Pensieve Header: The Algebra of Emergent Chord Diagrams.

Goal: Implement $\mathcal{O}_{red,ps:\{\},ss:\{\}}\Big[\mathcal{A}_0[\prod_{s\in ss} AW_s[\,..]]$,
$+\sum_{s1\le s2}\mathcal{A}_{c[s1,s2]}\Big[\prod_{s\in ss\cup\{\overline{s1},\overline{s2}\}} AW_s[\,..]\Big]\Big]$
including $\otimes$, $m_{i,j\to k}$ (only if $\{i,\ j\}$ are neighbors), $\mathcal{O}_{ss}$, CF (Canonical Form) and HCF (HOMFLYPT Canonical Form).

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Kuno"];
<< FreeLie.m
<< AwCalculus.m
<< FAA.m
```

```
FreeLie` implements / extends
 {*, +, **, $SeriesShowDegree, 〈〉, ∫, ≡, ad, Ad, adSeries, AllCyclicWords, AllLyndonWords,
  AllWords, Arbitrator, AS, ASeries, AW, b, BCH, BooleanSequence, BracketForm, BS, CC, Crop,
  cw, CW, CWS, CWSeries, D, Deg, DegreeScale, DerivationSeries, div, DK, DKS, DKSeries, EulerE,
  Exp, Inverse, j, J, JA, LieDerivation, LieMorphism, LieSeries, LS, LW, LyndonFactorization,
  Morphism, New, RandomCWSeries, Randomizer, RandomLieSeries, RC, SeriesSolve, Support,
  t, tb, TopBracketForm, tr, UndeterminedCoefficients, αMap, Γ, ι, Λ, σ, ħ, ⟌, ⟍}.

FreeLie` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150814.

AwCalculus` implements / extends {*, **, ≡, dA, dc, deg, dm, dS, dΔ, dη, dσ, El, Es, hA,
  hm, hS, hΔ, hη, hσ, RandomElSeries, RandomEsSeries, tA, tha, tm, tS, tΔ, tη, tσ, Γ, Λ}.

AwCalculus` is in the public domain. Dror Bar-Natan is committed
  to support it within reason until July 15, 2022. This is version 150909.
```

In[•]:=
```
X = LW@"x"; Y = LW@"y"; U = LW@"u";
{F = LS[{X, Y}, Fs], G = LS[{X, Y}, Gs]}; Fs["y"] = 1 / 2;
SeriesSolve[{F, G},
  ħ^-1 (LS[X + Y] - BCH[Y, X] ≡ F - G - Ad[-X][F] + Ad[Y][G]) ∧
   div_X[F] + div_Y[G] ≡ 1/2 tr_U[adSeries[ad/(e^ad - 1), X][U] +
       adSeries[ad/(e^ad - 1), Y][U] - adSeries[ad/(e^ad - 1), BCH[X, Y]][U]]];
{F, G}
```

Out[•]=
$$\left\{LS\left[\frac{\overline{y}}{2},\ \frac{\overline{xy}}{6},\ \frac{1}{24}\,\overline{\overline{xy}\,y},\ \ldots\right],\ LS\left[0,\ \frac{\overline{xy}}{12},\ \frac{1}{24}\,\overline{\overline{xy}\,y},\ \ldots\right]\right\}$$

```
In[•]:= Basis_d_[𝕆AR,ps_,ss_] := 𝕆AR,ps,ss /@ Flatten[{
          𝒜0 /@ Basis_d,ps[Product[AWs, {s, ss}]],
          Table[
           𝒜c[ss⟦i⟧] /@ (Basis_d-1,ps[Product[AWs, {s, ss}] AW⁻ss⟦i⟧] AWss̃⟦i⟧[]), {i, Length[ss]}],
          Table[𝒜c[ss⟦i⟧,ss⟦j⟧] /@ (Basis_d-1,ps[Product[AWs, {s, ss}] AW⁻ss⟦i⟧] AW⁻ss⟦j⟧[]),
           {i, Length[ss] - 1}, {j, i + 1, Length@ss}]
         }]
```

```
In[•]:= Basis2[𝕆AR,{x,y},{1,2}]
```

```
Out[•]=
{𝕆AR,{x,y},{1,2}[𝒜0[AW1[] AW2[x, x]]], 𝕆AR,{x,y},{1,2}[𝒜0[AW1[] AW2[x, y]]],
 𝕆AR,{x,y},{1,2}[𝒜0[AW1[] AW2[y, x]]], 𝕆AR,{x,y},{1,2}[𝒜0[AW1[] AW2[y, y]]],
 𝕆AR,{x,y},{1,2}[𝒜0[AW1[x] AW2[x]]], 𝕆AR,{x,y},{1,2}[𝒜0[AW1[x] AW2[y]]],
 𝕆AR,{x,y},{1,2}[𝒜0[AW1[y] AW2[x]]], 𝕆AR,{x,y},{1,2}[𝒜0[AW1[y] AW2[y]]],
 𝕆AR,{x,y},{1,2}[𝒜0[AW1[x, x] AW2[]]], 𝕆AR,{x,y},{1,2}[𝒜0[AW1[x, y] AW2[]]],
 𝕆AR,{x,y},{1,2}[𝒜0[AW1[y, x] AW2[]]], 𝕆AR,{x,y},{1,2}[𝒜0[AW1[y, y] AW2[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[1][AW1[] AW2[] AW1̄[x] AW1̃[]]], 𝕆AR,{x,y},{1,2}[𝒜c[1][AW1[] AW2[] AW1̄[y] AW1̃[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[1][AW1[] AW2[x] AW1̄[] AW1̃[]]], 𝕆AR,{x,y},{1,2}[𝒜c[1][AW1[] AW2[y] AW1̄[] AW1̃[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[1][AW1[x] AW2[] AW1̄[] AW1̃[]]], 𝕆AR,{x,y},{1,2}[𝒜c[1][AW1[y] AW2[] AW1̄[] AW1̃[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[2][AW1[] AW2[] AW2̄[x] AW2̃[]]], 𝕆AR,{x,y},{1,2}[𝒜c[2][AW1[] AW2[] AW2̄[y] AW2̃[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[2][AW1[] AW2[x] AW2̄[] AW2̃[]]], 𝕆AR,{x,y},{1,2}[𝒜c[2][AW1[] AW2[y] AW2̄[] AW2̃[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[2][AW1[x] AW2[] AW2̄[] AW2̃[]]], 𝕆AR,{x,y},{1,2}[𝒜c[2][AW1[y] AW2[] AW2̄[] AW2̃[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[1,2][AW1[] AW2[] AW1̄[x] AW2̄[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[1,2][AW1[] AW2[] AW1̄[y] AW2̄[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[1,2][AW1[] AW2[x] AW1̄[] AW2̄[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[1,2][AW1[] AW2[y] AW1̄[] AW2̄[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[1,2][AW1[x] AW2[] AW1̄[] AW2̄[]]],
 𝕆AR,{x,y},{1,2}[𝒜c[1,2][AW1[y] AW2[] AW1̄[] AW2̄[]]]}
```

```
In[•]:= 𝒜a_[A1_] + 𝒜a_[A2_] ^:= 𝒜a[A1 + A2];
        c_ * 𝒜a_[A_] ^:= 𝒜[Expand[c A]];
        𝒜_[0] = 0;
```

```
In[•]:= CF[𝕆red_,ps_,ss_[x_Plus]] := 𝕆red,ps,ss[red /@ x];
        CF[𝕆red_,ps_,ss_[x_]] := 𝕆red,ps,ss[red@x]
```

```
In[•]:= AR[0] = 0;
        AR[𝒜0[A_]] := 𝒜0[A];
        AR[𝒜c[s_][A_]] :=
          Module[{l, r}, 𝒜c[s][A // Δs̃→l,r // ms̄,l→s̄ // Δr→l,r // ms,r→s // S1→l // m1,s̄→s̄ // ηs̃]];
        AR[𝒜c[s1_,s2_][A_]] := Module[{l, r},
           𝒜c[s1,s2][A // Δs2̄→l,r // ms2,r→s2 // Δl→l,r // ms1,r→s1 // S1→l // m1,s1̄→s1̄ // ηs2̄]];
```

```
In[•]:= HR[0] = 0;
        HR[𝒜₀[A_]] := 𝒜₀[A];
        HR[𝒜c[s_][A_]] := Module[{l, r}, 𝒜c[s][A // m~s,s→s // tr~s→s // η~s]];
        HR[𝒜c[s1_,s2_][A_]] := 𝒜c[s1,s2][A // m~s1,s2→s1 // m~s2,s1→s2 // η~s1 // η~s2];

In[•]:= D1 = 𝕆AR,{x,y,z},{1,2}[
          𝒜₀[AW₁[x, y, x] AW₂[x, x, y]] +
           𝒜c[1,2][AW₁[x, y] AW₂[y, x] AW1̄[z] AW2̄[x, y]]
          ] // CF

Out[•]=
        𝕆AR,{x,y,z},{1,2}[𝒜₀[AW₁[x, y, x] AW₂[x, x, y]] +
          𝒜c[1,2][AW₁[x, y, x, y] AW₂[y, x] AW1̄[z] AW2̄[] + AW₁[x, y, y] AW₂[y, x, x] AW1̄[z] AW2̄[] +
            AW₁[x, y, x] AW₂[y, x, y] AW1̄[z] AW2̄[] + AW₁[x, y] AW₂[y, x, x, y] AW1̄[z] AW2̄[] -
            AW₁[x, y, y] AW₂[y, x] AW1̄[x, z] AW2̄[] - AW₁[x, y] AW₂[y, x, y] AW1̄[x, z] AW2̄[] -
            AW₁[x, y, x] AW₂[y, x] AW1̄[y, z] AW2̄[] - AW₁[x, y] AW₂[y, x, x] AW1̄[y, z] AW2̄[] +
            AW₁[x, y] AW₂[y, x] AW1̄[y, x, z] AW2̄[]]]

In[•]:= D2 = 𝕆HR,{x,y,z},{1,2}[
          𝒜₀[AW₁[x, y, x] AW₂[x, x, y]] +
           𝒜c[1,2][AW₁[x, y] AW₂[y, x] AW1̄[z] AW2̄[x, y]]
          ] // CF

Out[•]=
        𝕆HR,{x,y,z},{1,2}[𝒜₀[AW₁[x, y, x] AW₂[x, x, y]] + 𝒜c[1,2][AW₁[x, y, x, y] AW₂[y, x, z] AW1̄[] AW2̄[]]]
```

In[•]:=
```
𝒪ss_[𝒪red_,ps_,s0s_[y_]] /; FreeQ[y, 𝒜0] := CF@Module[{s1, s2},
    𝒪red,ps,ss[
     y /. 𝒜c[s1_,s2_][A1_] /;
        Position[ss, s1][[1, 1]] > Position[ss, s2][[1, 1]] :> red[𝒜c[s2,s1][A1]]
    ]];
𝒪ss_[𝒪red_,ps_,s0s_[𝒜0[A_] + y_.]] := CF@Module[{i, j, s1, s2, u1, u2},
   𝒪red,ps,ss[Plus[
     𝒜0[A],
     y /. 𝒜c[s1_,s2_][A1_] /;
        Position[ss, s1][[1, 1]] > Position[ss, s2][[1, 1]] :> red[𝒜c[s2,s1][A1]],
     Sum[
      If[Position[s0s, s1 = ss[[i]]][[1, 1]] < Position[s0s, s2 = ss[[j]]][[1, 1]], 0,
       Sum[
        red[𝒜c[s1,s2][Expand[A (AWu1[p] AWu2[] - AWu1[] AWu2[p])] // D[p]s1→s1,s1bar //
             D[p]s2→s2,s2bar // ms1,u1→s1 // ms2,u2→s2]],
        {p, ps}
       ]
      ],
      {i, Length[ss] - 1}, {j, i + 1, Length@ss}
     ]
    ]]]
```

In[•]:=
```
𝒪AR,{x,y},{1,2}[𝒜0[AW1[x, y, y] AW2[x]]] // 𝒪{2,1}
```

Out[•]=

$$\mathcal{O}_{AR,\{x,y\},\{2,1\}}\Big[\mathcal{A}_0[AW_1[x,y,y]\,AW_2[x]] + \mathcal{A}_{c[2,1]}[-AW_1[x,y,y]\,AW_2[]\,AW_{\bar1}[]\,AW_{\bar2}[] + AW_1[y,y]\,AW_2[x]\,AW_{\bar1}[]\,AW_{\bar2}[] - 2\,AW_1[x,y]\,AW_2[y]\,AW_{\bar1}[]\,AW_{\bar2}[] + 2\,AW_1[y]\,AW_2[x,y]\,AW_{\bar1}[]\,AW_{\bar2}[] - AW_1[x]\,AW_2[y,y]\,AW_{\bar1}[]\,AW_{\bar2}[] + AW_1[]\,AW_2[x,y,y]\,AW_{\bar1}[]\,AW_{\bar2}[] + 2\,AW_1[x,y]\,AW_2[]\,AW_{\bar1}[]\,AW_{\bar2}[y] - 2\,AW_1[y]\,AW_2[x]\,AW_{\bar1}[]\,AW_{\bar2}[y] + 2\,AW_1[x]\,AW_2[y]\,AW_{\bar1}[]\,AW_{\bar2}[y] - 2\,AW_1[]\,AW_2[x,y]\,AW_{\bar1}[]\,AW_{\bar2}[y] - AW_1[x]\,AW_2[]\,AW_{\bar1}[]\,AW_{\bar2}[y,y] + AW_1[]\,AW_2[x]\,AW_{\bar1}[]\,AW_{\bar2}[y,y]]\Big]$$

In[•]:=
```
Total@Table[B == (B // 𝒪{2,1,3} // 𝒪{1,2,3}), {B, Basis3[𝒪AR,{x,y},{1,2,3}]}]
```

Out[•]=

320 True

```
In[•]:= 𝕆red_,ps_,ss_[ℰ_] // m_{i_,j_→k_} :=
 CF@𝕆red,ps,{k}~Join~Complement[ss,{i,j}][First[𝕆red,ps,ss[ℰ] // 𝕆{i,j}~Join~Complement[ss,{i,j}]] /. {
     𝒜c[i,j][A_] :> 𝒜c[k][A // σ_{j→k̃} // m_{i,j→k̄} // σ_{i→k}],
     𝒜c[i][A_] :> 𝒜c[k][A // m_{i̲,j→k̃} // σ_{i→k} // σ_{i̲→k̄}],
     𝒜c[j][A_] :> 𝒜c[k][A // m_{i,j→k} // σ_{j→k̄} // σ_{j̲→k̃}],
     𝒜c[i,x_][A_] :> 𝒜c[k,x][A // m_{i̲,j→k̄} // σ_{i→k}],
     𝒜c[j,x_][A_] :> 𝒜c[k,x][A // m_{i,j→k} // σ_{j̲→k̄}],
     𝒜a_[A_] :> 𝒜a[A // m_{i,j→k}]
    }]
```

```
In[•]:= Total@Table[(B // m_{1,2→1} // m_{1,3→1}) == (B // m_{2,3→2} // m_{1,2→1}), {B, Basis3[𝕆AR,{x,y},{1,2,3}]}]
Out[•]=
320 True
```

```
In[•]:= Total@Table[(B // m_{3,2→2} // m_{2,1→1}) == (B // m_{2,1→1} // m_{3,1→1}), {B, Basis3[𝕆AR,{x,y},{1,2,3}]}]
Out[•]=
320 True
```

```
In[•]:= lhs = 𝕆AR,{x,y},{1}[𝒜0[Sum[AW1 @@ Table[x, k] / k!, {k, 0, 4}] // FA[x → x + y]]]
Out[•]=
```

$$\mathbb{O}_{AR,\{x,y\},\{1\}}\left[\mathcal{A}_0\left[\mathrm{AW}_1[\,]+\mathrm{AW}_1[x]+\mathrm{AW}_1[y]+\frac{1}{2}\mathrm{AW}_1[x,x]+\frac{1}{2}\mathrm{AW}_1[x,y]+\right.\right.$$
$$\frac{1}{2}\mathrm{AW}_1[y,x]+\frac{1}{2}\mathrm{AW}_1[y,y]+\frac{1}{6}\mathrm{AW}_1[x,x,x]+\frac{1}{6}\mathrm{AW}_1[x,x,y]+\frac{1}{6}\mathrm{AW}_1[x,y,x]+$$
$$\frac{1}{6}\mathrm{AW}_1[x,y,y]+\frac{1}{6}\mathrm{AW}_1[y,x,x]+\frac{1}{6}\mathrm{AW}_1[y,x,y]+\frac{1}{6}\mathrm{AW}_1[y,y,x]+\frac{1}{6}\mathrm{AW}_1[y,y,y]+$$
$$\frac{1}{24}\mathrm{AW}_1[x,x,x,x]+\frac{1}{24}\mathrm{AW}_1[x,x,x,y]+\frac{1}{24}\mathrm{AW}_1[x,x,y,x]+\frac{1}{24}\mathrm{AW}_1[x,x,y,y]+$$
$$\frac{1}{24}\mathrm{AW}_1[x,y,x,x]+\frac{1}{24}\mathrm{AW}_1[x,y,x,y]+\frac{1}{24}\mathrm{AW}_1[x,y,y,x]+\frac{1}{24}\mathrm{AW}_1[x,y,y,y]+$$
$$\frac{1}{24}\mathrm{AW}_1[y,x,x,x]+\frac{1}{24}\mathrm{AW}_1[y,x,x,y]+\frac{1}{24}\mathrm{AW}_1[y,x,y,x]+\frac{1}{24}\mathrm{AW}_1[y,x,y,y]+$$
$$\left.\left.\frac{1}{24}\mathrm{AW}_1[y,y,x,x]+\frac{1}{24}\mathrm{AW}_1[y,y,x,y]+\frac{1}{24}\mathrm{AW}_1[y,y,y,x]+\frac{1}{24}\mathrm{AW}_1[y,y,y,y]\right]\right]$$

```
In[•]:= AWExp_d_[ℒ_] := Module[{t = AW[]},
   AW[] + Sum[t = Expand[t ** ℒ / k] /. a_AW /; Length[a] > d :> 0, {k, d}]
  ]
```

In[•]:=
```
rhs1 = AWExp4[Plus @@ ι[Ad[F][X]]@{4}] /. AW → AW1
```

Out[•]=

$$AW_1[\,] + AW_1[x] + \frac{1}{2} AW_1[x, x] - \frac{1}{2} AW_1[x, y] + \frac{1}{2} AW_1[y, x] + \frac{1}{6} AW_1[x, x, x] - \frac{5}{12} AW_1[x, x, y] + \frac{1}{3} AW_1[x, y, x] + \frac{1}{8} AW_1[x, y, y] + \frac{1}{12} AW_1[y, x, x] - \frac{1}{4} AW_1[y, x, y] + \frac{1}{8} AW_1[y, y, x] + \frac{1}{24} AW_1[x, x, x, x] - \frac{1}{6} AW_1[x, x, x, y] + \frac{1}{12} AW_1[x, x, y, x] + \frac{1}{16} AW_1[x, x, y, y] + \frac{1}{12} AW_1[x, y, x, x] - \frac{1}{48} AW_1[x, y, y, y] - \frac{1}{8} AW_1[y, x, x, y] + \frac{1}{16} AW_1[y, x, y, y] + \frac{1}{16} AW_1[y, y, x, x] - \frac{1}{16} AW_1[y, y, x, y] + \frac{1}{48} AW_1[y, y, y, x]$$

In[•]:=
```
rhs2 = AWExp4[Plus @@ ι[Ad[G][Y]]@{4}] /. AW → AW2
```

Out[•]=

$$AW_2[\,] + AW_2[y] + \frac{1}{2} AW_2[y, y] + \frac{1}{12} AW_2[x, y, y] - \frac{1}{6} AW_2[y, x, y] + \frac{1}{12} AW_2[y, y, x] + \frac{1}{6} AW_2[y, y, y] + \frac{1}{12} AW_2[x, y, y, y] - \frac{1}{6} AW_2[y, x, y, y] + \frac{1}{12} AW_2[y, y, x, y] + \frac{1}{24} AW_2[y, y, y, y]$$

In[•]:=
```
(Expand[rhs1 * rhs2] // m1,2→1) /. AW1[w___] /; Length@{w} > 4 :→ 0
```

Out[•]=

$$AW_1[\,] + AW_1[x] + AW_1[y] + \frac{1}{2} AW_1[x, x] + \frac{1}{2} AW_1[x, y] + \frac{1}{2} AW_1[y, x] + \frac{1}{2} AW_1[y, y] + \frac{1}{6} AW_1[x, x, x] + \frac{1}{12} AW_1[x, x, y] + \frac{1}{3} AW_1[x, y, x] + \frac{5}{24} AW_1[x, y, y] + \frac{1}{12} AW_1[y, x, x] + \frac{1}{12} AW_1[y, x, y] + \frac{5}{24} AW_1[y, y, x] + \frac{1}{6} AW_1[y, y, y] + \frac{1}{24} AW_1[x, x, x, x] + \frac{1}{12} AW_1[x, x, y, x] - \frac{1}{48} AW_1[x, x, y, y] + \frac{1}{12} AW_1[x, y, x, x] + \frac{1}{6} AW_1[x, y, x, y] + \frac{1}{12} AW_1[x, y, y, x] + \frac{5}{48} AW_1[x, y, y, y] - \frac{1}{24} AW_1[y, x, x, y] - \frac{5}{48} AW_1[y, x, y, y] + \frac{1}{16} AW_1[y, y, x, x] + \frac{7}{48} AW_1[y, y, x, y] + \frac{1}{48} AW_1[y, y, y, x] + \frac{1}{24} AW_1[y, y, y, y]$$