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Knot polynomials which can be calculated in polynomial time

Knots are as old as mankind. The first mathematical paper which has mentioned knots is « Remarques sur les problèmes de situation » by A.T. Vandermonde in 1771. Since then, mathematicians try to solve the main problem in knot theory : for two given knots in 3-space, decide in an effective way whether or not they are isotopic, i.e. can be deformed one into the other in a continuous way ?
The main tool to achieve this are knot invariants. There is an enormous number of known knot invariants, but either they are not calculable, i.e. they do not reduce to numbers which can be calculated by an algorithm in polynomial time, or they evidently fail to distinguish all knots. Hence the problem is still very far from its solution.

There were two main advances in knot theory in the last decades : the discovery of quantum knot polynomials (and their upgradings to Topological Quantum Field Theories and categorifications), which come from representations of Lie algebras and more generally quantum groups, on one hand, and the discovery of finite type invariants (and the Kontsevich integral as their universal expression), which come from singularity theory on the other hand. The singularity theory under question was Vassiliev's study of the discriminant of all singular knots in 3-space. But it disappeared again rapidly because of Dror Bar-Natans paper « On the Vassiliev knot invariants » which has transformed the subject back into representation theory of Lie algebras.

Can history happen twice ?

There is a new approach which constructs knot polynomials by using singularity theory [1]. This time, the singularity theory is the study of the discriminant of all singular projections of a (non-singular) knot into the plane. The result are new knot polynomials, let us call them 1-cocycle polynomials. They are graded by natural integers n. Each of the polynomials can be calculated in at most quartic time and they can distinguish the orientations of knots. Greg Kuperberg had observed that if finite type invariants (and hence quantum polynomials) do not distinguish the orientations of knots then they fail to distinguish unoriented knots as well.
There are very many 1-cocycle polynomials for each fixed n and so there is at least a chance that they distinguish perhaps all knots in 3-space. But for the moment they are not related to representation theory at all.

On the other hand there is a new approach which constructs knot polynomials by using solvable Lie algebras [2]. Each of the new polynomials is calculable in polynomial time. They are also graded by natural integers n. It is conjectured that these polynomials are slices of the colored Jones polynomial and that in a similar way each colored quantum polynomial can be sliced into knot polynomials which are calculable in polynomial time.

For a very long time (in fact since 1923) only one polynomial time knot polynomial was known, the Alexander polynomial. Unexpectedly, new polynomial time knot polynomials arised from the very different new approaches mentioned above. We cannot expect that the two approaches are equivalent, but it could well be that there are relations between them. The Alexander polynomial is a very particular special case of the polynomials constructed from the solvable Lie algebra approach (in fact this was a highly non-trivial conjecture for some time). Hence, a first reasonable question would be :

Is the Alexander polynomial determined by 1-cocycle polynomials too?

It is known that the Alexander polynomial is calculable in quartic time exactly as the 1-cocycle polynomials. Hence we could expect a finite linear relation between them, if there is any at all. Unfortunately, there is not yet a computer program in order to calculate any 1-cocycle polynomials (the group working on the 1-cocycle approach is not enough qualified for doing that). So its difficult to get an idea what is going on and to make precise conjectures. On the other hand, the group working on the solvable Lie algebra approach is highly qualified to write such computer programs, as they have proven in the past many times.

Hence, the natural project is to bring these two very different working groups together, to learn from each other, to share ideas, to write computer programs together and perhaps to make connections between these two approaches, which are the only ones which produce knot polynomials calculable in polynomial time. This could turn out to be a major step towards a solution of the knot problem !

[1] Th. Fiedler: Knot polynomials from 1-cocycles, http://arxiv.org/abs/1709.10332

[2] D. Bar-Natan, R. van der Veen: A polynomial time knot polynomial, https://arxiv.org/abs/1708.04853