

## The Lie bracket on $\mathcal{P}$ ( $\#ss = 1$ )

$$\mathcal{P} = \text{Span} \left\{ \begin{array}{c} | \quad | \quad | \\ \diagup \quad \diagdown \\ | \end{array} , \begin{array}{c} | \quad | \quad | \\ \diagdown \quad \diagup \\ | \end{array} \right\} / \text{relations}$$

- grading by ( $\#$  of legs  $- 1$ )

- As a vector space,  $\mathcal{P} \cong \text{FL}\langle ps \rangle \oplus \text{FA}\langle ps \rangle[1]$ , where

$$[x, [x, y]] \mapsto \begin{array}{c} | \quad | \quad | \\ \text{wavy} \quad \text{wavy} \quad \diagup \\ | \quad | \quad | \\ x \quad y \end{array} = \frac{1}{2!} \left( \begin{array}{c} | \quad | \quad | \\ \diagup \quad \diagdown \\ | \end{array} + \begin{array}{c} | \quad | \quad | \\ \diagdown \quad \diagup \\ | \end{array} \right)$$

$$xyx\lambda \mapsto \begin{array}{c} | \quad | \quad | \\ \diagdown \quad \diagup \\ | \end{array}$$

⊙ Lie bracket

$$\left[ \begin{array}{|c|} \hline U \\ \hline \end{array}, \begin{array}{|c|} \hline V \\ \hline \end{array} \right] = \begin{array}{|c|} \hline V \\ \hline U \\ \hline \end{array} - \begin{array}{|c|} \hline U \\ \hline V \\ \hline \end{array}$$

$$(i) \left[ \begin{array}{|c|} \hline W_1 \\ \hline \end{array}, \begin{array}{|c|} \hline W_2 \\ \hline \end{array} \right] = 0 \quad \text{for } W_1, W_2 \in \text{FA}\langle ps \rangle[1]$$

$$(ii) \left[ \begin{array}{|c|} \hline T \\ \hline \end{array}, \begin{array}{|c|} \hline W \\ \hline \end{array} \right] = \begin{array}{|c|} \hline W \\ \hline T \\ \hline \end{array} - \begin{array}{|c|} \hline T \\ \hline W \\ \hline \end{array}$$

$$= \begin{array}{|c|} \hline W \\ \hline T \\ \hline \end{array} + \begin{array}{|c|} \hline W \\ \hline T \\ \hline \end{array} - \begin{array}{|c|} \hline T \\ \hline W \\ \hline \end{array} - \begin{array}{|c|} \hline T \\ \hline W \\ \hline \end{array} = \begin{array}{|c|} \hline TW - WT \\ \hline \end{array}$$

$$(T \in \text{FL}\langle ps \rangle, W \in \text{FA}\langle ps \rangle[1])$$

(iii)

Finally, let us compute  $\left[ \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \boxed{T} \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} , \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \boxed{S} \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right]$  for  $T, S \in \text{FL}\langle ps \rangle$

Notation:

$$\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \boxed{T} \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} = \text{average} = \frac{1}{m!n!} \left( \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \boxed{T} \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} + \text{all permutations} \right)$$

$x \quad y$        $m$  legs on  $x$   
 $n$  legs on  $y$

$$\left[ \text{Prop: } \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \boxed{T} \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} , \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \boxed{S} \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right] = \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \boxed{S} \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \boxed{T} \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} + \frac{1}{2} \sum_i \left( \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \boxed{S} \text{---} x_i \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right)^* - \left( \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \text{---} \boxed{T} \text{---} x_i \text{---} \begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} \right)^* \right]$$

$$\bullet \partial_2 [x_1, [x_2, x_1]] = \partial_2 \left( \begin{array}{c} x_1 \quad x_2 \quad x_1 \\ \diagdown \quad / \\ \text{---} \\ \diagup \quad \diagdown \\ x_1 \end{array} \right) = \begin{array}{c} x_1 \quad x_1 \\ \diagdown \quad / \\ \text{---} \\ \diagup \quad \diagdown \\ x_1 \end{array} = -x_1 x_1$$

$\bullet$   $*$ : the antipode on  $\text{FA}\langle ps \rangle$



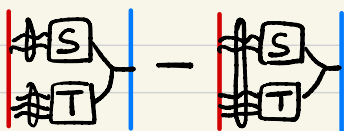
Let us compute the contribution of exchanges of legs along the ~~ith strand~~ <sup>pole</sup>.

• Set  $m := \#$  of legs of  $T$  on the ~~ith strand~~ <sup>pole</sup>

$n := \#$  of legs of  $S$  on the ~~ith strand~~ <sup>pole</sup>

• For  $(\sigma, \tau) \in \mathfrak{S}_m \times \mathfrak{S}_n$ , let  $A_{\sigma, \tau} \subset \mathfrak{S}_{m+n}$  be the set of permutations of  $\{z_1, \dots, z_{m+n}\} = \{x_1, \dots, x_m, y_1, \dots, y_n\}$  that induce  $\sigma$  on  $\{x_1, \dots, x_m\}$

(when forgetting  $y$ 's) and  $\tau$  on  $\{y_1, \dots, y_n\}$ .  $\left( \#A_{\sigma, \tau} = \frac{(m+n)!}{m!n!} \right)$



$$= \frac{1}{(m+n)!} \sum_{(\sigma, \tau) \in \mathfrak{S}_m \times \mathfrak{S}_n} \sum_{\nu \in A_{\sigma, \tau}} \left( \begin{array}{c} \tau \\ \sigma \end{array} \left| \begin{array}{c} S \\ T \end{array} \right. \right) - \nu \left( \begin{array}{c} S \\ T \end{array} \right) \Big| \dots \star$$

Now, fix  $\left\{ \begin{array}{l} \text{a leg } p \text{ of } T \\ \text{a leg } q \text{ of } S \end{array} \right.$  on the  $i$ th ~~strand~~ <sup>pole</sup>. Then,

$$\# \left\{ U: \begin{array}{c} q \\ | \\ \text{---} \\ | \\ p \end{array} \left| \begin{array}{c} S \\ | \\ T \end{array} \right| \right\} = \# \left\{ U: \begin{array}{c} p \\ | \\ \text{---} \\ | \\ q \end{array} \left| \begin{array}{c} S \\ | \\ T \end{array} \right| \right\} = \frac{(m+n)!}{2}$$

So, the contribution to  $\star$  of  $p \leftrightarrow q$  in the  $U$ -term amounts to

$$\frac{1}{2} \left( \begin{array}{c} q \\ | \\ \text{---} \\ | \\ p \end{array} \left| \begin{array}{c} S \\ | \\ T \end{array} \right| - \begin{array}{c} p \\ | \\ \text{---} \\ | \\ q \end{array} \left| \begin{array}{c} S \\ | \\ T \end{array} \right| \right) = \frac{-1}{2} \left| \begin{array}{c} S \\ | \\ T \end{array} \right| = \frac{-1}{2} \left( \left| \begin{array}{c} S \\ | \\ T \end{array} \right| - \left| \begin{array}{c} S \\ | \\ T \end{array} \right| \right)$$

which gives rise to  $\sum_i \frac{1}{2} \left( (\partial_i T) x_i (\partial_i S)^* + (\partial_i S) x_i (\partial_i T)^* \right)$ .

Therefore,

$$\left[ \left| \begin{array}{c} T \end{array} \right|, \left| \begin{array}{c} S \end{array} \right| \right] = \left| \begin{array}{c} S \\ | \\ T \end{array} \right| + \sum_i (\partial_i S) x_i (\partial_i T)^*$$

$$= \left| \begin{array}{c} S \\ T \end{array} \right| + \sum_i \frac{1}{2} \left( (\partial_i T) x_i (\partial_i S)^* + (\partial_i S) x_i (\partial_i T)^* \right) + \sum_i (\partial_i S) x_i (\partial_i T)^*$$

$$= \left| \begin{array}{c} S \\ T \end{array} \right| + \sum_i \frac{1}{2} \left( (\partial_i S) x_i (\partial_i T)^* - (\partial_i T) x_i (\partial_i S)^* \right) //$$