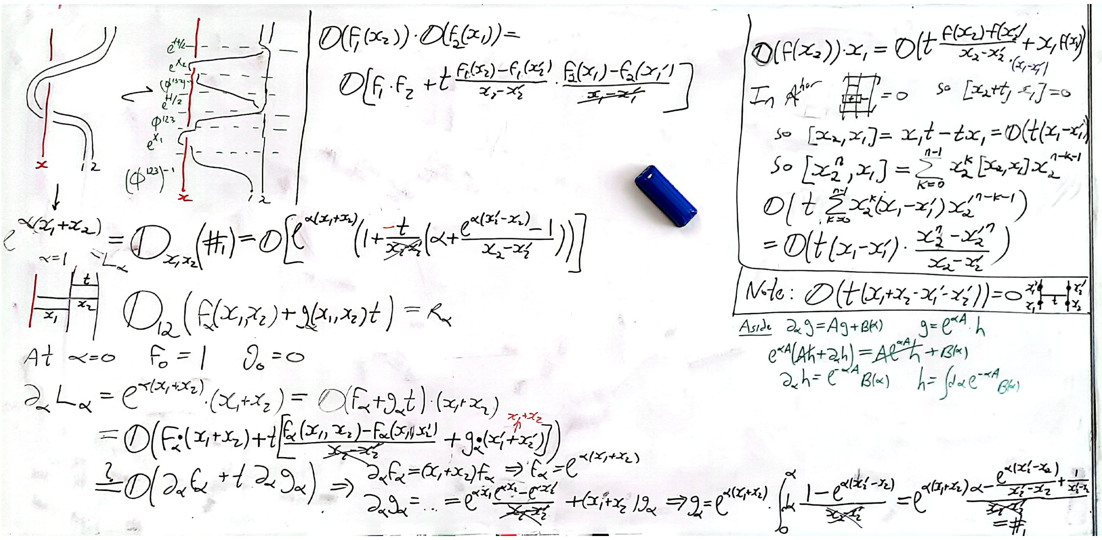
Pensieve header: Computations in chord diagrams mod 1ss. Continued pensieve://People/Kuno.



$\mathbb{O}[f, g]$ stands for $\mathbb{O}_{1,2}[f + t\,g]$.

```
In[•]:= CF[𝕆[f_, g_]] := 𝕆[Simplify[f], Simplify[g /. x̄₁ → x₁ + x₂ - x̄₂]]
```

```
In[•]:= 𝕆[f1_, g1_] ≡ 𝕆[f2_, g2_] := Simplify[(f1 == f2) ∧ (g1 == g2)]
```

```
In[•]:= 𝕆 /: 𝕆[f1_, g1_] ** 𝕆[f2_, g2_] :=
  CF@𝕆[f1 f2, ((f1 - (f1 /. x₂ → x̄₂)) ((f2 /. x₂ → x̄₂) - (f2 /. {x₁ → x̄₁, x₂ → x̄₂}))) / (x₂ - x̄₂) +
     f1 g2 + g1 (f2 /. {x₁ → x̄₁, x₂ → x̄₂})]
```

```
In[•]:= 𝕆[f[x₂], 0] ** 𝕆[x₁, 0]
Out[•]=
  𝕆[f[x₂] x₁, -f[x₂] + f[x̄₂]]
```

```
In[•]:= {h1, h2, h3} = {𝕆[f1[x₁, x₂], g1[x₁, x₂, x̄₁, x̄₂]],
    𝕆[f2[x₁, x₂], g2[x₁, x₂, x̄₁, x̄₂]], 𝕆[f3[x₁, x₂], g3[x₁, x₂, x̄₁, x̄₂]]}
Out[•]=
  {𝕆[f1[x₁, x₂], g1[x₁, x₂, x̄₁, x̄₂]],
   𝕆[f2[x₁, x₂], g2[x₁, x₂, x̄₁, x̄₂]], 𝕆[f3[x₁, x₂], g3[x₁, x₂, x̄₁, x̄₂]]}
```

```
In[•]:= h1 ** h2
Out[•]=
  𝕆[f1[x₁, x₂] f2[x₁, x₂],
   f2[x₁ + x₂ - x̄₂, x̄₂] g1[x₁, x₂, x₁ + x₂ - x̄₂, x̄₂] + f1[x₁, x₂] g2[x₁, x₂, x₁ + x₂ - x̄₂, x̄₂] +
    ((f1[x₁, x₂] - f1[x₁, x̄₂]) (f2[x₁, x̄₂] - f2[x₁ + x₂ - x̄₂, x̄₂])) / (x₂ - x̄₂)]
```

In[∘]:= **lhs = (h1 ** h2) ** h3**

Out[∘]=

$$\mathbb{O}\Big[f1[x_1,x_2]\,f2[x_1,x_2]\,f3[x_1,x_2],\; f1[x_1,x_2]\,f2[x_1,x_2]\,g3[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2]+f3[x_1+x_2-\overline{x}_2,\overline{x}_2]\Big(f2[x_1+x_2-\overline{x}_2,\overline{x}_2]\,g1[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2]+f1[x_1,x_2]\,g2[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2]+\frac{(f1[x_1,x_2]-f1[x_1,\overline{x}_2])\,(f2[x_1,\overline{x}_2]-f2[x_1+x_2-\overline{x}_2,\overline{x}_2])}{x_2-\overline{x}_2}\Big)+\frac{(f1[x_1,x_2]\,f2[x_1,x_2]-f1[x_1,\overline{x}_2]\,f2[x_1,\overline{x}_2])\,(f3[x_1,\overline{x}_2]-f3[x_1+x_2-\overline{x}_2,\overline{x}_2])}{x_2-\overline{x}_2}\Big]$$

In[∘]:= **rhs = h1 ** (h2 ** h3)**

Out[∘]=

$$\mathbb{O}\Big[f1[x_1,x_2]\,f2[x_1,x_2]\,f3[x_1,x_2],\; f2[x_1+x_2-\overline{x}_2,\overline{x}_2]\,f3[x_1+x_2-\overline{x}_2,\overline{x}_2]\,g1[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2]+f1[x_1,x_2]\Big(f3[x_1+x_2-\overline{x}_2,\overline{x}_2]\,g2[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2]+f2[x_1,x_2]\,g3[x_1,x_2,x_1+x_2-\overline{x}_2,\overline{x}_2]+\frac{(f2[x_1,x_2]-f2[x_1,\overline{x}_2])\,(f3[x_1,\overline{x}_2]-f3[x_1+x_2-\overline{x}_2,\overline{x}_2])}{x_2-\overline{x}_2}\Big)+\frac{(f1[x_1,x_2]-f1[x_1,\overline{x}_2])\,(f2[x_1,\overline{x}_2]\,f3[x_1,\overline{x}_2]-f2[x_1+x_2-\overline{x}_2,\overline{x}_2]\,f3[x_1+x_2-\overline{x}_2,\overline{x}_2])}{x_2-\overline{x}_2}\Big]$$

In[∘]:= **lhs ≡ rhs**

Out[∘]=

True

In[∘]:= **e12[α_] := 𝕆[e^(α (x1+x2)), e^(α (x1+x2)) g[α]]**

$$e12[\alpha\_] := \mathbb{O}\left[e^{\alpha\,(x_1+x_2)},\ e^{\alpha\,(x_1+x_2)}\,g[\alpha]\right]$$

In[∘]:= **lhs = (∂α #) & /@ e12[α]**

Out[∘]=

$$\mathbb{O}\left[e^{\alpha\,(x_1+x_2)}\,(x_1+x_2),\ e^{\alpha\,(x_1+x_2)}\,g[\alpha]\,(x_1+x_2)+e^{\alpha\,(x_1+x_2)}\,g'[\alpha]\right]$$

In[∘]:= **rhs = e12[α] ** 𝕆[x1 + x2, 0]**

Out[∘]=

$$\mathbb{O}\left[e^{\alpha\,(x_1+x_2)}\,(x_1+x_2),\ e^{\alpha\,x_1}\left(-e^{\alpha\,x_2}+e^{\alpha\,\overline{x}_2}+e^{\alpha\,x_2}\,g[\alpha]\,(x_1+x_2)\right)\right]$$

In[∘]:= **FullSimplify[e^(α (x1+x2)) g[α] /. DSolve[lhs ≡ rhs ∧ g[0] == 0, g[α], α]⟦1⟧]**

Out[∘]=

$$-\frac{e^{\alpha\,(x_1+x_2)}\left(-1+e^{\alpha\,(-x_2+\overline{x}_2)}+\alpha\,x_2-\alpha\,\overline{x}_2\right)}{x_2-\overline{x}_2}$$

In[∘]:=

$$e12[\alpha\_] := \mathbb{O}\left[e^{\alpha\,(x_1+x_2)},\ \frac{e^{\alpha\,(x_1+\overline{x}_2)}-e^{\alpha\,(x_1+x_2)}}{\overline{x}_2-x_2}-\alpha\,e^{\alpha\,(x_1+x_2)}\right]$$

In[∘]:=
```
lhs = CF[(∂α #) & /@ e12[α]]
```
Out[∘]=

$$\mathbb{O}\left[e^{\alpha\,(x_1+x_2)}\,(x_1+x_2),\ -e^{\alpha\,(x_1+x_2)}-e^{\alpha\,(x_1+x_2)}\,\alpha\,(x_1+x_2)+\frac{-e^{\alpha\,(x_1+x_2)}\,(x_1+x_2)+e^{\alpha\,(x_1+\bar{x}_2)}\,(x_1+\bar{x}_2)}{-x_2+\bar{x}_2}\right]$$

In[∘]:=
```
rhs = e12[α] ** 𝕆[x1 + x2, 0]
```
Out[∘]=

$$\mathbb{O}\left[e^{\alpha\,(x_1+x_2)}\,(x_1+x_2),\ -e^{\alpha\,(x_1+x_2)}+e^{\alpha\,(x_1+\bar{x}_2)}+\frac{(x_1+x_2)\left(e^{\alpha\,(x_1+x_2)}-e^{\alpha\,(x_1+\bar{x}_2)}-e^{\alpha\,(x_1+x_2)}\,\alpha\,(x_2-\bar{x}_2)\right)}{x_2-\bar{x}_2}\right]$$

In[∘]:=
```
lhs ≡ rhs
```
Out[∘]=

True

In[∘]:=
```
FullSimplify[g2[x1, x2, x̄2] /.
  Solve[𝕆[f[x1, x2], g[x1, x2, x̄1, x̄2]] ** 𝕆[f[x1, x2]^-1, g2[x1, x2, x̄2]] ≡ 𝕆[1, 0],
    g2[x1, x2, x̄2]][[1]]]
```
Out[∘]=

$$-\frac{\frac{g[x_1,x_2,x_1+x_2-\bar{x}_2,\bar{x}_2]}{f[x_1+x_2-\bar{x}_2,\bar{x}_2]}+\frac{(f[x_1,x_2]-f[x_1,\bar{x}_2])\left(\frac{1}{f[x_1,\bar{x}_2]}-\frac{1}{f[x_1+x_2-\bar{x}_2,\bar{x}_2]}\right)}{x_2-\bar{x}_2}}{f[x_1,x_2]}$$

In[∘]:=

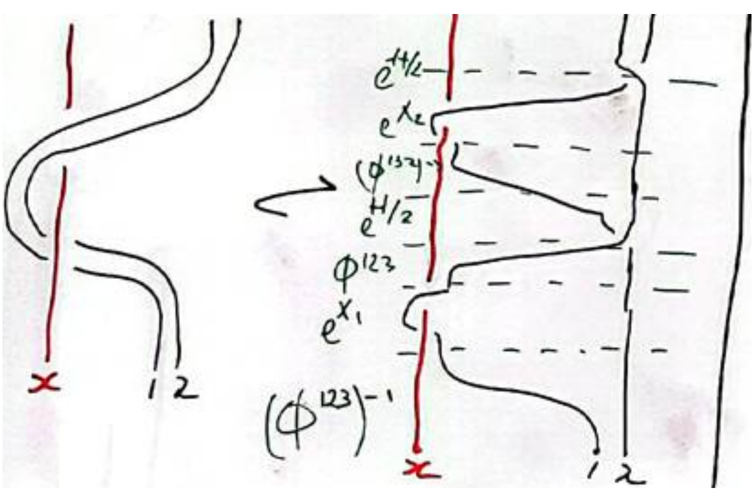
$$\mathbb{O}\ /:\ \mathbb{O}[f\_,\ g\_]^{-1} := \mathbb{O}\left[f^{-1},\ -\frac{\frac{g\ /.\ \bar{x}_1\to x_1+x_2-\bar{x}_2}{f\ /.\ \{x_1\to x_1+x_2-\bar{x}_2,\ x_2\to\bar{x}_2\}}+\frac{(f-(f\ /.\ x_2\to\bar{x}_2))\left(\frac{1}{f\ /.\ x_2\to\bar{x}_2}-\frac{1}{f\ /.\ \{x_1\to x_1+x_2-\bar{x}_2,\ x_2\to\bar{x}_2\}}\right)}{x_2-\bar{x}_2}}{f}\right]$$

In[∘]:=
```
𝕆[1, g]^-1
```
Out[∘]=

$\mathbb{O}[1,\ -g]$

In[∘]:=
```
h3 ** h3^-1
```
Out[∘]=

$\mathbb{O}[1,\ 0]$



In[∘]:= $\Phi = \mathbb{O}[1, \phi[x_1, x_2, \overline{x}_2]]$

Out[∘]=

$\mathbb{O}[1, \phi[x_1, x_2, \overline{x}_2]]$

In[∘]:= $\text{rhslist} = \{\Phi^{-1}, \mathbb{O}[e^{x_1}, 0], \Phi, \mathbb{O}[1, 1/2], \text{CF}[\Phi^{-1} \text{ /. } \{x_1 \to x_2, x_2 \to x_1, \overline{x}_1 \to \overline{x}_2, \overline{x}_2 \to \overline{x}_1\}], \mathbb{O}[e^{x_2}, 0], \text{CF}[\Phi \text{ /. } \{x_1 \to x_2, x_2 \to x_1, \overline{x}_1 \to \overline{x}_2, \overline{x}_2 \to \overline{x}_1\}], \mathbb{O}[1, -1/2]\}$

Out[∘]=

$\left\{\mathbb{O}[1, -\phi[x_1, x_2, \overline{x}_2]], \mathbb{O}[e^{x_1}, 0], \mathbb{O}[1, \phi[x_1, x_2, \overline{x}_2]], \mathbb{O}\left[1, \frac{1}{2}\right], \mathbb{O}[1, -\phi[x_2, x_1, x_1 + x_2 - \overline{x}_2]], \mathbb{O}[e^{x_2}, 0], \mathbb{O}[1, \phi[x_2, x_1, x_1 + x_2 - \overline{x}_2]], \mathbb{O}\left[1, -\frac{1}{2}\right]\right\}$

In[∘]:= rhs = NonCommutativeMultiply @@ rhslist

Out[∘]=

$\mathbb{O}\left[e^{x_1+x_2}, -\frac{1}{2} e^{x_1} \left(e^{x_2} - e^{\overline{x}_2}\right) (1 + 2\,\phi[x_1, x_2, \overline{x}_2] - 2\,\phi[x_2, x_1, x_1 + x_2 - \overline{x}_2])\right]$

In[∘]:= lhs = e12[1]

Out[∘]=

$\mathbb{O}\left[e^{x_1+x_2}, -e^{x_1+x_2} + \frac{-e^{x_1+x_2} + e^{x_1+\overline{x}_2}}{-x_2 + \overline{x}_2}\right]$

In[∘]:= lhs == rhs

Out[∘]=

$\frac{1}{2} e^{x_1} \left(-2\,e^{x_2} + \frac{2\left(e^{x_2} - e^{\overline{x}_2}\right)}{x_2 - \overline{x}_2} + \left(e^{x_2} - e^{\overline{x}_2}\right) (1 + 2\,\phi[x_1, x_2, \overline{x}_2] - 2\,\phi[x_2, x_1, x_1 + x_2 - \overline{x}_2])\right) == 0$

In[∘]:= $\text{Apart}\left[g \text{ /. First@Solve}\left[-2\,e^{x_2} + \frac{2\left(e^{x_2} - e^{\overline{x}_2}\right)}{x_2 - \overline{x}_2} + \left(e^{x_2} - e^{\overline{x}_2}\right)(1 + 2\,g) == 0, g\right]\right]$

Out[∘]=

$-\frac{e^{x_2}}{-e^{x_2} + e^{\overline{x}_2}} - \frac{2 + x_2 - \overline{x}_2}{2\,(x_2 - \overline{x}_2)}$

In[∘]:= $\text{Apart}\left[-\frac{e^{x_2}}{-e^{x_2} + e^{\overline{x}_2}} - \frac{2 + x_2 - \overline{x}_2}{2\,(x_2 - \overline{x}_2)} \text{ /. } \{x_1 \to x_2, x_2 \to x_1, \overline{x}_1 \to \overline{x}_2, \overline{x}_2 \to \overline{x}_1\} \text{ /. } \overline{x}_1 \to x_1 + x_2 - \overline{x}_2\right]$

Out[∘]=

$\frac{e^{x_2}}{-e^{x_2} + e^{\overline{x}_2}} + \frac{2 + x_2 - \overline{x}_2}{2\,(x_2 - \overline{x}_2)}$

In[∘]:= $\text{Simplify}\left[(\text{lhs} == \text{rhs}) \text{ /. } \phi[\_, x2\_, x2b\_] :\to \frac{-1}{2}\left(\frac{e^{x2}}{-e^{x2} + e^{x2b}} + \frac{2 + x2 - x2b}{2\,(x2 - x2b)}\right)\right]$

Out[∘]=

True

In[∘]:=
```
Φ = 𝕆[1, -(e^{x2}/2)/(-e^{x2} + e^{x̄2}) - (2 + x2 - x̄2)/(4 (x2 - x̄2)) + φ[x1, x2, x̄2] +
    (φ[x1, x2, x̄2] /. {x1 → x2, x2 → x1, x̄1 → x̄2, x̄2 → x̄1} /. x̄1 → x1 + x2 - x̄2)];
lhs = e12[1]
rhs = Φ^-1 ** 𝕆[e^{x1}, 0] ** Φ ** 𝕆[1, 1 / 2] ** (Φ^-1 /. {x1 → x2, x2 → x1, x̄1 → x̄2, x̄2 → x̄1}) **
  𝕆[e^{x2}, 0] ** (Φ /. {x1 → x2, x2 → x1, x̄1 → x̄2, x̄2 → x̄1}) ** 𝕆[1, -1 / 2]
lhs ≡ rhs
```

Out[∘]=

$$\mathbb{O}\left[e^{x_1+x_2}, -e^{x_1+x_2} + \frac{-e^{x_1+x_2} + e^{x_1+\overline{x}_2}}{-x_2 + \overline{x}_2}\right]$$

Out[∘]=

$$\mathbb{O}\left[e^{x_1+x_2}, \frac{e^{x_1}\left(e^{x_2} - e^{\overline{x}_2} - e^{x_2}\, x_2 + e^{x_2}\, \overline{x}_2\right)}{x_2 - \overline{x}_2}\right]$$

Out[∘]=

True

In[∘]:=
```
lhs = e12[-1]
rhs = Φ^-1 ** 𝕆[e^{-x1}, 0] ** Φ ** 𝕆[1, -1 / 2] ** (Φ^-1 /. {x1 → x2, x2 → x1, x̄1 → x̄2, x̄2 → x̄1}) **
  𝕆[e^{-x2}, 0] ** (Φ /. {x1 → x2, x2 → x1, x̄1 → x̄2, x̄2 → x̄1}) ** 𝕆[1, 1 / 2]
lhs ≡ rhs
```

Out[∘]=

$$\mathbb{O}\left[e^{-x_1-x_2}, e^{-x_1-x_2} + \frac{-e^{-x_1-x_2} + e^{-x_1-\overline{x}_2}}{-x_2 + \overline{x}_2}\right]$$

Out[∘]=

$$\mathbb{O}\left[e^{-x_1-x_2}, \frac{e^{-x_1-x_2-\overline{x}_2}\left(-e^{x_2} + e^{\overline{x}_2} + e^{\overline{x}_2}\, x_2 - e^{\overline{x}_2}\, \overline{x}_2\right)}{x_2 - \overline{x}_2}\right]$$

Out[∘]=

True

In[∘]:=
```
Φ ** (MapAt[-# &, Φ, 2] /. {x1 → -x1, x2 → -x2, x̄1 → -x̄1, x̄2 → -x̄2})
```

Out[∘]=

$$\mathbb{O}\left[1, \frac{e^{x_2}}{2\, e^{x_2} - 2\, e^{\overline{x}_2}} + \frac{e^{\overline{x}_2}}{2\, e^{x_2} - 2\, e^{\overline{x}_2}} + \frac{1}{-x_2 + \overline{x}_2} - \varphi[-x_1, -x_2, -\overline{x}_2] + \varphi[x_1, x_2, \overline{x}_2] - \varphi[-x_2, -x_1, -x_1 - x_2 + \overline{x}_2] + \varphi[x_2, x_1, x_1 + x_2 - \overline{x}_2]\right]$$