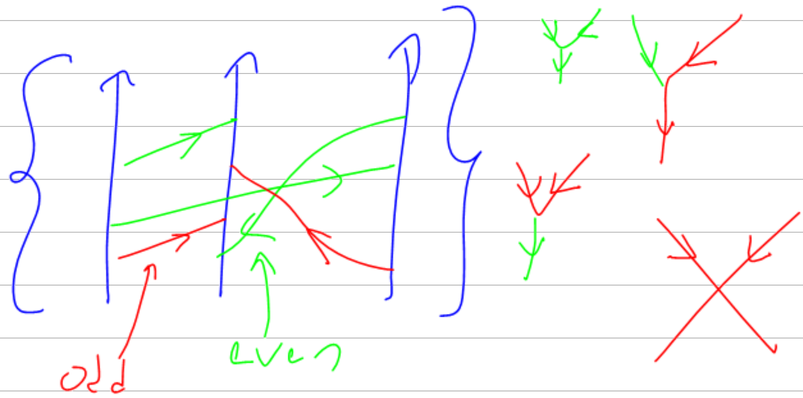
$$A^{w}$$

$$\downarrow$$

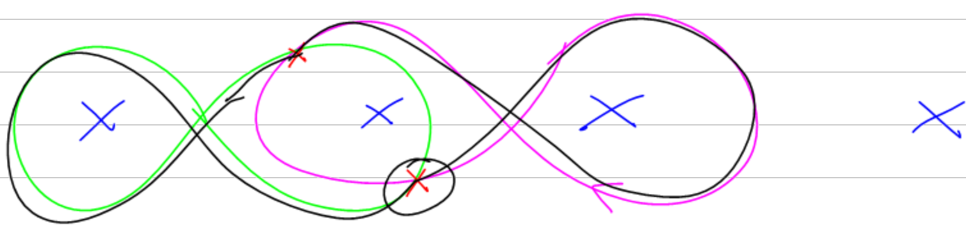
$$A^{\text{sup-}w} =$$



$A \in M_{n\times n}(R)$ $\qquad \begin{pmatrix} 1+\epsilon & \epsilon & \\ \epsilon & 1 & \\ & 1 & 1, \end{pmatrix}$

$\det(1-\epsilon(1-A))$ using

Gaussian elimination in $R[[\epsilon]]/\epsilon^{n+1}$



$$
\begin{tikzcd}
K^u \arrow[r] \arrow[d, "Z^u"] & K^w \arrow[d, "Z^w"] \\
A^u \arrow[r] & A^w
\end{tikzcd}
$$

$$
\begin{tikzcd}
A \arrow[r] \arrow[dr] & B \arrow[r] \arrow[d] & C \arrow[dl] \\
& D &
\end{tikzcd}
$$

$$
\begin{tikzcd}
\mathbb{Z}_\Phi \arrow[r, "\cdot 2"] \arrow[dr, "u"'] & \mathbb{Z}_V \arrow[r] \arrow[d, "w"] & \left\{\frac{a}{2^b}\right\} \arrow[dl, "p"] \\
& R &
\end{tikzcd}
$$

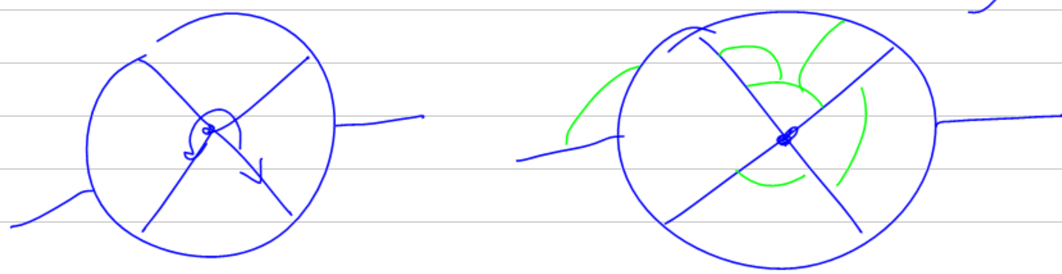
$V = \frac{1}{2}\Phi$

"$\Phi = 4V$"

R ✓ w well-defined

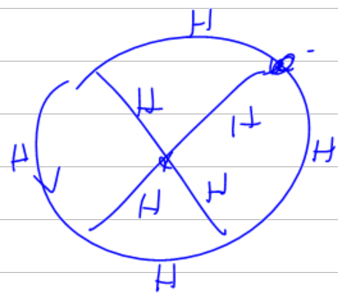
$r \mapsto r/2$ op.

$$\Gamma \longmapsto A(\Gamma)$$

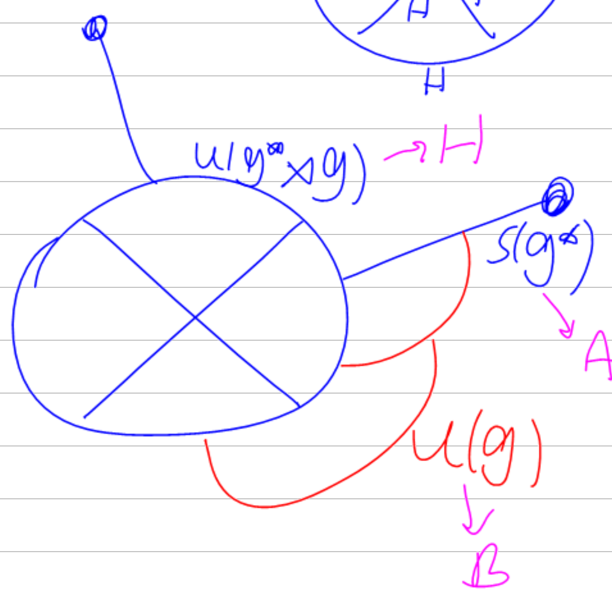


Given a Hopf algebra $\left(\begin{array}{c}\text{involutive} \\ \text{co-commutative} \\ S^2 = I\end{array}\right)$

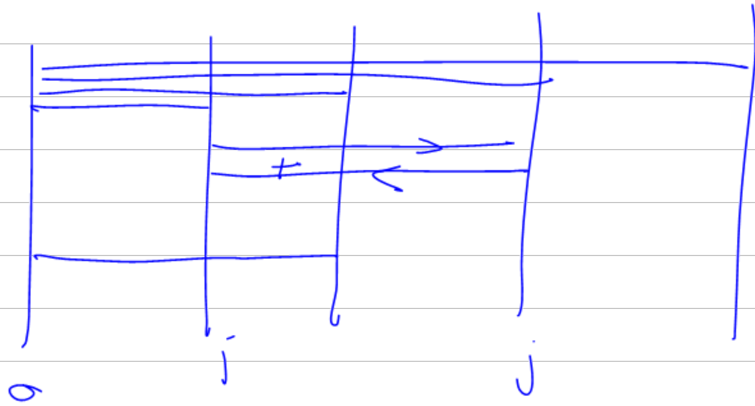
$$G \longmapsto H(G)$$



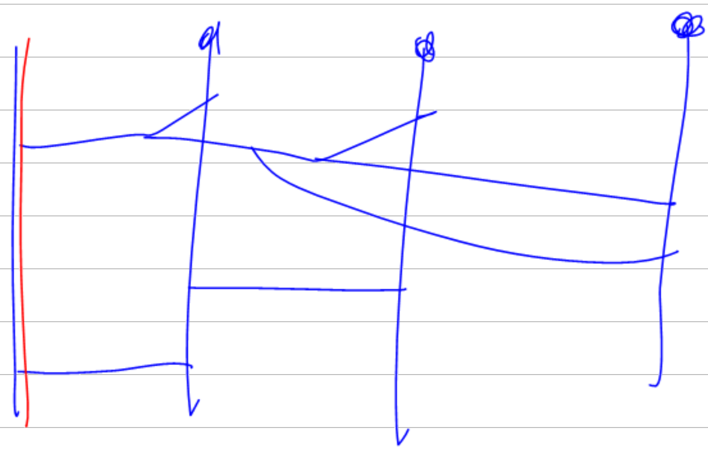
$$= H^{\otimes E} \big/ H^{\otimes V}$$

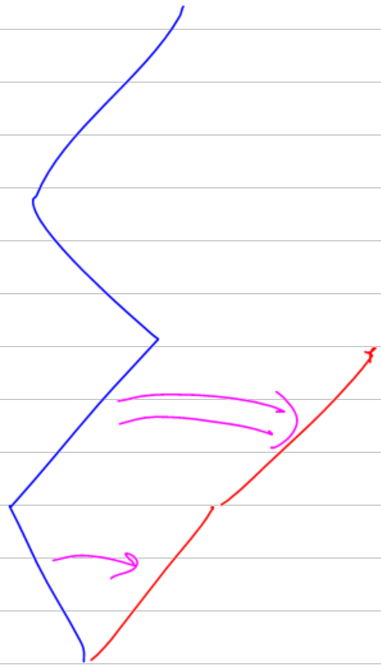
$G =$ 

$\longrightarrow A^\omega(G)$



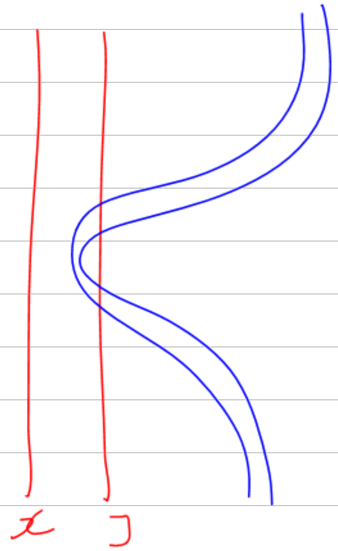
$f_j = FL\langle t_{oj} \rangle$

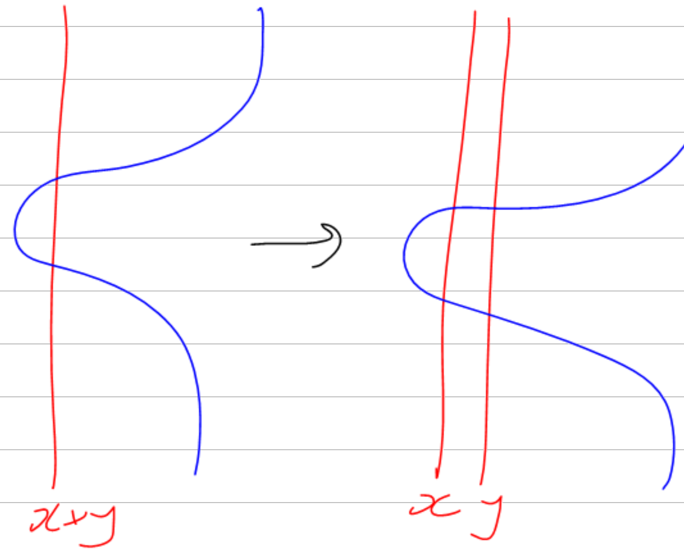




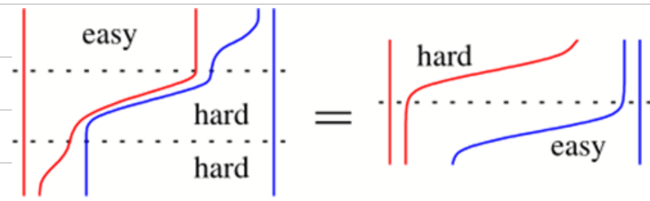
$GRT_7 \twoheadrightarrow GRT_6$

$grt_7 \overset{?}{\twoheadrightarrow} grt_6$





$$A_0 \oplus A_1 \otimes_A A$$



easy
hard
hard
=
hard
easy

$\varphi_a : x \to [x, a]$

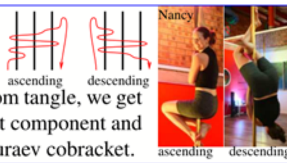
$\varphi_b : x \longmapsto [x, b]$

$$[\varphi_a, \varphi_b](x) = \varphi_a(\varphi_b(x)) - \varphi_b(\varphi_a(x))$$
$$= [[x, b], a] - [[x, a], b]$$
$$= -[[b, a], x] = -[x, [a, b]] = -\varphi_{[a,b]}(x)$$

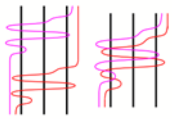
$$[D, D'](x_i) = (DD' - D'D)(x_i) = \underline{D[x_i, a_i']} - D'[x_i, a_i] =$$
$$= [[x_i, a_i], a_i'] + [x_i, Da_i'] - [[x_i, a_i'], a_i] - [x_i, D'a_i] = [x_i, Da_i' - D'a_i + [a_i, a_i']].$$

$[Dx_i, a_i'] + [x_i, Da_i']$

Example 3. With $\gamma \in \bar{\pi}$ and $\lambda_0(\gamma)$ its ascending realization as a bottom tangle and $\lambda_1(\gamma)$ its descending realization as a bottom tangle, we get $\eta_3\colon \bar{\pi} \to \bar{\pi} \otimes |\bar{\pi}|$. Closing the first component and anti-symmetrizing, this is the Turaev cobracket.

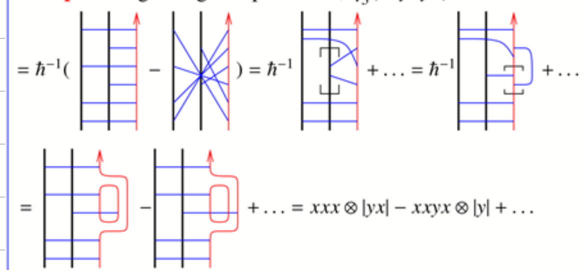


Example 4 [Ma]. With $\gamma \in \bar{\pi}$ and $\lambda_0(\gamma)$ its ascending outer double and $\lambda_1(\gamma)$ its ascending inner double we get $\eta_4\colon \bar{\pi} \to \bar{\pi} \otimes \bar{\pi}$. After some massaging, it too becomes the Turaev cobracket.



Applying gr

Example 3^a. ~~Ignoring complications,~~ $\eta_3^a(xxyxyx) =$



$\eta_3 : \bar{\pi}_{\geq t} \longrightarrow (\bar{\pi} \otimes |\bar{\pi}|)_{\geq t-1}$ degree decreasing filtered.

$$\mathrm{gr}_{-1}\eta_3 : \prod \frac{\bar{\pi}_{\geq t}}{\bar{\pi}_{\geq t+1}} \longrightarrow \prod \frac{(\bar{\pi} \otimes |\bar{\pi}|)_{\geq t-1}}{(\bar{\pi} \otimes |\bar{\pi}|)_{\geq t}}$$

$\wr\mathrm{S}$

$A \xrightarrow{\eta^3_a} A \otimes |A|$ of degree (-1)

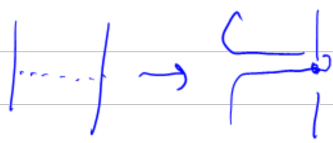
$$\begin{array}{ccc} K & \xrightarrow{\eta} & K \\ \downarrow Z_{hrd} & \Downarrow \mathrm{soft} & \downarrow Z_{hrd} \\ \mathcal{A} & \xrightarrow{\eta^a} & \mathcal{A} \end{array}$$

$$\begin{array}{ccc} K_1 & \xrightarrow{u} & K_2 \\ & \Downarrow & \\ \mathrm{gr}\,K_1 & \xrightarrow{\mathrm{gr}\,u} & \mathrm{gr}\,K_2 \\ \uparrow s & & \uparrow \\ \mathcal{A} & \xrightarrow{u^a} & \mathcal{A} \\ \downarrow & & \\ D & & \end{array}$$

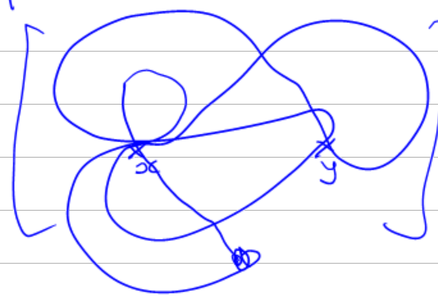
$\mathcal{A} \longrightarrow \mathrm{gr}\,K$

$D \longmapsto [K_D]$

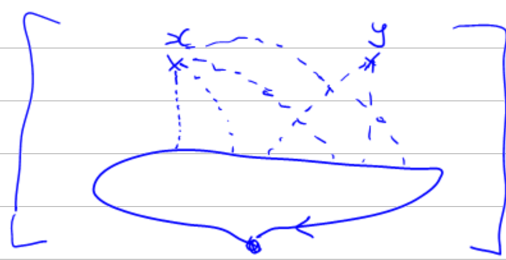
$\forall D \quad u[K_D] = [K_{u^a D}]$



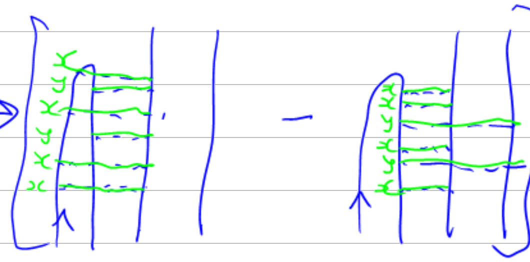
$D \rightsquigarrow xxyxyx \longrightarrow K_{xxyxyx} =$



$=$



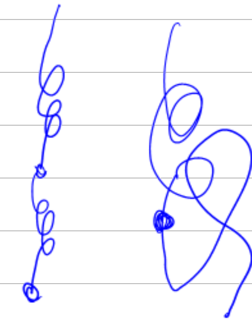
$\xrightarrow{\eta_3}$



$\cdots$ \cdots

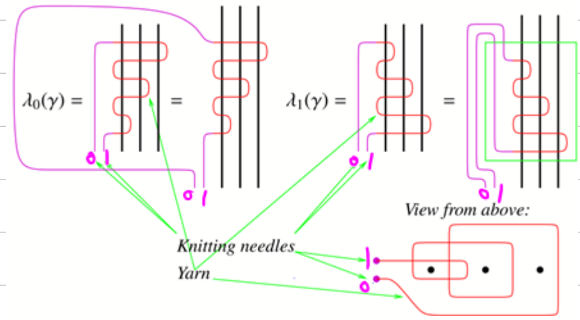
$$\begin{array}{ccc} \bar{\pi} & \xrightarrow{\eta} & \bar{\pi} \otimes |\bar{\pi}| \\ \downarrow tr & & \downarrow tr//Alt \\ |\bar{\pi}| & \xrightarrow{\delta} & |\bar{\pi}| \otimes |\bar{\pi}| \end{array}$$

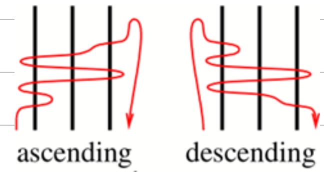
$$\begin{array}{ccc} \bar{\pi} & \xrightarrow{\eta} & \bar{\pi} \otimes |\bar{\pi}| \\ \downarrow tr & & \downarrow tr \\ |\bar{\pi}| & & |\bar{\pi}| \otimes |\bar{\pi}| \end{array}$$



Unignoring the Complications. We need λ_0 and λ_1 such that:

1. $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by flipping all self-intersections from ascending to descending.
2. ~~(Up to conjugation)~~ $\lambda_1(\gamma)$ is obtained from $\lambda_0(\gamma)$ by a global flip. temporarily
3. $Z(\lambda_i(\gamma))$ is computable from $W(\gamma)$ and $Z^{/1}(\lambda_i(\gamma)) = W(\gamma)$.





$\eta = \hbar^{-1}(\lambda_0 - \lambda_1)$

$\exists \zeta_i$ s.t. $Z(\lambda_i(\gamma)) = \zeta_i(W(\gamma))$

$\overline{\pi} \xrightarrow{\tau \text{ (for Turaev)}} \overline{\pi} \otimes |\overline{\pi}|$

$\eta = \hbar^{-1}(\lambda_0 - \lambda_1) : \overline{\pi} \longrightarrow \overline{\pi} \otimes |\overline{\pi}|$ the Turaev operation

$\delta = \eta // tr_1 // Alt$ $\qquad tr_1 : \overline{\pi} \otimes |\overline{\pi}| \longrightarrow |\overline{\pi}| \otimes |\overline{\pi}|$

$W // \delta^a = W // \eta^a // tr_1 // Alt = W // \frac{\lambda_0^a - \lambda_1^a}{\hbar} // tr_1 // Alt$

$\delta // W = \delta // Z^{/1} = \frac{\lambda_0 - \lambda_1}{\hbar} // tr_1 // Alt // Z^{/1} = (\lambda_0 - \lambda_1) // tr_1 // Alt // Z^{/2} // \hbar^{-1}$

$= (\lambda_0 - \lambda_0 // F // C) // tr_1 // Alt // Z^{/2} // \hbar^{-1}$

$= (\lambda_0 - \lambda_0 // F) // tr_1 // Alt // Z^{/2} // \hbar^{-1} = (\lambda_0 - \lambda_0 // F) // Z^{/2} // tr_1 // Alt // \hbar^{-1}$

$= \frac{1}{\hbar} \lambda_0 // Z^{/2} // (1-F) // tr_1 // Alt = \frac{1}{\hbar} W // (1-F) //$

Key 1. $W\colon |\bar{\pi}| \to |A|$ is $Z_H^{/1}\colon \mathcal{K}_H^{/1}(\bigcirc) \to \mathcal{A}_H^{/1}(\bigcirc)$.

Key 2 (Schematic). Suppose $\lambda_0, \lambda_1\colon |\bar{\pi}| \to \mathcal{K}(\bigcirc)$ are two ways of lifting plane curves into knots in PDS_p (namely, $P \circ \lambda_i = I$). Then for $\gamma \in |\bar{\pi}|$,

Lemma 1. "Division by $\hbar$" is well-defined.

$$\eta(\gamma) := (\lambda_0(\gamma) - \lambda_1(\gamma))/\hbar \in \mathcal{K}_H^{/1}(\bigcirc\bigcirc) = |\bar{\pi}| \otimes |\bar{\pi}|$$

and we get an operation η on plane curves. If Kontsevich likes λ_0 and λ_1 (namely if there are λ_i^a with $Z^{/2}(\lambda_i(\gamma)) = \lambda_i^a(W(\gamma))$), then η will have a compatible algebraic companion η^a:

$$\eta^a(\alpha) := (\lambda_0^a(\alpha) - \lambda_1^a(\alpha))/\hbar \in \mathcal{A}_H^{/1}(\bigcirc\bigcirc) = |A| \otimes |A|.$$

For indeed, in $\mathcal{A}_H^{/2}$ we have $\hbar W(\eta(\gamma)) = \hbar Z(\eta(\gamma)) = Z(\lambda_0(\gamma)) - Z(\lambda_1(\gamma)) = \lambda_0^a(W(\gamma)) - \lambda_1^a(W(\gamma)) = \hbar\eta^a(W(\gamma))$.

$$\begin{tikzcd} K(S) \arrow[d] \\ F(S) \end{tikzcd}$$

$K(OO)$ $K(/O)$

$|\bar{\pi}| \times |\bar{\pi}|$

$K(S)$ knots

$C(S)$ curves $C(OO) = |\bar{\pi}| \otimes |\bar{\pi}|$

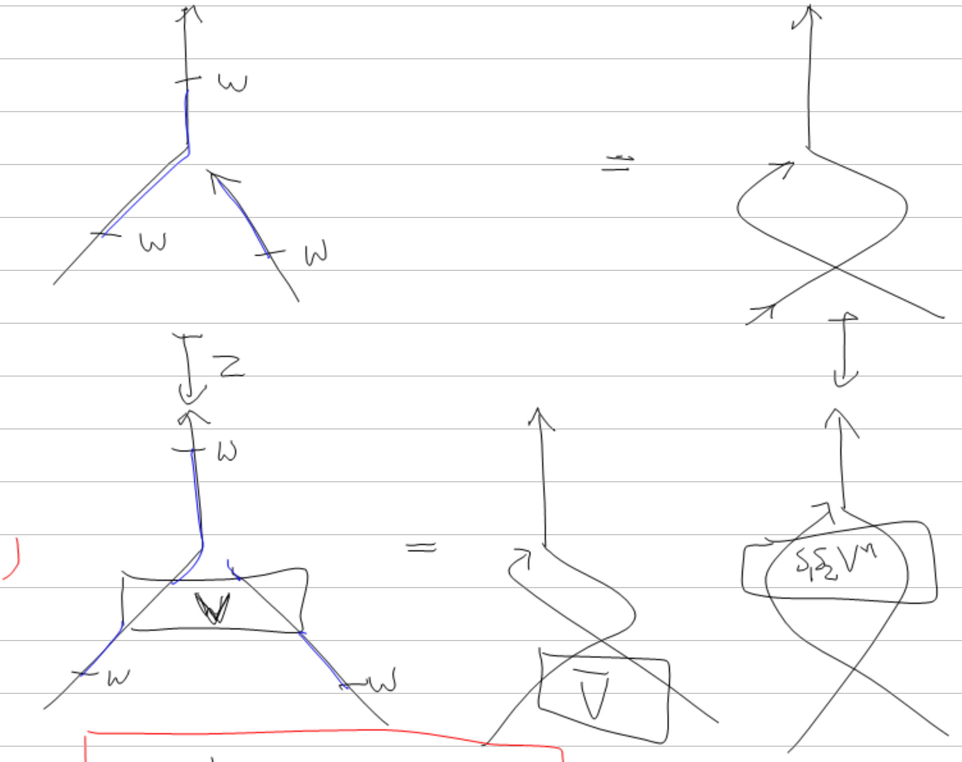
$$\begin{tikzcd} K(S) \arrow[d, "P"] \\ C(S) \end{tikzcd}$$

Example 1

$$\begin{tikzcd} & K(OO) \arrow[d, "P"] \\ C(O\,O) \arrow[ur, "\lambda_0, \lambda_1"] \arrow[r, "id"] & C(OO) \end{tikzcd}$$

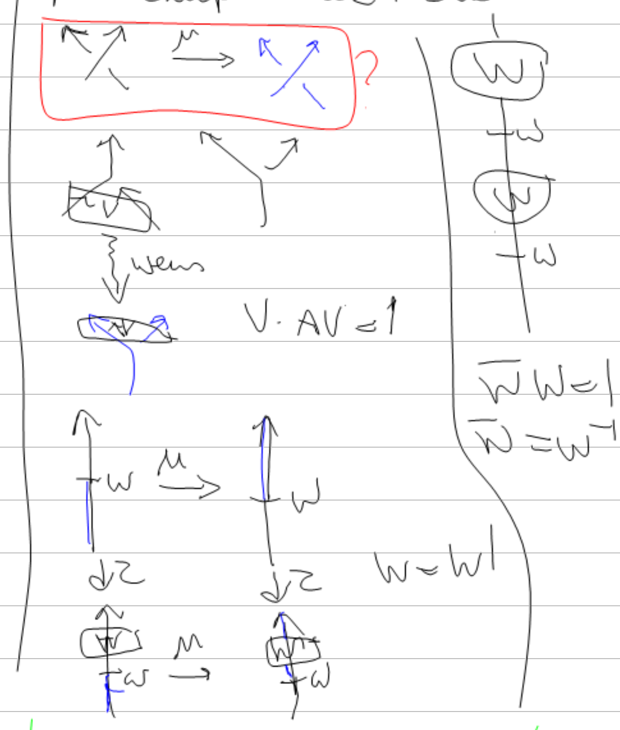
$$\begin{tikzcd} K(OO) \arrow[r, "\lambda_0, \lambda_1"] \arrow[d] & K(OO) \arrow[d, "P"] \\ C(OO) \arrow[r, "I"] & C(OO) \end{tikzcd}$$

$\bar{D} = (-1)^{\#\text{tails}}(D)$



$$\boxed{\bar{V}^{21} = S_1 S_2 V^{-1}}$$

M: swap black / blue

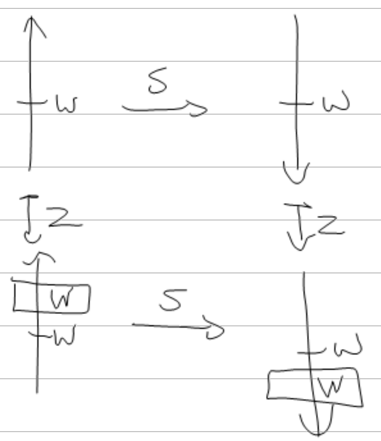


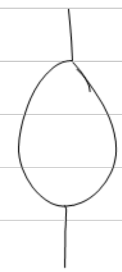
$V \cdot AV = 1$

$\bar{W} W = 1$

$\bar{W} = W^{-1}$

$W = W'$

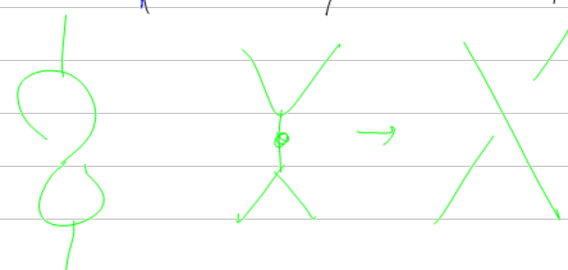


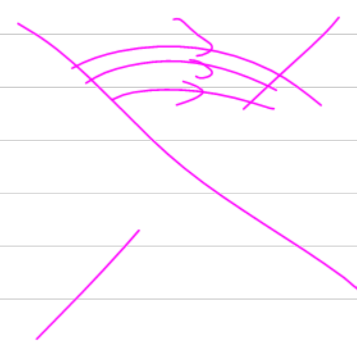


$W = \bar{W}$

$e^{a+w^0+w'} \qquad e^{-a+w^0-w'}$

$\Rightarrow$ W is even wheels

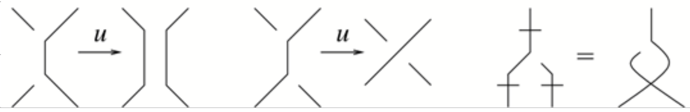


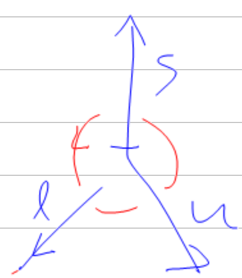


a b c d

231009 Best *w*-practices:

1. Trivalent tangles are end-labeled, to make a circuit algebra.
2. Tubes are bare (no colours and/or orientations).
3. Crossings have no signs; filtration is by comparison with virtual crossings.
4. Vertices are oriented and have marked legs: stem (s), upper (u), and lower (l). They are classical: they satisfy both R4s.
5. Wenjugating interchanges the two vertex types, and adds a virtual $l \leftrightarrow u$ crossing.
6. Only stems can be unzipped. Unzipping untwisted edges connects u to u above a connection of l to l. Unzipping through a wen is defined by wenjugating it out.





"R4 type"

$$slu = usl = lus = l^{-1}s^{-1}u^{-1}$$

$slu \quad sul$

$s^{-1}lu \quad s^{-1}ul$

$sl^{-1}u \quad sl^{-1}u$

$slu^{-1} \quad slu^{-1}$

$$slu \to s^{-1}l^{-1}u^{-1} = uls$$