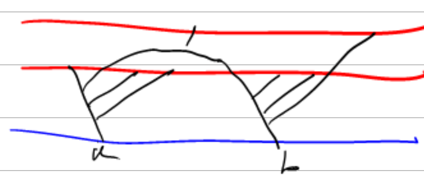
$\partial_i x_j = \delta_{ij}$

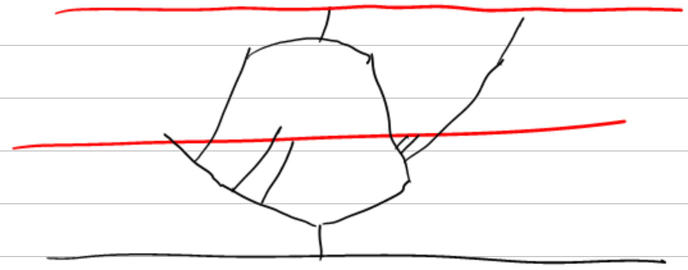
$\partial_i [a,b] = a\partial_i b - b\partial_i a$

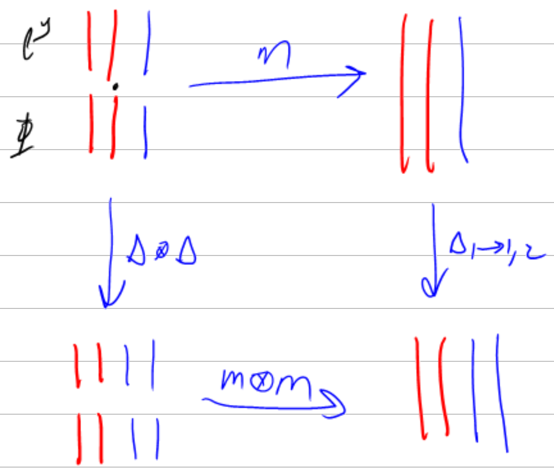


$$[a,b]_F := [a,b]_L + \sum_i (\partial_i a) x_i (\partial_i b)^* - (\partial_i b) x_i (\partial_i a)^*$$









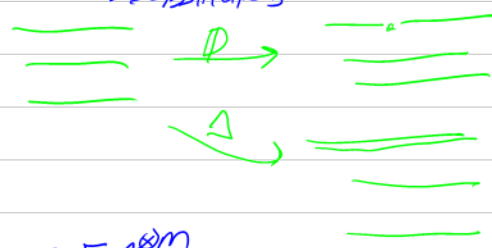

$\mathcal{D}$:

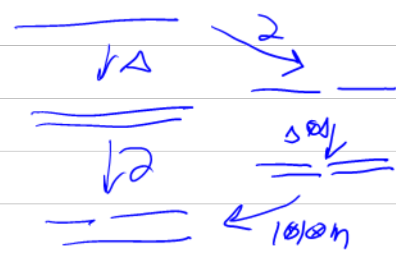
```
Sum[
 Expand[A (AWu1[p] AWu2[] - AWu1[] AWu2[p])] // 𝔻[p]i→i,ī //
        𝔻[p]ī→ī,ĩ // Δi→j,k // Δī→j̄,k̄ // Δĩ→j̃,k̃ // mj,j̄→j // σj̃→j̄ //
   mk̄,k̃→k̄ // mj,u1→j // mu2,j̄→j̄,
 {p, ps}]
```

230406 The AKKN operational envelope: Cuts, then doubles (and reversals), then infusion of constants, then merges.

substitutions

$x \mapsto x+y$
$y \mapsto 0$




$\mathcal{D}/\!/\mathcal{D} = \mathcal{D}/\!/\mathcal{D}/\!/\mathcal{D}$

$\partial\Delta m /\!/ \partial\Delta m$

$\partial\partial\Delta\Delta mm = \partial\partial\Delta\Delta mm$

$FA^{\otimes n} \longrightarrow FA^{\otimes m}$

$\partial\partial .. \Delta\Delta SS .. CCC\, mmm$

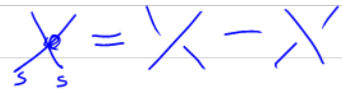
Theorem (FA-level) Any composition of the operations $\partial_x, \Delta, S, FA_{\text{substitutions}}, C, m$ can be written uniquely in this order:

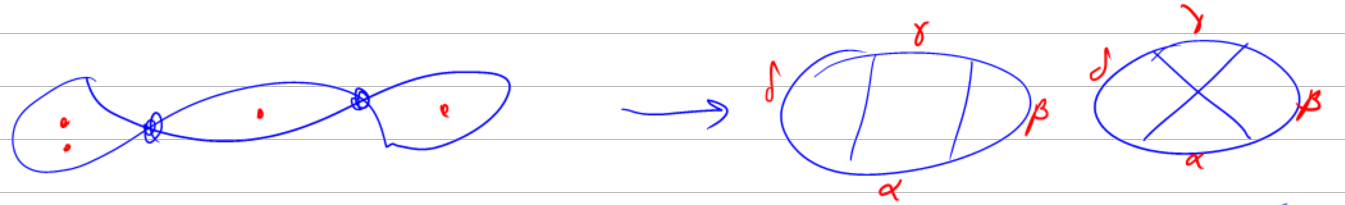
$$(\partial_x \partial_y|_{i\to j,k} \cdots)(\Delta\Delta\Delta \,..\, SS)(\text{substitutions})\, C\, (mmmm)$$

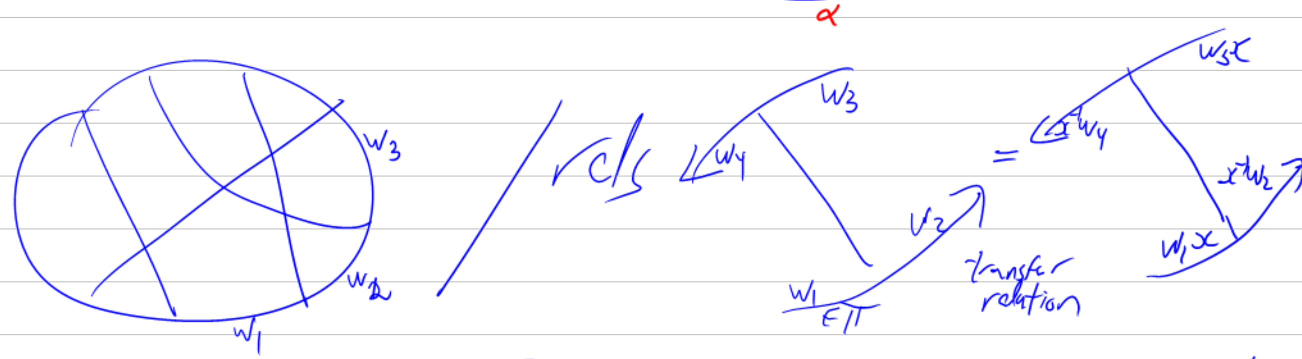
$$FA \xrightarrow{\text{subs}} FA \xrightarrow{\partial} FA\otimes FA$$

$\underline{x\,y\,y\,x\,x} \longrightarrow \underline{a\,b\,b\,a\,a} \longrightarrow$

$K^{(n)}$

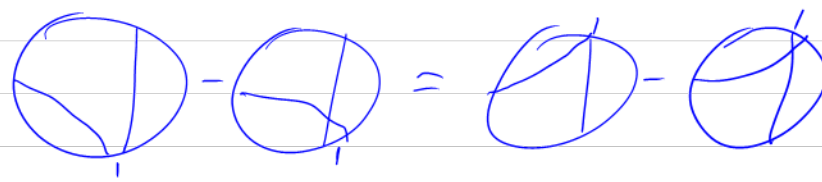


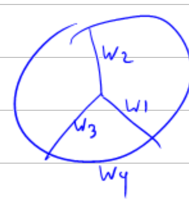


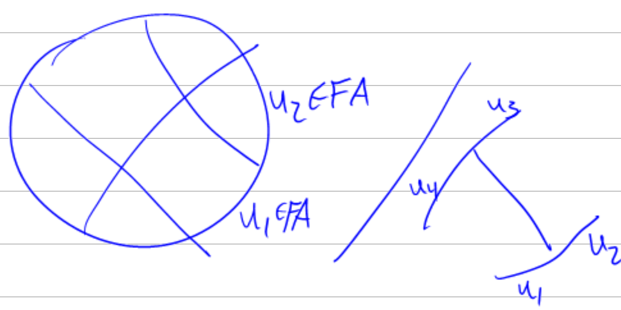
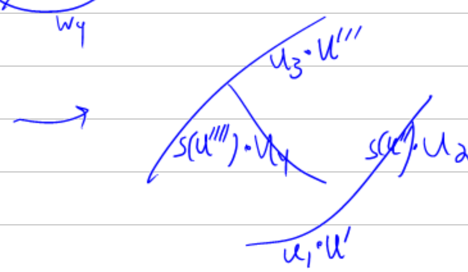



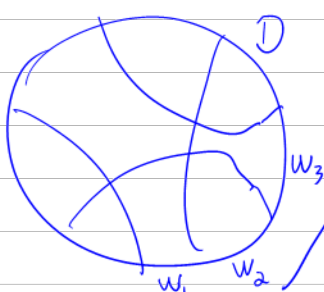

Q) Is there an expansion

$K(PDS_n) \longrightarrow A^{\pi}$






$\pi\Big($  $/\text{transfer}\Big) = \pi^{d+1}/\mathrm{Ad}\,\pi$



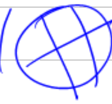
d-chords

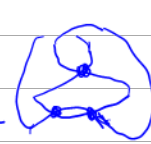
$\|$

punctured disk
n punctures

$[D : D_n]$

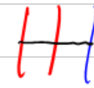
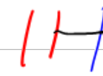
$D \sim$  $^{d+1}$

$\pi\Big($  $\Big)/\text{transfer} \cong$

$[$  $: D_n] = [$  $: D_n]$

$\mathcal{P} = \mathrm{span}\{$  ,  $\}/\text{rels}$

$\Psi : FL \longrightarrow \mathcal{P}$ ... symmetrization ...

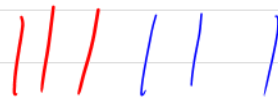
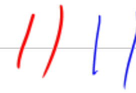
$\phi : FL(x,y) \longrightarrow \mathcal{P}$ Lie morphism $x \mapsto$  $y \mapsto$ 

$\phi(u) = u^{hor}$ $\Psi(u) = u^{sym}$

$\phi(u) - \Psi(u) =$ $u \in FL$ $u = [u_1, u_2]$

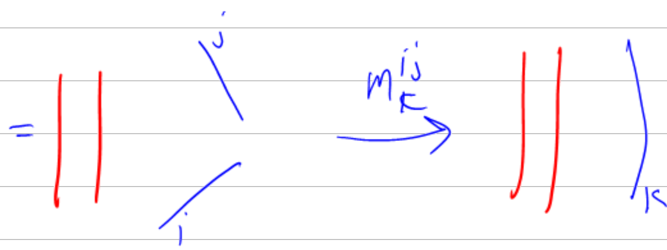
$\phi(u) = [\phi(u_1), \phi(u_2)]_F$

---

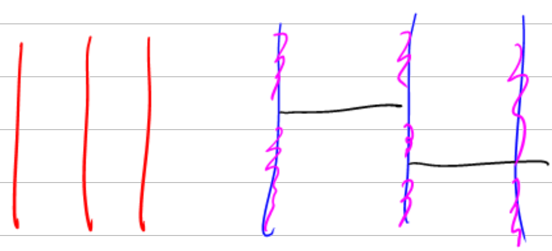
Do $[\ ]_F$ for   $\in$ 

---

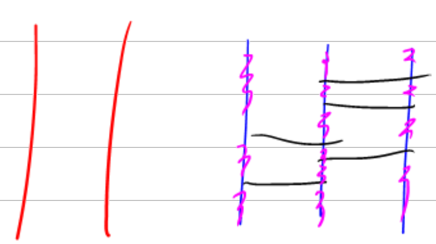
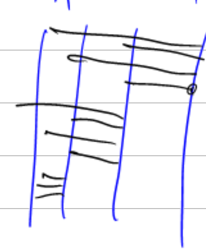
All order ops: $\Delta_p$ p2S, ..., $\Delta_s$, ...



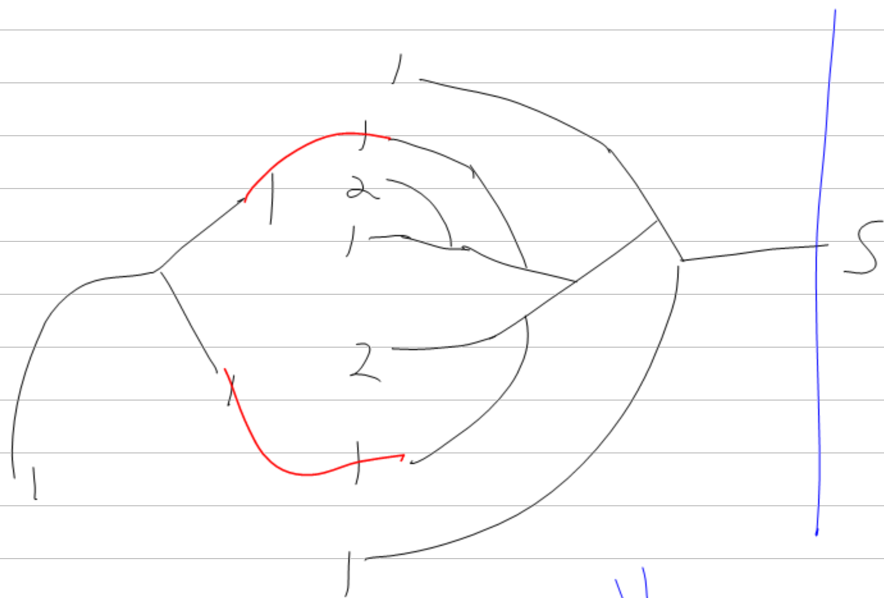

---

 $/3SS$

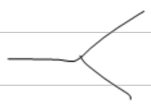
$DK_n = FA_{n-1} \boxtimes FA_{n-2} \boxtimes FA_{n-3} \ldots$

 $DK_{FA}$ 

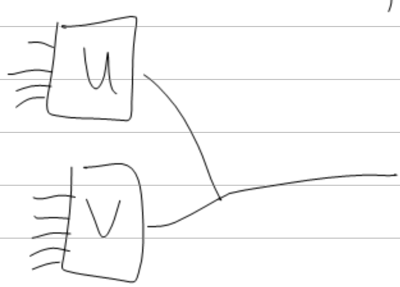
$$R(u) = \frac{1}{2}\sum_i \langle Y^{i\ i}_{x_i}, u \rangle$$

$\frac{1}{2}$ 


$=$ 



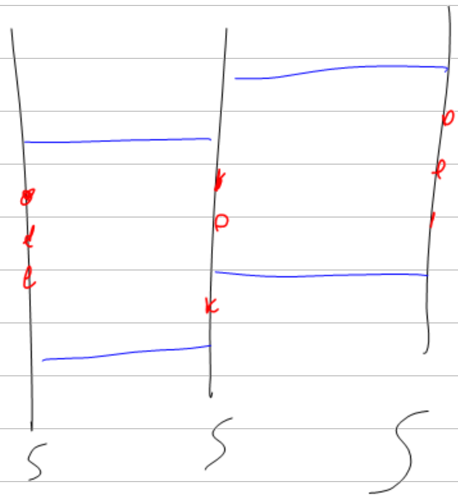
fork



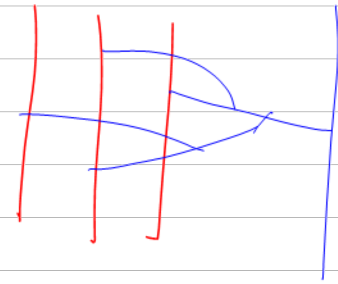

$$\Psi \mapsto \Psi_1 \qquad R \mapsto R_{11}$$

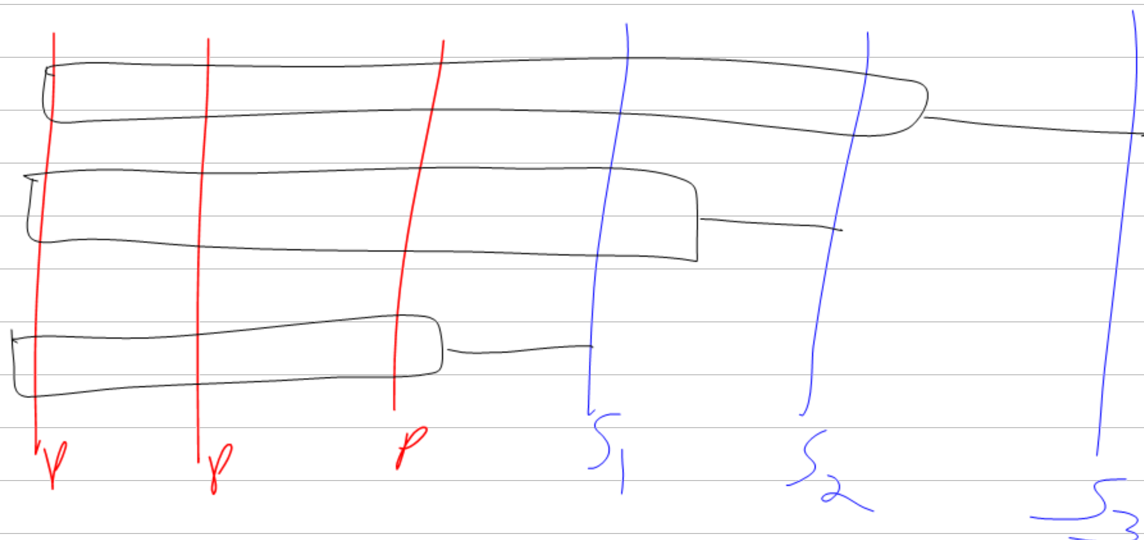
$$L_1 \quad L_2 \quad A_{11} \quad A_{12} \quad A_{22}$$

"Computations for emergent knots in a pole dancing studio".




$FL \to P(pps)$



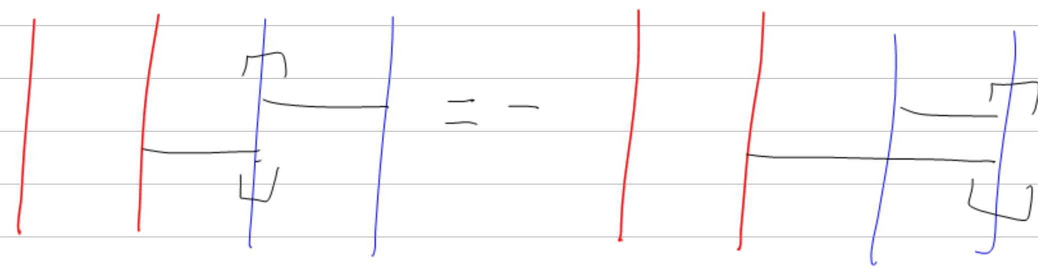


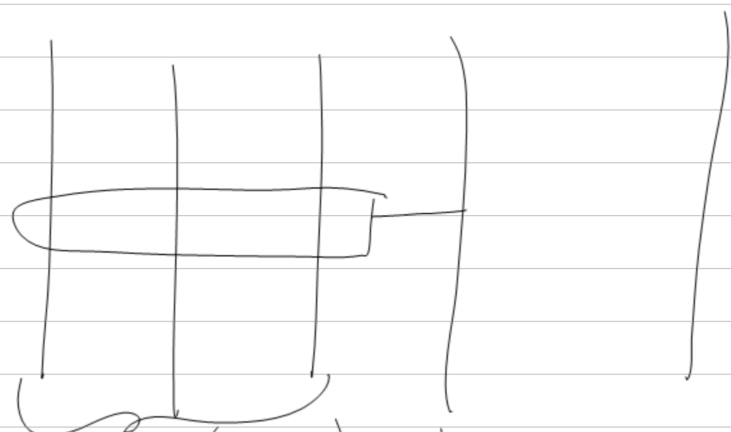

$$FL(p_1 p_2 p_3) \ltimes FL(p_1 p_2 p_3 s_1)_{\deg s_1 \leq 1}$$

$$\ltimes FL(p_1 p_2 p_3 s_1 s_2)_{\deg s_1 + \deg s_2 \leq 1}$$

$$[\underbrace{a_{12}}_{FL_2}, \underbrace{a_{13}}_{FL_3}] = -[a_{23}, a_{13}] \in FL_3$$

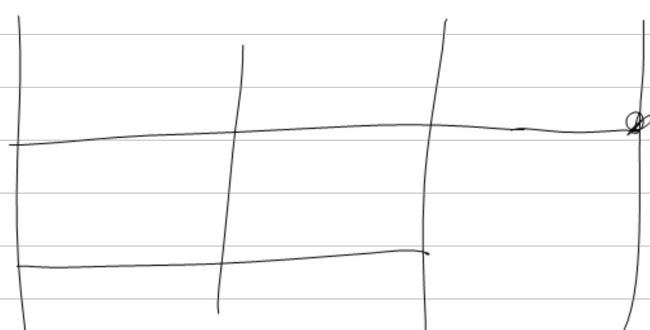
$$[\underbrace{a_{12}}_{FL_2}, \underbrace{a_{23}}_{FL_3}] = -[a_{13}, a_{23}] \in FL_3$$





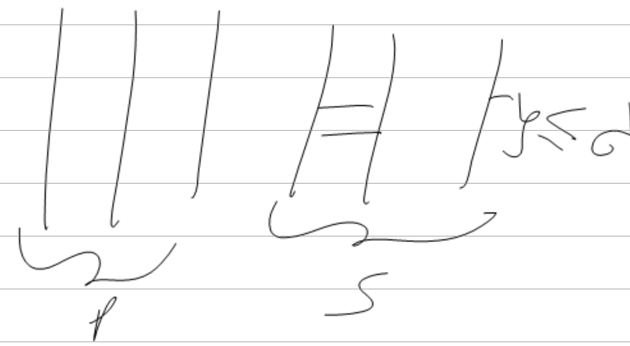
$FL(k-1) \qquad k$

$$DK_n \underset{v.s.}{=} \bigoplus_{k=2}^{n} FL_{k-1}$$



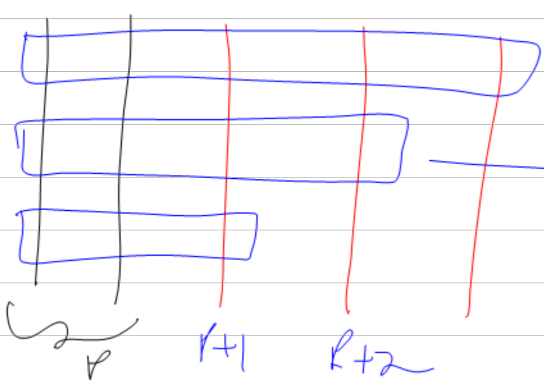
$i_1 \qquad i_2 \qquad k_1 < k_2$

$DK^{d,t}_{p,s}$



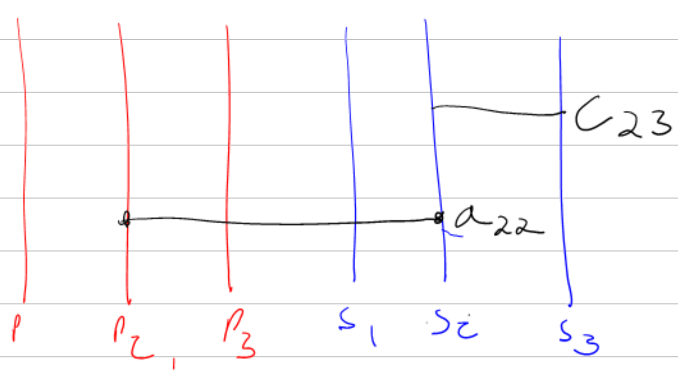

$$DK^{t,t} \to \cdots \to DK^{2,*} \to DK^{1,*} \to DK^{0,*}$$
$$DK^{0,*} = (FL_p)^s$$

$$DK^{1,*}_{p,s} \overset{?}{=} FL_p \ltimes \cancel{FA_p^{s-1}[1]}$$



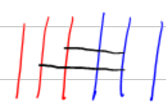

$\to FL_p \oplus (FA_p[1])^{\oplus 2}$

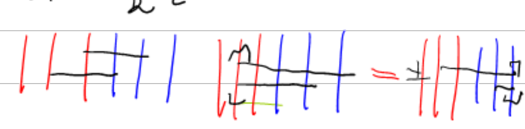
$\to FL_p \oplus FA_p[1]$



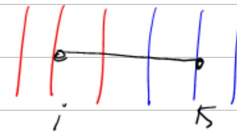

$$DK^{1,*}_{p,s} = FL\langle a_{ij}\ \ 1\le i\le p\ \ 1\le j\le s,\ \ c_{ij}\ \ 1\le i,j\le s\rangle$$
$$/\,4T,\ \deg_c \le 1$$

$[a_{i_1j_1}, a_{i_2j_2}] =$





$L_k : FL(x_1 \dots x_p) \longrightarrow DK_k \qquad x_i \to a_{ik}$



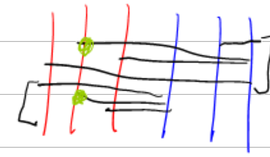

$A_{jk} : FA(x_1 \dots x_p)[1] \longrightarrow DK \qquad W = x_{i_1} x_{i_2} \dots x_{i_\alpha}$

$\longrightarrow [a_{i_1 k} \dots [a_{i_\alpha k} c_{jk}]]$

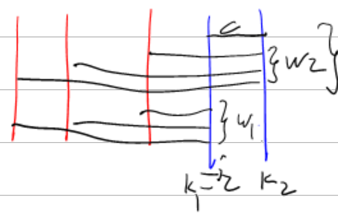
<u>Claim</u> $DK = \bigoplus_k \operatorname{im} L_k \oplus \bigoplus_{j<k} \operatorname{im} A_{jk} = \bigoplus_k \left[ \operatorname{im} L_k \oplus \bigoplus_{j<k} \operatorname{im} A_{jk} \right]$

<u>Claim</u> $[A_{j_1 k_1}(W_1), A_{j_2 k_2}(W_2)] = 0$ obvious

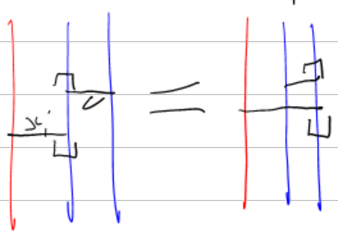
$[L_{k_1}(W_1), A_{j_2 k_2}(W_2)] =$



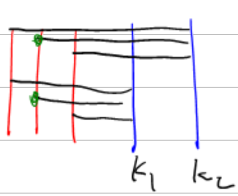
$$\begin{cases} A_{j_2 k_2}(W_2 W_1) & k_1 = j_2 < k_2 \\ 0 & k_1 \neq j_2 < k_2 \\ A_{j_2 k_2}(W_1 W_2) & k_1 = k_2 \\ 0 & k_1 > k_2 \end{cases}$$

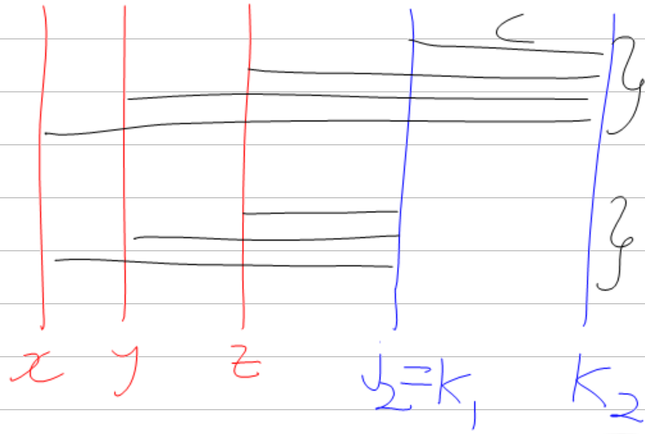



$A_{j_2 k_2}(w_2)$



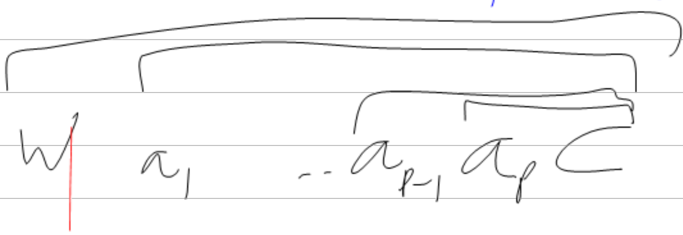

$k_1 \leq k_2 \qquad [L_{k_1}(W_1), L_{k_2}(W_2)] = \begin{cases} L_{k_1}([W_1, W_2]) & k_1 = k_2 \\ A_{k_1 k_2}\left( \sum_i (\partial_i W_1) x_i (\partial_i W_2) \right) & k_1 < k_2 \end{cases}$

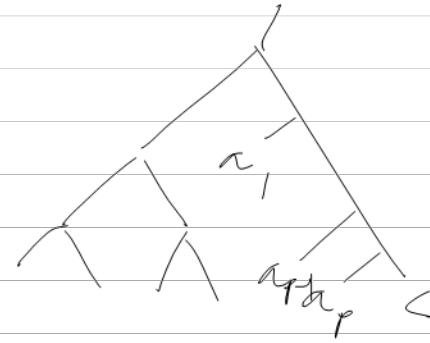




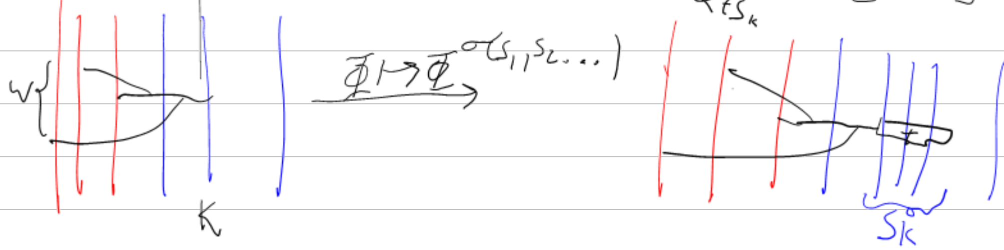

$\} A_{j_2k_2}[a_1 a_2 \dots a_p]$

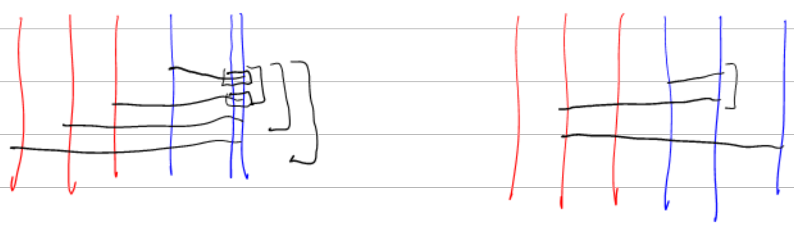
$\} L_{k_1}[\underbrace{b_1 \dots b_q}_{w}]$

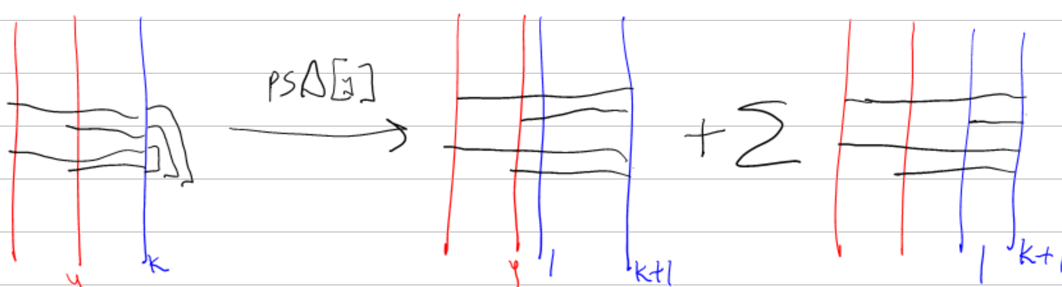



$EDK[k, w]$ $\qquad\qquad$ $\sum_{\alpha \in S_k} EDK[\alpha, W]$






$L_k(W) \longrightarrow L_k(W) \quad \pm \quad A_{1k+1}(T_y W)$

$A_{jk}(W) \longrightarrow A_{jk+1}(W)$