On the linearized pentagon in emergent $\mathcal{P}$

$$\psi \in \mathrm{FL} = \mathrm{FL}\langle x,y\rangle \overset{\text{hor}}{\subset} \mathcal{A}\Big(\underset{x\ \ y\ \ \ }{|\,|\,|}\Big) \supset \mathcal{P} = \mathrm{FL} \oplus \mathrm{FA}[1]$$

$$\psi \longmapsto \psi + R \qquad R = R(\psi)$$

$$\left[\begin{array}{ll} (\mathrm{P1}) & \psi(y,0) - \psi(x+y,0) = 0 \\ (\mathrm{P2}) & R(y,0) - R(x+y,0) = 0 \\ (\mathrm{P3}) & \partial_y \psi(x,y) + \partial_y \psi(y,0) - \partial_y \psi(x+y,0) - \big(R(x,y) + R^*(x,y)\big) = 0 \end{array}\right]$$

R^* : the antipode

<u>Rem</u>

(P1) : the coeff. of x in ψ $= 0$

(P2) : the coeff. of x^n in R $= 0$ $\quad (n = 1, 2, 3, \cdots)$

[**Lem** $R^* = R$]

<u>proof</u> Suppose $R^*(u) = R(u)$ & $R^*(v) = v$.

* is anti-homom

$$R^*([u,v]) = [u, R(v)]^* + [R(u), v]^* + \frac{1}{2}\sum_i \Big((\partial_i v)x_i(\partial_i u)^* - (\partial_i u)x_i(\partial_i v)^*\Big)^*$$

$$= [R^*(v), u^*] + [v^*, R^*(u)] + \frac{1}{2}\sum_i \Big(-(\partial_i u)x_i(\partial_i v)^* + (\partial_i v)x_i(\partial_i u)^*\Big)$$

$R^*(u)=u$

$R^*(v)=v$

$u^* = -u$, $v^* = -v$

$$= [u, R(v)] + [R(u), v] + \text{———— '' ————}$$

$$= R([u,v])$$

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Therefore, (P3) is equivalent to

[(P3)' $\quad \partial_y \psi(x,y) + \partial_y \psi(y,0) - \partial_y \psi(x+y,0) - 2R(x,y) = 0$]

Recall: Alekseev–Torossian's map $\nu : \mathfrak{grt}_1 \to \mathfrak{krv}_2$, $\psi \mapsto (\psi(-x-y,x), \psi(-x-y,y))$

Question: Let $\psi \in FL\langle x,y \rangle$ be a solution to (P1)(P2)(P3). Does

$$(u,v) = (\psi(-x-y,x), \psi(-x-y,y))$$

satisfy KV equations?

$$\text{(KV I)} \quad [x,u] + [y,v] = 0$$

$$\text{(KV II)} \quad |x\partial_x u + y\partial_y u| = |f(x) + f(y) - f(x+y)|$$

For the moment, assume further that ψ satisfy the 2-cycle relation

$$\psi(x,y) + \psi(y,x) = 0$$

$$\rightsquigarrow \begin{cases} (\partial_x\psi)(x,y) + (\partial_y\psi(y,x)) = 0 \\ (\partial_y\psi)(x,y) + \partial_x\psi(y,x) = 0 \end{cases} \quad \text{—— (0)}$$

Let's investigate (KV2). May assume that ψ is homogeneous of deg ≥ 2.

◎ Since $u = \psi(-x-y, x)$ & $v = \psi(-x-y, y)$,

$$\left\{\begin{array}{l} \partial_x u = -\partial_x \psi(-x-y, x) + \partial_y \psi(-x-y, x) \\ \partial_y v = -\partial_x \psi(-x-y, y) + \partial_y \psi(-x-y, y) \end{array}\right. \quad \text{—— (1)}$$

◎ Using $\psi(x, y) = (\partial_x \psi) x + (\partial_y \psi) y$,

$$\left.\begin{array}{l} \psi(-x-y, x) = \big(\partial_x \psi(-x-y, x)\big)(-x-y) + \big(\partial_y \psi(-x-y, x)\big) x \\ \qquad \underset{(1)}{=} -\big(\partial_x \psi(-x-y, x)\big) y + (\partial_x u) x \\ \psi(-x-y, y) = \big(\partial_x \psi(-x-y, y)\big)(-x-y) + \big(\partial_y \psi(-x-y, y)\big) y \\ \qquad \underset{(1)}{=} -\big(\partial_x \psi(-x-y, y)\big) x + (\partial_y v) y \end{array}\right\} \text{—— (2)}$$

⓪ Hence

$$\left|(\partial_x u)x + (\partial_y v)y\right| \underset{(2)}{=} \left|\begin{aligned}&\psi(-x-y,x) + \big(\partial_x\psi(-x-y,x)\big)y \\ &+ \psi(-x-y,y) + \big(\partial_x\psi(-x-y,y)\big)x\end{aligned}\right|$$

$\deg\psi \geq 2$

$$= \left|\big(\partial_x\psi(-x-y,x)\big)y + \big(\partial_x\psi(-x-y,y)\big)x\right|$$

$|FL_{22}| = 0$

$$\underset{(0)}{=} -\left|\big(\partial_y\psi(x,-x-y)\big)y + \big(\partial_y\psi(y,-x-y)\big)x\right|$$

$$\left[(P3)' \quad \partial_y\psi(x,y) + \partial_y\psi(y,0) - \partial_y\psi(x+y,0) - 2R(x,y) = 0\right]$$

$$\rightsquigarrow \begin{cases} -\partial_y\psi(x,-x-y) = \partial_y\psi(-x-y,0) - \partial_y\psi(-y,0) - 2R(x,-x-y) \\ -\partial_y\psi(y,-x-y) = \partial_y\psi(-x-y,0) - \partial_y\psi(-x,0) - 2R(y,-x-y) \end{cases} (3)$$

Using (3),

$$\left|(\partial_x u)x+(\partial_y v)y\right| = \left|\begin{array}{l}\left(\partial_y\psi(-x-y,0)\right)y-\left(\partial_y\psi(-y,0)\right)y-2R(x,-x-y)y\\ +\left(\partial_y\psi(-x-y,0)\right)x-\left(\partial_y\psi(-x,0)\right)x-2R(y,-x-y)x\end{array}\right|$$

$$= \left|\begin{array}{l}(x+y)\,\partial_y\psi(-x-y,0)-x\,\partial_y\psi(-x,0)-y\,\partial_y\psi(-y,0)\\ \quad -2\left(R(x,-x-y)y+R(y,-x-y)x\right)\end{array}\right|$$

Therefore, (KV2) for (u,v) is reduced to

$$\left[\quad \left|R(x,-x-y)y+R(y,-x-y)x\right| = \left|f(x)+f(y)-f(x+y)\right| \quad\right]$$