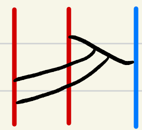
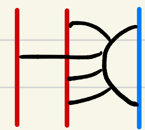
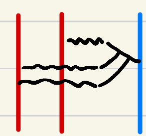
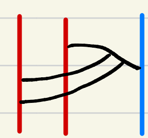
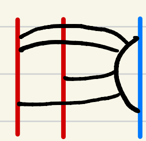
The Lie bracket on $\mathcal{P}$ ($\#\,ss = 1$)

$\mathcal{P} = \mathrm{Span}\Big\{$  ,  $\Big\}$ / relations

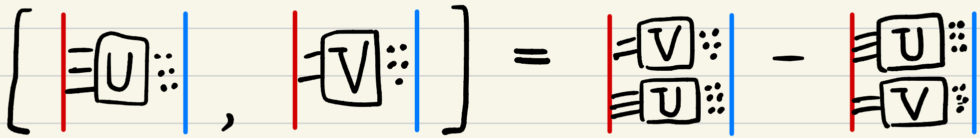
- grading by ($\#$ of legs $- 1$)
- As a vector space, $\mathcal{P} \cong FL\langle ps \rangle \oplus FA\langle ps \rangle[1]$, where

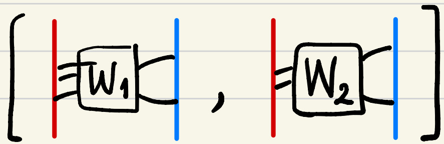
$[x,[x,y]] \longmapsto$ 

$= \frac{1}{2!}\Big($  $+$  $\Big)$

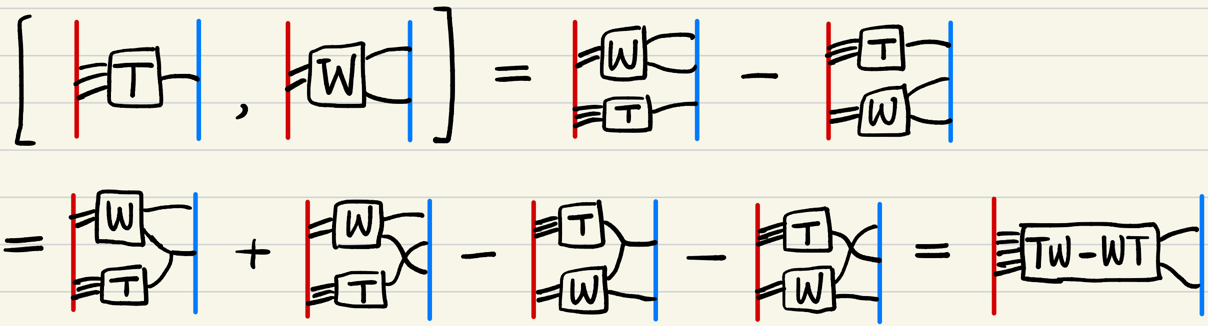
$xyxX \longmapsto$ 

◎ Lie bracket



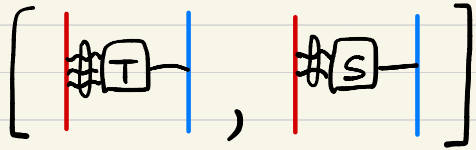
(i) 

$= 0$ for $W_1, W_2 \in FA\langle ps\rangle[1]$

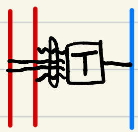
(ii) 

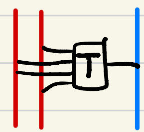
$(T \in FL\langle ps\rangle, \ W \in FA\langle ps\rangle[1])$

(iii)

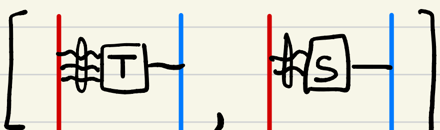
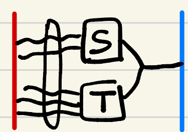
Finally, let us compute  for $T, S \in FL\langle ps \rangle$

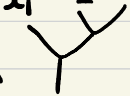
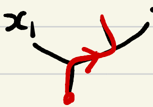
Notation:



$= \text{average} = \dfrac{1}{m!\,n!}\Big($  $+$ all permutations $\Big)$

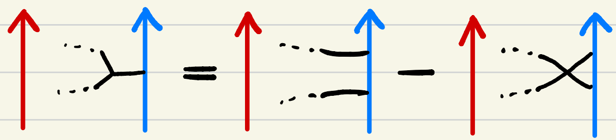
m legs on x
n legs on y

Prop:  $=$  $+ \dfrac{1}{2} \sum_i \begin{array}{l} (\partial_i S)\, x_i\, (\partial_i T)^* \\ -(\partial_i T)\, x_i\, (\partial_i S)^* \end{array}$

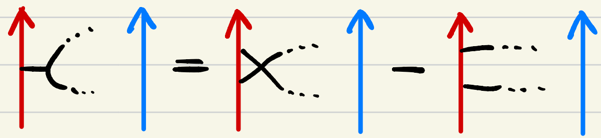
- $\partial_2 \, [x_1, [x_2, x_1]] = \partial_2 \Big($  $\Big) =$  $= -x_1 x_1$

- $*$: the antipode on $FA\langle ps \rangle$

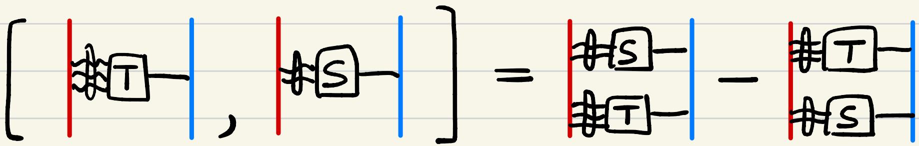
Recall: the STU relation

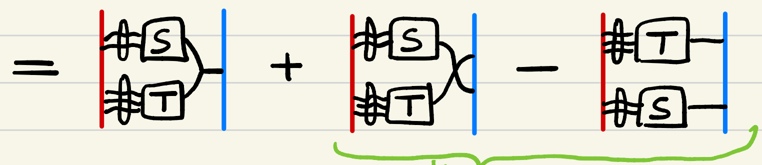


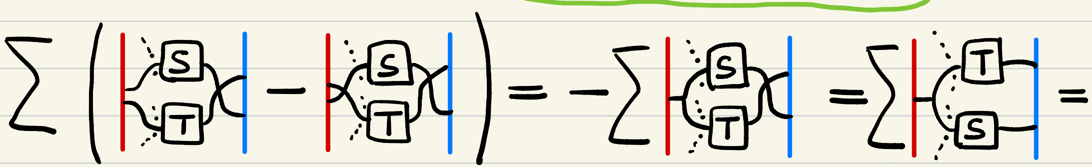
&



proof of Prop







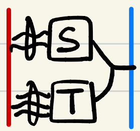
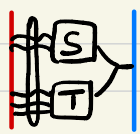
$$= \sum_i (\partial_i S)\, x_i\, (\partial_i T)^*$$

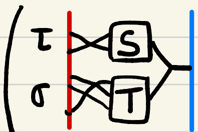
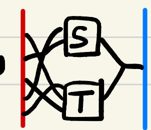
Let us compute the contribution of exchanges of legs along the ith ~~strand~~ pole.

- Set $m :=$ # of legs of T on the ith ~~strand~~ pole

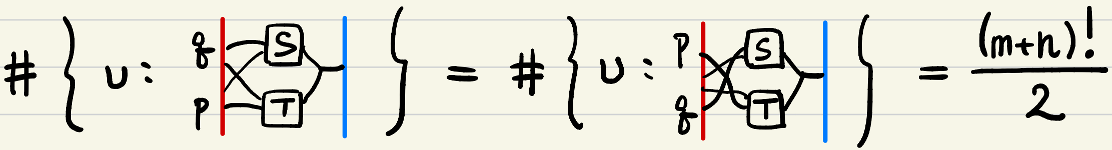
  $n :=$ — " — S — " ——

- For $(\sigma, \tau) \in \mathfrak{S}_m \times \mathfrak{S}_n$, let $A_{\sigma,\tau} \subset \mathfrak{S}_{m+n}$ be the set of permutations of $\{z_1, \cdots, z_{m+n}\} = \{x_1, \cdots, x_m, y_1, \cdots, y_n\}$ that induce σ on $\{x_1, \cdots, x_m\}$ (when forgetting y's) and τ on $\{y_1, \cdots, y_n\}$. $\left(\# A_{\sigma,\tau} = \frac{(m+n)!}{m!\,n!} \right)$

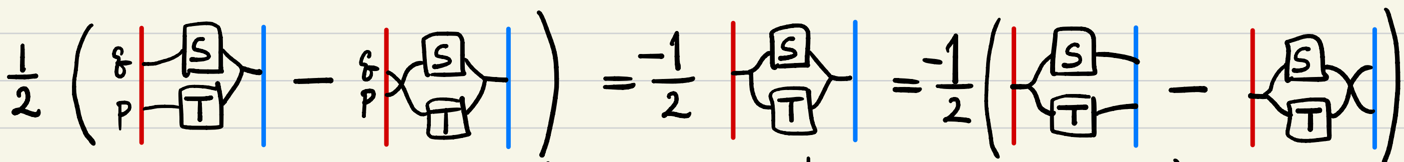
 $-$ 

$$= \frac{1}{(m+n)!} \sum_{(\sigma,\tau) \in \mathfrak{S}_m \times \mathfrak{S}_n} \sum_{u \in A_{\sigma,\tau}} \Bigg($$  $- \; u$  $\Bigg) \quad \cdots \star$

Now, fix $\begin{cases} \text{a leg } p \text{ of } T \\ \text{a leg } q \text{ of } S \end{cases}$ on the ith ~~strand~~ pole. Then,

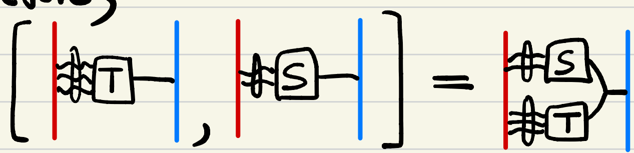


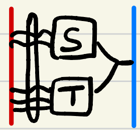
So, the contribution to ✩ of $p \leftrightarrow q$ in the υ-term amounts to



which gives rise to $\sum_i \frac{-1}{2}\Big((\partial_i T)\, x_i\, (\partial_i S)^* + (\partial_i S)\, x_i\, (\partial_i T)^*\Big)$.

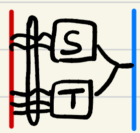
Therefore,

 $+ \sum_i (\partial_i S)\, x_i\, (\partial_i T)^*$

$=$ 

$$+ \sum_i \frac{-1}{2}\Big((\partial_i T)\, x_i\, (\partial_i S)^* + (\partial_i S)\, x_i\, (\partial_i T)^* \Big)$$

$$+ \sum_i (\partial_i S)\, x_i\, (\partial_i T)^*$$

$=$ 

$$+ \sum_i \frac{1}{2}\Big((\partial_i S)\, x_i\, (\partial_i T)^* - (\partial_i T)\, x_i\, (\partial_i S)^* \Big)$$

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