Pensieve Header: The Algebra of Emergent Chord Diagrams.

Goal: Implement $\mathcal{O}_{\text{red,ps:\{\},ss:\{\}}}\Big[\mathcal{A}_0[\prod_{s\in ss} AW_s[\,..]]$,
$+\sum_{s1\le s2}\mathcal{A}_{t[s1,s2]}\Big[\prod_{s\in ssU\{\overline{s1},\overline{s2}\}} AW_s[\,..]\Big]\Big]$
including $\otimes$, $m_{i,j\to k}$ (only if $\{i,\,j\}$ are neighbors), $\mathcal{O}_{ss}$, CF (Canonical Form) and HCF (HOMFLYPT Canonical Form).

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Kuno"];
<< FAA.m
```

In[•]:=
```
𝒜a_[A1_] + 𝒜a_[A2_] ^:= 𝒜a[A1 + A2];
c_ * 𝒜a_[A_] ^:= 𝒜[Expand[c A]]
```

In[•]:=
```
CF[𝒪red_,ps_,ss_[x_Plus]] := 𝒪red,ps,ss[red /@ x];
CF[𝒪red_,ps_,ss_[x_]] := 𝒪red,ps,ss[red@x]
```

In[•]:=
```
AR[𝒜0[A_]] := 𝒜0[A];
AR[𝒜t[s_][A_]] :=
  Module[{l, r}, 𝒜t[s][A // Δs̃→l,r // ms̄,l→s̄ // Δr→l,r // ms,r→s // S1→l // ml,s̄→s̄ // ηs̃]];
AR[𝒜t[s1_,s2_][A_]] := Module[{l, r},
   𝒜t[s1,s2][A // Δs2̄→l,r // ms2,r→s2 // Δl→l,r // ms1,r→s1 // S1→l // ml,s1̄→s1̄ // ηs2̄]];
```

In[•]:=
```
HR[𝒜0[A_]] := 𝒜0[A];
HR[𝒜t[s_][A_]] := Module[{l, r}, 𝒜t[s][A // ms̃,s→s // trs̄→s̄ // ηs̃]];
HR[𝒜t[s1_,s2_][A_]] := 𝒜t[s1,s2][A // ms1,s2̄→s1 // ms2,s1̄→s2 // ηs1̄ // ηs2̄];
```

In[•]:=
```
D1 = 𝒪AR,{x,y,z},{1,2}[
    𝒜0[AW1[x, y, x] AW2[x, x, y]] +
     𝒜t[1,2][AW1[x, y] AW2[y, x] AW1̄[z] AW2̄[x, y]]
  ] // CF
```

Out[•]=
```
𝒪AR,{x,y,z},{1,2}[𝒜0[AW1[x, y, x] AW2[x, x, y]] +
  𝒜t[1,2][AW1[x, y, x, y] AW2[y, x] AW1̄[z] AW2̄[] + AW1[x, y, y] AW2[y, x, x] AW1̄[z] AW2̄[] +
    AW1[x, y, x] AW2[y, x, y] AW1̄[z] AW2̄[] + AW1[x, y] AW2[y, x, x, y] AW1̄[z] AW2̄[] -
    AW1[x, y, y] AW2[y, x] AW1̄[x, z] AW2̄[] - AW1[x, y] AW2[y, x, y] AW1̄[x, z] AW2̄[] -
    AW1[x, y, x] AW2[y, x] AW1̄[y, z] AW2̄[] - AW1[x, y] AW2[y, x, x] AW1̄[y, z] AW2̄[] +
    AW1[x, y] AW2[y, x] AW1̄[y, x, z] AW2̄[]]]
```

```
In[•]:= D2 = 𝕆HR,{x,y,z},{1,2}[
           𝒜0[AW1[x, y, x] AW2[x, x, y]] +
            𝒜t[1,2][AW1[x, y] AW2[y, x] AW1‾[z] AW2‾[x, y]]
          ] // CF
```

```
Out[•]=
𝕆HR,{x,y,z},{1,2}[𝒜0[AW1[x, y, x] AW2[x, x, y]] + 𝒜t[1,2][AW1[x, y, x, y] AW2[y, x, z] AW1‾[] AW2‾[]]]
```

```
In[•]:= 𝕆ss_[𝕆red_,ps_,s0s_[y_]] := CF@Module[{s1, s2},
           𝕆red,ps,ss[
            y /. 𝒜t[s1_,s2_][A1_] /;
                Position[ss, s1][[1, 1]] > Position[ss, s2][[1, 1]] :> red[𝒜t[s2,s1][A1]]
           ]];
         𝕆ss_[𝕆red_,ps_,s0s_[𝒜0[A_] + y_]] := CF@Module[{i, j, s1, s2, u1, u2},
           𝕆red,ps,ss[Plus[
            𝒜0[A],
            y /. 𝒜t[s1_,s2_][A1_] /;
                Position[ss, s1][[1, 1]] > Position[ss, s2][[1, 1]] :> red[𝒜t[s2,s1][A1]],
            Sum[
             If[Position[s0s, s1 = ss[[i]]][[1, 1]] < Position[s0s, s2 = ss[[j]]][[1, 1]], 0,
              Sum[
               red[𝒜t[s1,s2][Expand[A (AWu1[p] AWu2[] - AWu1[] AWu2[p])] // D[p]s1→s1,s1‾ //
                     D[p]s2→s2,s2‾ // ms1,u1→s1 // ms2,u2→s2]],
               {p, ps}
              ]
             ],
             {i, Length[ss] - 1}, {j, i + 1, Length@ss}
            ]
           ]]]
```

```
In[•]:= D1
```

```
Out[•]=
𝕆AR,{x,y,z},{1,2}[𝒜0[AW1[x, y, x] AW2[x, x, y]] +
  𝒜t[1,2][AW1[x, y, x, y] AW2[y, x] AW1‾[z] AW2‾[] + AW1[x, y, y] AW2[y, x, x] AW1‾[z] AW2‾[] +
    AW1[x, y, x] AW2[y, x, y] AW1‾[z] AW2‾[] + AW1[x, y] AW2[y, x, x, y] AW1‾[z] AW2‾[] -
    AW1[x, y, y] AW2[y, x] AW1‾[x, z] AW2‾[] - AW1[x, y] AW2[y, x, y] AW1‾[x, z] AW2‾[] -
    AW1[x, y, x] AW2[y, x] AW1‾[y, z] AW2‾[] - AW1[x, y] AW2[y, x, x] AW1‾[y, z] AW2‾[] +
    AW1[x, y] AW2[y, x] AW1‾[y, x, z] AW2‾[]]]
```

In[•]:=
```
D1 // 𝕆{2,1}
```

Out[•]=
```
𝕆AR,{x,y,z},{2,1}[𝒜0[AW1[x, y, x] AW2[x, x, y]] +
  𝒜t[2,1][-AW1[x, y, x] AW2[x, x] AW1‾[] AW2‾[] - AW1[x, y] AW2[x, x, x] AW1‾[] AW2‾[] +
    AW1[x, x] AW2[x, x, y] AW1‾[] AW2‾[] + AW1[x] AW2[x, x, y, x] AW1‾[] AW2‾[] +
    AW1[x, y] AW2[x, x] AW1‾[] AW2‾[x] - AW1[x] AW2[x, x, y] AW1‾[] AW2‾[x] -
    2 AW1[x, y, x] AW2[x] AW1‾[] AW2‾[y] + AW1[y, x] AW2[x, x] AW1‾[] AW2‾[y] -
    AW1[x, x] AW2[x, y] AW1‾[] AW2‾[y] + AW1[y] AW2[x, x, x] AW1‾[] AW2‾[y] +
    AW1[x] AW2[x, x, y] AW1‾[] AW2‾[y] - AW1[x] AW2[x, y, x] AW1‾[] AW2‾[y] +
    AW1[] AW2[x, x, y, x] AW1‾[] AW2‾[y] - 2 AW1[x, y, x] AW2[] AW1‾[] AW2‾[x, y] +
    AW1[x, y] AW2[x] AW1‾[] AW2‾[x, y] + AW1[y, x] AW2[x] AW1‾[] AW2‾[x, y] -
    AW1[x, x] AW2[y] AW1‾[] AW2‾[x, y] + 2 AW1[x] AW2[x, y] AW1‾[] AW2‾[x, y] -
    AW1[x] AW2[y, x] AW1‾[] AW2‾[x, y] + AW1[x, y, z] AW2[y, x] AW1‾[] AW2‾[x, y] -
    AW1[] AW2[x, x, y] AW1‾[] AW2‾[x, y] + AW1[] AW2[x, y, x] AW1‾[] AW2‾[x, y] +
    AW1[x, y] AW2[y, x, z] AW1‾[] AW2‾[x, y] + AW1[x, x] AW2[x] AW1‾[] AW2‾[y, y] -
    AW1[] AW2[x, x, x] AW1‾[] AW2‾[y, y] + AW1[x, y] AW2[] AW1‾[] AW2‾[x, x, y] -
    AW1[y] AW2[x] AW1‾[] AW2‾[x, x, y] + AW1[x] AW2[y] AW1‾[] AW2‾[x, x, y] -
    AW1[] AW2[x, y] AW1‾[] AW2‾[x, x, y] - AW1[x] AW2[x] AW1‾[] AW2‾[x, y, y] +
    AW1[] AW2[x, x] AW1‾[] AW2‾[x, y, y] + AW1[x, x] AW2[] AW1‾[] AW2‾[y, x, y] -
    AW1[] AW2[x, x] AW1‾[] AW2‾[y, x, y] - AW1[x, y] AW2[y, x] AW1‾[] AW2‾[z, x, y] -
    AW1[x] AW2[] AW1‾[] AW2‾[x, y, x, y] + AW1[] AW2[x] AW1‾[] AW2‾[x, y, x, y]]]
```

In[•]:=
```
D1 // 𝕆{2,1} // 𝕆{1,2}
```

Out[•]=
```
𝕆AR,{x,y,z},{1,2}[𝒜0[AW1[x, y, x] AW2[x, x, y]] +
  𝒜t[1,2][AW1[x, y, x, y] AW2[y, x] AW1‾[z] AW2‾[] + AW1[x, y, y] AW2[y, x, x] AW1‾[z] AW2‾[] +
    AW1[x, y, x] AW2[y, x, y] AW1‾[z] AW2‾[] + AW1[x, y] AW2[y, x, x, y] AW1‾[z] AW2‾[] -
    AW1[x, y, y] AW2[y, x] AW1‾[x, z] AW2‾[] - AW1[x, y] AW2[y, x, y] AW1‾[x, z] AW2‾[] -
    AW1[x, y, x] AW2[y, x] AW1‾[y, z] AW2‾[] - AW1[x, y] AW2[y, x, x] AW1‾[y, z] AW2‾[] +
    AW1[x, y] AW2[y, x] AW1‾[y, x, z] AW2‾[]]]
```

In[•]:=
```
(D1 // 𝕆{2,1} // 𝕆{1,2}) - D1
```

Out[•]=
```
0
```

In[•]:=
```
{D2, D2 // 𝕆{2,1}, D2 // 𝕆{2,1} // 𝕆{1,2}, (D2 // 𝕆{2,1} // 𝕆{1,2}) - D2}
```

Out[•]=
```
{𝕆HR,{x,y,z},{1,2}[𝒜0[AW1[x, y, x] AW2[x, x, y]] + 𝒜t[1,2][AW1[x, y, x, y] AW2[y, x, z] AW1‾[] AW2‾[]]],
 𝕆HR,{x,y,z},{2,1}[𝒜0[AW1[x, y, x] AW2[x, x, y]] + 𝒜t[2,1][-AW1[x, y, x, x, y] AW2[] AW1‾[] AW2‾[] +
     AW1[x, y, y] AW2[x, x] AW1‾[] AW2‾[] - AW1[x, x, y] AW2[y, x] AW1‾[] AW2‾[] -
     AW1[x, y] AW2[x, x, x] AW1‾[] AW2‾[] + AW1[x, y, x, y] AW2[y, x, z] AW1‾[] AW2‾[] +
     AW1[x] AW2[x, x, y, x] AW1‾[] AW2‾[] + AW1[y] AW2[x, x, y, x] AW1‾[] AW2‾[]]],
 𝕆HR,{x,y,z},{1,2}[𝒜0[AW1[x, y, x] AW2[x, x, y]] + 𝒜t[1,2][AW1[x, y, x, y] AW2[y, x, z] AW1‾[] AW2‾[]]],
 0}
```

```
In[•]:= 𝕆red_,ps_,ss_[ℰ_] // m_{i_,j_→k_} := CF@𝕆red,ps,{k}~Join~Complement[ss,{i,j}][
    Echo@First@Echo[𝕆red,ps,ss[ℰ] // 𝕆Echo[{i,j}~Join~Complement[ss,{i,j}]]] /. {
      𝒜t[i,j][A_] :↦ 𝒜t[k][A // σ_{j→k̃} // m_{i,j→k̄} // σ_{i→k}],
      𝒜t[i][A_] :↦ 𝒜t[k][A // m_{i,j→k̃} // σ_{i→k} // σ_{i→k̄}],
      𝒜t[j][A_] :↦ 𝒜t[k][A // m_{i,j→k} // σ_{j→k̄} // σ_{j→k̃}],
      𝒜t[i,x_][A_] :↦ 𝒜t[k,x][A // m_{i,j→k̄} // σ_{i→k}],
      𝒜t[j,x_][A_] :↦ 𝒜t[k,x][A // m_{i,j→k} // σ_{j→k̄}],
      𝒜a_[A_] :↦ 𝒜a[A // m_{i,j→k}]
     }]
```

```
In[•]:= D1 = 𝕆AR,{x,y},{1,2,3,4}[𝒜t[2,3][AW1[x] AW2[x, y] AW2̄[y] AW3[x] AW3̄[y, x] AW4[x, y]]]
```

```
Out[•]=
```

$$\mathbb{O}_{AR,\{x,y\},\{1,2,3,4\}}\left[\mathcal{A}_{t[2,3]}\left[AW_1[x]\,AW_2[x,y]\,AW_3[x]\,AW_4[x,y]\,AW_{\bar{2}}[y]\,AW_{\bar{3}}[y,x]\right]\right]$$

```
In[•]:= D1 // 𝕆{3,4,1,2}
```

```
Out[•]=
```

$$\mathbb{O}_{AR,\{x,y\},\{3,4,1,2\}}\Big[\mathcal{A}_{t[3,2]}\left[AW_1[x]\,AW_2[x,y,y]\,AW_3[x]\,AW_4[x,y]\,AW_{\bar{2}}[]\,AW_{\bar{3}}[y,x]+AW_1[x]\,AW_2[x,y]\,AW_3[x,y]\,AW_4[x,y]\,AW_{\bar{2}}[]\,AW_{\bar{3}}[y,x]-AW_1[x]\,AW_2[x,y]\,AW_3[x]\,AW_4[x,y]\,AW_{\bar{2}}[]\,AW_{\bar{3}}[y,y,x]\right]\Big]$$

```
In[•]:= D1 // m{3,4→5}
```

```
» {3, 4, 1, 2}
```

$$\mathbb{O}_{AR,\{x,y\},\{3,4,1,2\}}\Big[\mathcal{A}_{t[3,2]}\left[AW_1[x]\,AW_2[x,y,y]\,AW_3[x]\,AW_4[x,y]\,AW_{\bar{2}}[]\,AW_{\bar{3}}[y,x]+AW_1[x]\,AW_2[x,y]\,AW_3[x,y]\,AW_4[x,y]\,AW_{\bar{2}}[]\,AW_{\bar{3}}[y,x]-AW_1[x]\,AW_2[x,y]\,AW_3[x]\,AW_4[x,y]\,AW_{\bar{2}}[]\,AW_{\bar{3}}[y,y,x]\right]\Big]$$

$$\mathcal{A}_{t[3,2]}\left[AW_1[x]\,AW_2[x,y,y]\,AW_3[x]\,AW_4[x,y]\,AW_{\bar{2}}[]\,AW_{\bar{3}}[y,x]+AW_1[x]\,AW_2[x,y]\,AW_3[x,y]\,AW_4[x,y]\,AW_{\bar{2}}[]\,AW_{\bar{3}}[y,x]-AW_1[x]\,AW_2[x,y]\,AW_3[x]\,AW_4[x,y]\,AW_{\bar{2}}[]\,AW_{\bar{3}}[y,y,x]\right]$$

```
Out[•]=
```

$$\mathbb{O}_{AR,\{x,y\},\{5,1,2\}}\Big[\mathcal{A}_{t[5,2]}\left[AW_1[x]\,AW_2[x,y,y]\,AW_5[x]\,AW_{\bar{2}}[]\,AW_{\bar{5}}[y,x,x,y]+AW_1[x]\,AW_2[x,y]\,AW_5[x,y]\,AW_{\bar{2}}[]\,AW_{\bar{5}}[y,x,x,y]-AW_1[x]\,AW_2[x,y]\,AW_5[x]\,AW_{\bar{2}}[]\,AW_{\bar{5}}[y,y,x,x,y]\right]\Big]$$

```
Basis_d_[]
```