

Pensieve Header: The Algebra of Emergent Chord Diagrams.

Goal: Implement $\mathcal{O}_{red,ps:\{\},ss:\{\}}$ $\left[\mathcal{A}_0[\prod_{s \in SS} AW_s[\dots]], \right.$
 $\left. + \sum_{s1 \leq s2} \mathcal{A}_{t[s1,s2]} \left[\prod_{s \in SSU\{\overline{s1},\overline{s2}\}} AW_s[\dots] \right] \right]$

including $\otimes, m_{i,j \rightarrow k}$ (only if $\{i, j\}$ are neighbors), \mathcal{O}_{SS} , CF (Canonical Form) and HCF (HOMFLYPT Canonical Form).

```
In[*]:= SetDirectory["C:\\drorbn\\AcademicPensieve\\People\\Kuno"];
<< FAA.m
```

```
In[*]:= A_a[A1_] + A_a[A2_] ^:= A_a[A1 + A2];
c_ * A_a[A_] ^:= A[Expand[c A]]
```

```
In[*]:= CF[O_red,ps,ss[x_Plus]] := O_red,ps,ss[red/@x]
```

```
In[*]:= AR[A_0[A_]] := A_0[A];
AR[A_t[s_][A_]] :=
Module[{l, r}, A_t[s][A // Delta_s-1,r // m_s,1->s // m_t,r->t // Delta_t-1,r // m_s,r->s // S_1->1 // m_1,s->s]];
AR[A_t[s1,s2][A_]] := Module[{l, r},
A_t[s1,s2][A // Delta_s1-1,r // m_s1,1->s1 // S_r->r // m_r,t->t // Delta_s2-1,r // m_t,1->t // m_s2,r->s2]]];
```

In the cell below we are confusing the projection A->H with the canonical form in H!

```
HR[A_0[A_]] := A_0[A];
HR[A_t[s_][A_]] :=
Module[{l, r}, A_t[s][A // Delta_t-1,r // m_s,r->s // m_s,s->s // S_1->1 // m_1,s->s // tr_s->s]];
HR[A_t[s1,s2][A_]] :=
Module[{l, r}, A_t[s1,s2][A // Delta_t-1,r // m_s1,1->s1 // S_r->r // m_s2,r->s2 // m_s1,s2->s1 // m_s2,s1->s2]]];
```

```
In[*]:= D1 = O_AR,{x,y,z},{1,2} [
A_0[AW_1[x, y, x] AW_2[x, x, y]] +
A_t[1,2][AW_t[x] AW_1[x, y] AW_2[y, x] AW_t[z] AW_2[x, y]]
] // CF
```

```
Out[*]= O_AR,{x,y,z},{1,2} [A_0[AW_1[x, y, x] AW_2[x, x, y]] +
A_t[1,2][AW_1[x, y, z] AW_2[y, x, x, y] AW_t[x] + AW_1[x, y, z] AW_2[y, x, y] AW_t[x] +
AW_1[x, y, z] AW_2[y, x, x] AW_t[y] - AW_1[x, y] AW_2[y, x, x, y] AW_t[z] +
AW_1[x, y, z] AW_2[y, x] AW_t[x, y] - AW_1[x, y] AW_2[y, x, y] AW_t[z, x] -
AW_1[x, y] AW_2[y, x, x] AW_t[z, y] - AW_1[x, y] AW_2[y, x] AW_t[z, x, y]]]
```

```
In[*]:= D2 = OHR, {x,y,z}, {1,2} [
  A0[AW1[x, y, x] AW2[x, x, y]] +
  At[1,2][AW1[x, y] AW2[y, x] AW1[z] AW2[x, y]]
] // CF
```

Out[*]=

O_{HR, {x,y,z}, {1,2}} [A₀[AW₁[x, y, x] AW₂[x, x, y]] + A_t[1,2][AW₁[x, y, x, y] AW₂[y, x, z]]]

```
In[*]:= OSS[Ored,ps,s0s[A0[A_] + y_]] := CF@Module[{i, j, s1, s2, u1, u2},
  Ored,ps,ss[Plus[
    A0[A],
    y /. At[s1, s2][A1_] /;
    Position[ss, s1][[1, 1]] > Position[ss, s2][[1, 1]] => At[s2, s1][A1 // St->t],
    Sum[
      If[Position[s0s, s1 = ss[[i]]][[1, 1]] < Position[s0s, s2 = ss[[j]]][[1, 1]], 0,
      Sum[
        At[s1, s2][Expand[A (AWu1[p] AWu2[ ] - AWu1[ ] AWu2[p]) AWt[ ]]] // D[p]s1->s1, s1 //
        D[p]s2->s2, s2 // ms1, u1->s1 // ms2, u2->s2],
        {p, ps}
      ]
    ],
    {i, Length[ss] - 1}, {j, i + 1, Length@ss}
  ]
]]
```

In[*]:= D1

Out[*]=

O_{AR, {x,y,z}, {1,2}} [A₀[AW₁[x, y, x] AW₂[x, x, y]] +
 A_t[1,2][AW₁[x, y, z] AW₂[y, x, x, y] AW_t[] + AW₁[x, y, z] AW₂[y, x, y] AW_t[x] +
 AW₁[x, y, z] AW₂[y, x, x] AW_t[y] - AW₁[x, y] AW₂[y, x, x, y] AW_t[z] +
 AW₁[x, y, z] AW₂[y, x] AW_t[x, y] - AW₁[x, y] AW₂[y, x, y] AW_t[z, x] -
 AW₁[x, y] AW₂[y, x, x] AW_t[z, y] - AW₁[x, y] AW₂[y, x] AW_t[z, x, y]]]

In[*]:= **D1** // $\mathcal{O}_{\{2,1\}}$

Out[*]=

$$\begin{aligned} & \mathcal{O}_{AR, \{x,y,z\}, \{2,1\}} [\mathcal{A}_0 [AW_1 [x, y, x] AW_2 [x, x, y]] + \\ & \mathcal{A}_{t[2,1]} [-AW_1 [x, y, x] AW_2 [x, x] AW_t [] - 4 AW_1 [x, y, x] AW_2 [x, y] AW_t [] + \\ & AW_1 [x, x] AW_2 [x, x, y] AW_t [] + 2 AW_1 [x, y] AW_2 [x, x, y] AW_t [] + 2 AW_1 [y, x] AW_2 [x, x, y] AW_t [] + \\ & AW_1 [x, y, z] AW_2 [y, x, x, y] AW_t [] + 2 AW_1 [x, y, x] AW_2 [y] AW_t [x] - \\ & AW_1 [x, y] AW_2 [x, x] AW_t [x] - 3 AW_1 [x, y] AW_2 [x, y] AW_t [x] - AW_1 [y, x] AW_2 [x, y] AW_t [x] + \\ & AW_1 [x] AW_2 [x, x, y] AW_t [x] + 2 AW_1 [y] AW_2 [x, x, y] AW_t [x] - AW_1 [x, y, z] AW_2 [y, x, y] AW_t [x] + \\ & 4 AW_1 [x, y, x] AW_2 [x] AW_t [y] - 2 AW_1 [x, y] AW_2 [x, x] AW_t [y] - 2 AW_1 [y, x] AW_2 [x, x] AW_t [y] - \\ & 2 AW_1 [x, x] AW_2 [x, y] AW_t [y] + 2 AW_1 [x] AW_2 [x, x, y] AW_t [y] - AW_1 [x, y, z] AW_2 [y, x, x] AW_t [y] + \\ & AW_1 [x, y] AW_2 [y, x, x, y] AW_t [z] + AW_1 [x, y] AW_2 [y] AW_t [x, x] - AW_1 [y] AW_2 [x, y] AW_t [x, x] + \\ & AW_1 [x, x] AW_2 [y] AW_t [x, y] - AW_1 [x] AW_2 [x, y] AW_t [x, y] - AW_1 [x, y] AW_2 [y, x, y] AW_t [x, z] - \\ & 2 AW_1 [x, y, x] AW_2 [] AW_t [y, x] + 3 AW_1 [x, y] AW_2 [x] AW_t [y, x] + AW_1 [y, x] AW_2 [x] AW_t [y, x] - \\ & 2 AW_1 [y] AW_2 [x, x] AW_t [y, x] - 2 AW_1 [x] AW_2 [x, y] AW_t [y, x] + AW_1 [x, y, z] AW_2 [y, x] AW_t [y, x] + \\ & 2 AW_1 [] AW_2 [x, x, y] AW_t [y, x] + 2 AW_1 [x, x] AW_2 [x] AW_t [y, y] - 2 AW_1 [x] AW_2 [x, x] AW_t [y, y] - \\ & AW_1 [x, y] AW_2 [y, x, x] AW_t [y, z] + AW_1 [x] AW_2 [y] AW_t [x, y, x] - AW_1 [] AW_2 [x, y] AW_t [x, y, x] - \\ & AW_1 [x, y] AW_2 [] AW_t [y, x, x] + AW_1 [y] AW_2 [x] AW_t [y, x, x] - AW_1 [x, x] AW_2 [] AW_t [y, x, y] + \\ & AW_1 [x] AW_2 [x] AW_t [y, x, y] + AW_1 [x, y] AW_2 [y, x] AW_t [y, x, z] + 2 AW_1 [x] AW_2 [x] AW_t [y, y, x] - \\ & 2 AW_1 [] AW_2 [x, x] AW_t [y, y, x] - AW_1 [x] AW_2 [] AW_t [y, x, y, x] + AW_1 [] AW_2 [x] AW_t [y, x, y, x]]] \end{aligned}$$

In[*]:= **D1** // $\mathcal{O}_{\{2,1\}}$ // $\mathcal{O}_{\{1,2\}}$

Out[*]=

$$\begin{aligned} & \mathcal{O}_{AR, \{x,y,z\}, \{1,2\}} [\mathcal{A}_0 [AW_1 [x, y, x] AW_2 [x, x, y]] + \\ & \mathcal{A}_{t[1,2]} [AW_1 [x, y, z] AW_2 [y, x, x, y] AW_t [] + AW_1 [x, y, z] AW_2 [y, x, y] AW_t [x] + \\ & AW_1 [x, y, z] AW_2 [y, x, x] AW_t [y] - AW_1 [x, y] AW_2 [y, x, x, y] AW_t [z] + \\ & AW_1 [x, y, z] AW_2 [y, x] AW_t [x, y] - AW_1 [x, y] AW_2 [y, x, y] AW_t [z, x] - \\ & AW_1 [x, y] AW_2 [y, x, x] AW_t [z, y] - AW_1 [x, y] AW_2 [y, x] AW_t [z, x, y]]] \end{aligned}$$

In[*]:= (**D1** // $\mathcal{O}_{\{2,1\}}$ // $\mathcal{O}_{\{1,2\}}$) - **D1**

Out[*]=

0

In[*]:= {**D2**, **D2** // $\mathcal{O}_{\{2,1\}}$, **D2** // $\mathcal{O}_{\{2,1\}}$ // $\mathcal{O}_{\{1,2\}}$ }

Out[*]=

$$\begin{aligned} & \{ \mathcal{O}_{HR, \{x,y,z\}, \{1,2\}} [\mathcal{A}_0 [AW_1 [x, y, x] AW_2 [x, x, y]] + \mathcal{A}_{t[1,2]} [AW_1 [x, y, x, y] AW_2 [y, x, z]]], \\ & \mathcal{O}_{HR, \{x,y,z\}, \{2,1\}} [\mathcal{A}_0 [AW_1 [x, y, x] AW_2 [x, x, y]] + \mathcal{A}_{t[2,1]} [-AW_1 [x, y, x, x, y] AW_2 [] + \\ & AW_1 [x, y, y] AW_2 [x, x] - AW_1 [x, x, y] AW_2 [y, x] - AW_1 [x, y] AW_2 [x, x, x] + \\ & AW_1 [x, y, x, y] AW_2 [y, x, z] + AW_1 [x] AW_2 [x, x, y, x] + AW_1 [y] AW_2 [x, x, y, x]]], \\ & \mathcal{O}_{HR, \{x,y,z\}, \{1,2\}} [\mathcal{A}_0 [AW_1 [x, y, x] AW_2 [x, x, y]] + \mathcal{A}_{t[1,2]} [AW_1 [x, y, x, y] AW_2 [y, x, z]]] \} \end{aligned}$$

$\mathcal{O}_{red_ps_ss_} [] // m_{i_ , j_ \rightarrow k_} :=$

In[*]:= **CF** [$\mathcal{A}_{ps_ , ss_} [A0_ , A1_] := \text{Module} [\{ 1, r, u \},$

$\mathcal{A}_{ps_ , ss_} [A0_ , A1_ / . t_{i_ , j_} _ \mathcal{E} \rightarrow \text{Expand} [$
 $t_{i, j} (\mathcal{E} // \Delta_{\vec{j} \rightarrow u, 1} // S_{u \rightarrow u} // m_{u, \vec{i} \rightarrow \vec{i}} // \Delta_{1 \rightarrow 1, r} // m_{i, 1 \rightarrow i} // m_{j, r \rightarrow j})$
 $]]$
 $]$

```

In[*]:=  $\mathcal{A}_{\{x,y,z\},\{1,2\}}$  [
   $AW_1[x, y, x] AW_2[x, x, y]$ ,
   $t_{1,2} AW_1[x, y] AW_2[y, x] AW_{\bar{1}}[z] AW_{\bar{2}}[x, y]$ 
] // CF

Out[*]=
 $\mathcal{A}_{\{x,y,z\},\{1,2\}}$  [ $AW_1[x, y, x] AW_2[x, x, y]$ ,  $t_{1,2} AW_1[x, y, x, y] AW_2[y, x] AW_{\bar{1}}[z]$  +
 $t_{1,2} AW_1[x, y, y] AW_2[y, x, x] AW_{\bar{1}}[z]$  +  $t_{1,2} AW_1[x, y, x] AW_2[y, x, y] AW_{\bar{1}}[z]$  +
 $t_{1,2} AW_1[x, y] AW_2[y, x, x, y] AW_{\bar{1}}[z]$  -  $t_{1,2} AW_1[x, y, y] AW_2[y, x] AW_{\bar{1}}[x, z]$  -
 $t_{1,2} AW_1[x, y] AW_2[y, x, y] AW_{\bar{1}}[x, z]$  -  $t_{1,2} AW_1[x, y, x] AW_2[y, x] AW_{\bar{1}}[y, z]$  -
 $t_{1,2} AW_1[x, y] AW_2[y, x, x] AW_{\bar{1}}[y, z]$  +  $t_{1,2} AW_1[x, y] AW_2[y, x] AW_{\bar{1}}[y, x, z]$ ]

In[*]:= HCF[ $\mathcal{A}_{ps,ss}[A0_, A1_]$ ] :=  $\mathcal{A}_{ps,ss}[A0, A1 / . t_{i,j} \mathcal{E}_\rightarrow \Rightarrow \text{Expand}[$ 
   $t_{i,j} (\mathcal{E} // m_{i,\bar{j} \rightarrow i} // m_{j,\bar{i} \rightarrow j})$ 
]]

In[*]:=  $\mathcal{A}_{\{x,y,z\},\{1,2\}}$  [
   $AW_1[x, y, x] AW_2[x, x, y]$ ,
   $t_{1,2} AW_1[x, y] AW_2[y, x] AW_{\bar{1}}[z] AW_{\bar{2}}[x, y]$ 
] // HCF

Out[*]=
 $\mathcal{A}_{\{x,y,z\},\{1,2\}}$  [ $AW_1[x, y, x] AW_2[x, x, y]$ ,  $t_{1,2} AW_1[x, y, x, y] AW_2[y, x, z]$ ]

Unprotect[NonCommutativeMultiply];
 $\mathcal{A}_{ps,ss}[A0_, A1_] ** \mathcal{A}_{ps,ss}[B0_, B1_] := \text{Module}[\{v, T\}, \mathcal{A}_{ps,ss}[$ 
   $T = B0$ ; Do[ $T = T // \sigma_{s \rightarrow v \circ s}$ , {s, ss}];
   $T = \text{Expand}[A0 T]$ ; Do[ $T = T // m_{s, v \circ s \rightarrow s}$ , {s, ss}];
  T,
  Plus[
     $T = B1$ ; Do[ $T = T // \sigma_{s \rightarrow v \circ s}$ , {s, ss}];
     $T = \text{Expand}[A0 T]$ ; Do[ $T = T // m_{s, v \circ s \rightarrow s}$ , {s, ss}];
    T,
    A1B0,
    A0B0
  ]
]]

In[*]:=  $\mathcal{A}_{\{x,y,z\},\{1,2\}}$  [ $AW_1[x, y, x] AW_2[x, x, y]$ , 777] **  $\mathcal{A}_{\{x,y,z\},\{1,2\}}$  [ $AW_1[z, z, x] AW_2[x, z, z]$ , 888]

Out[*]=
 $\mathcal{A}_{\{x,y,z\},\{1,2\}}$  [ $AW_1[x, y, x, z, z, x] AW_2[x, x, y, x, z, z]$ , 666]

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