The Turbo-Gassner Representation

Pensieve header: The turbo Gassner representation. Continues pensieve://2016-06/.

```
In[•]:= Kδ /: Kδ_is__ := KroneckerDelta[1, Length[Union[{is}]]];
```

The Burau Representation

```
In[•]:= Inverse[{{0, t}, {1, 1 - t}}] /. t → t^-1 // Simplify // Transpose // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1-t & t \\ 1 & 0 \end{pmatrix}$$

```
In[•]:= B_i_,j_[ξ_] := ξ /. v_j :↦ (1 - t) v_i + t v_j;
        B̄_i_,j_[ξ_] := ξ /. v_j :↦ (1 - t^-1) v_i + t^-1 v_j;

In[•]:= {{v1, v2, v3} // B1,3, {v1, v2, v3} // B̄1,3}

Out[•]= {{v1, v2, (1 - t) v1 + t v3}, {v1, v2, (1 - 1/t) v1 + v3/t}}

In[•]:= {v1, v2, v3} // B1,3 // B̄1,3 // Expand

Out[•]= {v1, v2, v3}

In[•]:= Column@{R3l = {v1, v2, v3} // B1,2 // B1,3 // B2,3,
          R3r = {v1, v2, v3} // B2,3 // B1,3 // B1,2 ,
          R3l - R3r // Expand}

Out[•]= {v1, (1 - t) v1 + t v2, (1 - t) v1 + t ((1 - t) v2 + t v3)}
        {v1, (1 - t) v1 + t v2, (1 - t) ((1 - t) v1 + t v2) + t ((1 - t) v1 + t v3)}
        {0, 0, 0}
```

The Gassner Representation

```
In[•]:= G_i_,j_[ξ_] := ξ /. v_j :↦ (1 - t_i) v_i + t_i v_j;
        Ḡ_i_,j_[ξ_] := ξ /. v_j :↦ (1 - t_i^-1) v_i + t_i^-1 v_j;

In[•]:= {v1, v2, v3} // G1,3 // Ḡ1,3 // Expand

Out[•]= {v1, v2, v3}

In[•]:= Column@{R3l = {v1, v2, v3} // G1,2 // G1,3 // G2,3,
          R3r = {v1, v2, v3} // G2,3 // G1,3 // G1,2,
          R3l - R3r // Expand}

Out[•]= {v1, (1 - t1) v1 + t1 v2, (1 - t1) v1 + t1 ((1 - t2) v2 + t2 v3)}
        {v1, (1 - t1) v1 + t1 v2, (1 - t2) ((1 - t1) v1 + t1 v2) + t2 ((1 - t1) v1 + t1 v3)}
        {0, 0, 0}
```

```
In[•]:= Column@{OCl = {v1, v2, v3} // G1,2 // G1,3,
          OCr = {v1, v2, v3} // G1,3 // G1,2 ,
          OCl - OCr // Expand}
```

$$\left\{v_1,\ \left(1-t_1\right) v_1+t_1 v_2,\ \left(1-t_1\right) v_1+t_1 v_3\right\}$$

Out[•]=

$$\begin{array}{l}\left\{v_1,\ \left(1-t_1\right) v_1+t_1 v_2,\ \left(1-t_1\right) v_1+t_1 v_3\right\}\\ \{0,\ 0,\ 0\}\end{array}$$

```
In[•]:= Column@{UCl = {v1, v2, v3} // G1,3 // G2,3,
          UCr = {v1, v2, v3} // G2,3 // G1,3 ,
          UCl - UCr // Expand}
```

$$\left\{v_1,\ v_2,\ \left(1-t_1\right) v_1+t_1\left(\left(1-t_2\right) v_2+t_2 v_3\right)\right\}$$

Out[•]=

$$\begin{array}{l}\left\{v_1,\ v_2,\ \left(1-t_2\right) v_2+t_2\left(\left(1-t_1\right) v_1+t_1 v_3\right)\right\}\\ \{0,\ 0,\ v_1-t_1 v_1-t_2 v_1+t_1 t_2 v_1-v_2+t_1 v_2+t_2 v_2-t_1 t_2 v_2\}\end{array}$$

The Gassner-Plus Representation

```
In[•]:= GPi_,j_[ℰ_] := Expand[ℰ /. {uj :> (1 - ti) ui + ti uj,
              f_. vj :> f (1 - ti) vi + f ti vj + (ti - 1) (ti ∂ti f - tj ∂tj f) ui + f ti ui }];
        GP‾i_,j_[ℰ_] := Expand[ℰ /. {uj :> (1 - ti^-1) ui + ti^-1 uj,
              f_. vj :> f (1 - ti^-1) vi + f ti^-1 vj + (ti^-1 - 1) (ti ∂ti f - tj ∂tj f) ui - f ti^-1 ui }];

In[•]:= GPchecks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3};
        GPchecks // GP1,3 // GP‾1,3

Out[•]= {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3}

In[•]:= R3l = GPchecks // GP1,2 // GP1,3 // GP2,3

Out[•]= {f[t1, t2, t3] v1, f[t1, t2, t3] t1 u1 + f[t1, t2, t3] v1 -
           f[t1, t2, t3] t1 v1 + f[t1, t2, t3] t1 v2 + t2 u1 f^(0,1,0)[t1, t2, t3] -
           t1 t2 u1 f^(0,1,0)[t1, t2, t3] - t1 u1 f^(1,0,0)[t1, t2, t3] + t1^2 u1 f^(1,0,0)[t1, t2, t3],
         f[t1, t2, t3] t1 u1 + f[t1, t2, t3] t1 t2 u2 + f[t1, t2, t3] v1 - f[t1, t2, t3] t1 v1 +
           f[t1, t2, t3] t1 v2 - f[t1, t2, t3] t1 t2 v2 + f[t1, t2, t3] t1 t2 v3 + t3 u1 f^(0,0,1)[t1, t2, t3] -
           t1 t3 u1 f^(0,0,1)[t1, t2, t3] + t1 t3 u2 f^(0,0,1)[t1, t2, t3] - t1 t2 t3 u2 f^(0,0,1)[t1, t2, t3] -
           t1 t2 u2 f^(0,1,0)[t1, t2, t3] + t1 t2^2 u2 f^(0,1,0)[t1, t2, t3] - t1 u1 f^(1,0,0)[t1, t2, t3] +
           t1^2 u1 f^(1,0,0)[t1, t2, t3], u1, u1 - t1 u1 + t1 u2, u1 - t1 u1 + t1 u2 - t1 t2 u2 + t1 t2 u3}

In[•]:= R3r = GPchecks // GP2,3 // GP1,3 // GP1,2; R3l - R3r

Out[•]= {0, 0, 0, 0, 0, 0}

In[•]:= OCl = GPchecks // GP1,2 // GP1,3

Out[•]= {f[t1, t2, t3] v1, f[t1, t2, t3] t1 u1 + f[t1, t2, t3] v1 -
           f[t1, t2, t3] t1 v1 + f[t1, t2, t3] t1 v2 + t2 u1 f^(0,1,0)[t1, t2, t3] -
           t1 t2 u1 f^(0,1,0)[t1, t2, t3] - t1 u1 f^(1,0,0)[t1, t2, t3] + t1^2 u1 f^(1,0,0)[t1, t2, t3],
         f[t1, t2, t3] t1 u1 + f[t1, t2, t3] v1 - f[t1, t2, t3] t1 v1 + f[t1, t2, t3] t1 v3 +
           t3 u1 f^(0,0,1)[t1, t2, t3] - t1 t3 u1 f^(0,0,1)[t1, t2, t3] - t1 u1 f^(1,0,0)[t1, t2, t3] +
           t1^2 u1 f^(1,0,0)[t1, t2, t3], u1, u1 - t1 u1 + t1 u2, u1 - t1 u1 + t1 u3}
```

```
In[•]:= OCr = GPchecks // GP1,3 // GP1,2; OCl - OCr
Out[•]= {0, 0, 0, 0, 0, 0}
```

Question. Does GP factor through G? How?

The End(G) Representation

```
In[•]:= EGi_,j_[ℰ_] := Expand[ℰ /. {uj → (1 - ti) ui + ti uj, wi → wi + (1 - ti^-1) wj, wj → ti^-1 wj}];
        EGi_,j_[ℰ_] := Expand[ℰ /. {uj → (1 - ti^-1) ui + ti^-1 uj, wi → wi + (1 - ti) wj, wj → ti wj}];
        EGchecks = Flatten@Table[ui wj, {i, 3}, {j, 3}];
        EGchecks // EG1,3 // EG1,3
Out[•]= {u1 w1, u1 w2, u1 w3, u2 w1, u2 w2, u2 w3, u3 w1, u3 w2, u3 w3}
```

```
Short[R3l = EGchecks // EG1,2 // EG1,3 // EG2,3, 10]
```

$$\Big\{u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1}, \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2}, \frac{u_1 w_3}{t_1 t_2},$$
$$u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + 2 u_1 w_3 - \frac{u_1 w_3}{t_1} -$$
$$t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3, -u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},$$
$$-\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} + \frac{u_2 w_3}{t_2}, u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 -$$
$$\frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 +$$
$$2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$
$$-u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$
$$\frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3, -\frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3\Big\}$$

```
R3r = EGchecks // EG2,3 // EG1,3 // EG1,2; R3l - R3r
{0, 0, 0, 0, 0, 0, 0, 0, 0}
```

```
(# → Collect[EGi,j[#], u_ w_, Simplify]) & /@ {uk wj, uk wi, uj wk, uj wi}
```

$$\Big\{u_k w_j \to \frac{u_k w_j}{t_i}, u_k w_i \to u_k w_i + \left(1 - \frac{1}{t_i}\right) u_k w_j, u_j w_k \to (1 - t_i) u_i w_k + t_i u_j w_k,$$
$$u_j w_i \to (1 - t_i) u_i w_i + t_i u_j w_i - \frac{(-1 + t_i)^2 u_i w_j}{t_i} + (-1 + t_i) u_j w_j\Big\}$$

The End(G)+c Representation

Is there topology behind this representation?

In[•]:=
```
EGc_{i_,j_}[ξ_] := Expand[ξ /. {
      c_i → c_i - (1 - t_i^-1) u_i w_j, c_j → c_j + (1 - t_i^-1) u_i w_j,
      u_j → (1 - t_i) u_i + t_i u_j,
      w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
EGc̄_{i_,j_}[ξ_] := Expand[ξ /. {
      c_i → c_i + (t_i - 1) u_i w_j, c_j → c_j + (1 - t_i) u_i w_j,
      u_j → (1 - t_i^-1) u_i + t_i^-1 u_j,
      w_i → w_i + (1 - t_i) w_j, w_j → t_i w_j}];
EGcchecks = {c1, c2, c3, u1 w1, u1 w2, u1 w3, u2 w1, u2 w2, u2 w3, u3 w1, u3 w2, u3 w3};
EGcchecks // EGc_{1,3} // EGc̄_{1,3}
```

Out[•]=
$$\{c_1, c_2, c_3, u_1 w_1, u_1 w_2, u_1 w_3, u_2 w_1, u_2 w_2, u_2 w_3, u_3 w_1, u_3 w_2, u_3 w_3\}$$

```
(# → Collect[EGc_{1,2}[#], u_ w_, Simplify]) & /@ EGcchecks
```

$$\left\{c_1 \to c_1 + \left(-1 + \frac{1}{t_1}\right) u_1 w_2,\ c_2 \to c_2 + \left(1 - \frac{1}{t_1}\right) u_1 w_2,\right.$$
$$c_3 \to c_3,\ u_1 w_1 \to u_1 w_1 + \left(1 - \frac{1}{t_1}\right) u_1 w_2,\ u_1 w_2 \to \frac{u_1 w_2}{t_1},\ u_1 w_3 \to u_1 w_3,$$
$$u_2 w_1 \to \left(1 - t_1\right) u_1 w_1 + t_1 u_2 w_1 - \frac{\left(-1 + t_1\right)^2 u_1 w_2}{t_1} + \left(-1 + t_1\right) u_2 w_2,\ u_2 w_2 \to \left(-1 + \frac{1}{t_1}\right) u_1 w_2 + u_2 w_2,$$
$$\left.u_2 w_3 \to \left(1 - t_1\right) u_1 w_3 + t_1 u_2 w_3,\ u_3 w_1 \to u_3 w_1 + \left(1 - \frac{1}{t_1}\right) u_3 w_2,\ u_3 w_2 \to \frac{u_3 w_2}{t_1},\ u_3 w_3 \to u_3 w_3\right\}$$

```
u1 w2 + c2 // EGc_{1,2}
```

$$c_2 + u_1 w_2$$

```
(# → Simplify[EGc_{1,2}[#] /. {u_i_ w_i_ → 1, u_i_ w_j_ /; i ≠ j :> 0}]) & /@ EGcchecks
```

$$\{c_1 \to c_1,\ c_2 \to c_2,\ c_3 \to c_3,\ u_1 w_1 \to 1,\ u_1 w_2 \to 0,\ u_1 w_3 \to 0,$$
$$u_2 w_1 \to 0,\ u_2 w_2 \to 1,\ u_2 w_3 \to 0,\ u_3 w_1 \to 0,\ u_3 w_2 \to 0,\ u_3 w_3 \to 1\}$$

```
Short[R3l = EGcchecks // EGc1,2 // EGc1,3 // EGc2,3, 10]
```

$$\left\{c_1 - u_1 w_2 + \frac{u_1 w_2}{t_1} - u_1 w_3 + \frac{u_1 w_3}{t_1},\ c_2 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2},\right.$$
$$c_3 + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},\ u_1 w_1 + u_1 w_2 - \frac{u_1 w_2}{t_1} + u_1 w_3 - \frac{u_1 w_3}{t_1},\ \frac{u_1 w_2}{t_1} + \frac{u_1 w_3}{t_1} - \frac{u_1 w_3}{t_1 t_2},$$
$$\frac{u_1 w_3}{t_1 t_2},\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 + 2 u_1 w_2 - \frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + 2 u_1 w_3 - \frac{u_1 w_3}{t_1} -$$
$$t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3,\ -u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + u_2 w_3 - \frac{u_2 w_3}{t_2},$$
$$- \frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} + \frac{u_2 w_3}{t_2},\ u_1 w_1 - t_1 u_1 w_1 + t_1 u_2 w_1 - t_1 t_2 u_2 w_1 + t_1 t_2 u_3 w_1 + 2 u_1 w_2 -$$
$$\frac{u_1 w_2}{t_1} - t_1 u_1 w_2 - u_2 w_2 + t_1 u_2 w_2 + t_2 u_2 w_2 - t_1 t_2 u_2 w_2 - t_2 u_3 w_2 + t_1 t_2 u_3 w_2 +$$
$$2 u_1 w_3 - \frac{u_1 w_3}{t_1} - t_1 u_1 w_3 - u_2 w_3 + t_1 u_2 w_3 + t_2 u_2 w_3 - t_1 t_2 u_2 w_3 - t_2 u_3 w_3 + t_1 t_2 u_3 w_3,$$
$$- u_1 w_2 + \frac{u_1 w_2}{t_1} + u_2 w_2 - t_2 u_2 w_2 + t_2 u_3 w_2 - u_1 w_3 + \frac{u_1 w_3}{t_1} + \frac{u_1 w_3}{t_2} - \frac{u_1 w_3}{t_1 t_2} + 2 u_2 w_3 -$$
$$\left.\frac{u_2 w_3}{t_2} - t_2 u_2 w_3 - u_3 w_3 + t_2 u_3 w_3,\ - \frac{u_1 w_3}{t_2} + \frac{u_1 w_3}{t_1 t_2} - u_2 w_3 + \frac{u_2 w_3}{t_2} + u_3 w_3\right\}$$

```
R3r = EGcchecks // EGc2,3 // EGc1,3 // EGc1,2; R3l - R3r
```

$$\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$$

The Turbo-Gassner Representation

```
In[•]:= TGi_,j_[ℰ_] := Expand[ℰ /. {
          f_. vk_ :> Plus[f vk /. vj → (1 - ti) vi + ti vj,
            (1 - ti^-1) (ti ∂ti f - tj ∂tj f) (uk /. uj → (1 - ti) ui + ti uj) ui wj,
            Kδk,i f (uj - ui) ui wj],
          uj → (1 - ti) ui + ti uj,
          wi → wi + (1 - ti^-1) wj, wj → ti^-1 wj}];
        TG̅i_,j_[ℰ_] := Expand[ℰ /. {
          f_. vk_ :> Plus[f vk /. vj → (1 - ti^-1) vi + ti^-1 vj,
            (1 - ti) (ti ∂ti f - tj ∂tj f) (uk /. uj → (1 - ti^-1) ui + ti^-1 uj) ui wj,
            Kδk,i f (ui - uj) ui wj],
          uj → (1 - ti^-1) ui + ti^-1 uj,
          wi → wi + (1 - ti) wj, wj → ti wj}];
        TGchecks = {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3};
        TGchecks // TG1,3 // TG̅1,3
```

```
Out[•]= {f[t1, t2, t3] v1, f[t1, t2, t3] v2, f[t1, t2, t3] v3, u1, u2, u3, w1, w2, w3}
```

```
In[•]:= Short[R3l = TGchecks // TG1,2 // TG1,3 // TG2,3, 10]
```

Out[•]//Short=
$$\Big\{f[t_1,t_2,t_3]\,v_1 - f[t_1,t_2,t_3]\,u_1^2\,w_2 + f[t_1,t_2,t_3]\,u_1\,u_2\,w_2 - f[t_1,t_2,t_3]\,u_1^2\,w_3 + f[t_1,t_2,t_3]\,u_1\,u_3\,w_3 - \frac{t_3\,u_1^2\,w_3\,f^{(0,0,1)}[t_1,t_2,t_3]}{t_2} + \text{<<15>>} + t_2\,u_1\,u_2\,w_3\,f^{(0,1,0)}[t_1,t_2,t_3] - u_1^2\,w_2\,f^{(1,0,0)}[t_1,t_2,t_3] + t_1\,u_1^2\,w_2\,f^{(1,0,0)}[t_1,t_2,t_3] - u_1^2\,w_3\,f^{(1,0,0)}[t_1,t_2,t_3] + t_1\,u_1^2\,w_3\,f^{(1,0,0)}[t_1,t_2,t_3],\ f[t_1,t_2,t_3]\,v_1 - \text{<<1>>}\,t_1\,v_1 + \text{<<63>>} + t_1^2\,u_1\,u_2\,w_3\,f^{(1,0,0)}[t_1,t_2,t_3],\ \text{<<5>>},\ \text{<<1>>},\ \frac{w_3}{t_1\,t_2}\Big\}$$

```
In[•]:= R3r = TGchecks // TG2,3 // TG1,3 // TG1,2; R3l - R3r
```

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0}

```
In[•]:= Short[OCl = TGchecks // TG1,2 // TG1,3]
```

Out[•]//Short=
$$\Big\{\text{<<18>>} + t_1\,\text{<<2>>}\,f^{\text{<<1>>}}[t_1,t_2,t_3],\ \text{<<1>>},\ \text{<<6>>},\ \frac{\text{<<1>>}}{\text{<<1>>}}\Big\}$$

```
In[•]:= OCr = TGchecks // TG1,3 // TG1,2; OCl - OCr
```

Out[•]= {0, -f[t1, t2, t3] u1 u2 w3 + f[t1, t2, t3] t1 u1 u2 w3 + f[t1, t2, t3] u1 u3 w3 - f[t1, t2, t3] t1 u1 u3 w3, -f[t1, t2, t3] u1 u2 w2 + f[t1, t2, t3] t1 u1 u2 w2 + f[t1, t2, t3] u1 u3 w2 - f[t1, t2, t3] t1 u1 u3 w2, 0, 0, 0, 0, 0, 0}

A Finite-Rank Turbo-Gassner Representation

```
In[•]:= η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
FTG_{i_,j_}[ℰ_] := Expand[ℰ /. {
      f_. v_k_ :> Plus[f v_k /. v_j → (1 - t_i - η[i]) v_i + (t_i + η[i]) v_j,
         (1 - t_i^-1) (t_i ∂_η[i] f - t_j ∂_η[j] f) (u_k /. u_j → (1 - t_i) u_i + t_i u_j) u_i w_j,
         Kδ_k,i (f /. _η → 0) (u_j - u_i) u_i w_j],
      u_j → (1 - t_i) u_i + t_i u_j,
      w_i → w_i + (1 - t_i^-1) w_j, w_j → t_i^-1 w_j}];
FTG‾_{i_,j_}[ℰ_] := Expand[ℰ /. {
      f_. v_k_ :> Plus[f v_k /. v_j → (1 - t_i^-1 + t_i^-2 η[i]) v_i + (t_i^-1 - t_i^-2 η[i]) v_j,
         (1 - t_i) (t_i ∂_η[i] f - t_j ∂_η[j] f) (u_k /. u_j → (1 - t_i^-1) u_i + t_i^-1 u_j) u_i w_j,
         Kδ_k,i (f /. _η → 0) (u_i - u_j) u_i w_j],
      u_j → (1 - t_i^-1) u_i + t_i^-1 u_j,
      w_i → w_i + (1 - t_i) w_j, w_j → t_i w_j}];
ff = f0 + f1 η[1] + f2 η[2] + f3 η[3];
FTGchecks = {ff v1, ff v2, ff v3, u1, u2, u3, w1, w2, w3};
Expand[(FTGchecks // FTG1,3 // FTG‾1,3) - FTGchecks]
```

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0}

```
In[•]:= {v1, v2} // FTG1,2 // Column
```

Out[•]=
$$v_1 - u_1^2\,w_2 + u_1\,u_2\,w_2$$
$$v_1 - t_1\,v_1 + t_1\,v_2 - v_1\,\eta[1] + v_2\,\eta[1]$$

In[•]:= **Short[R3l = FTGchecks // FTG$_{1,2}$ // FTG$_{1,3}$ // FTG$_{2,3}$, 10]**

Out[•]//Short= $\left\{f_0\, v_1 - f_0\, u_1^2\, w_2 - f_1\, u_1^2\, w_2 + f_1\, t_1\, u_1^2\, w_2 - f_2\, t_2\, u_1^2\, w_2 + \frac{f_2\, t_2\, u_1^2\, w_2}{t_1} + f_0\, u_1\, u_2\, w_2 - f_0\, u_1^2\, w_3 - f_1\, u_1^2\, w_3 + f_2\, u_1^2\, w_3 - \frac{f_2\, u_1^2\, w_3}{t_1} + f_1\, t_1\, u_1^2\, w_3 - f_2\, t_2\, u_1^2\, w_3 + \frac{f_2\, t_2\, u_1^2\, w_3}{t_1} - \frac{f_3\, t_3\, u_1^2\, w_3}{t_2} + \frac{f_3\, t_3\, u_1^2\, w_3}{t_1\, t_2} - f_2\, u_1\, u_2\, w_3 + f_2\, t_2\, u_1\, u_2\, w_3 - f_3\, t_3\, u_1\, u_2\, w_3 + \frac{f_3\, t_3\, u_1\, u_2\, w_3}{t_2} + f_0\, u_1\, u_3\, w_3 + f_1\, v_1\, \eta[1] + f_2\, v_1\, \eta[2] + f_3\, v_1\, \eta[3],\ \text{<<7>>},\ \frac{w_3}{t_1\, t_2}\right\}$

In[•]:= **R3r = FTGchecks // FTG$_{2,3}$ // FTG$_{1,3}$ // FTG$_{1,2}$; R3l - R3r**

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0}

In[•]:= **Short[OCl = FTGchecks // FTG$_{1,2}$ // FTG$_{1,3}$]**

Out[•]//Short= $\left\{\text{<<20>>} + f_2\, v_1\, \eta[2] + f_3\, v_1\, \eta[3],\ \text{<<7>>},\ \frac{w_3}{t_1}\right\}$

In[•]:= **Short[OCr = FTGchecks // FTG$_{1,3}$ // FTG$_{1,2}$]**

Out[•]//Short= $\left\{\text{<<20>>} + f_2\, v_1\, \eta[2] + f_3\, v_1\, \eta[3],\ \text{<<7>>},\ \frac{w_3}{t_1}\right\}$

In[•]:= **OCl - OCr**

Out[•]= {0, $-f_0\, u_1\, u_2\, w_3 + f_0\, t_1\, u_1\, u_2\, w_3 + f_0\, u_1\, u_3\, w_3 - f_0\, t_1\, u_1\, u_3\, w_3$,
$-f_0\, u_1\, u_2\, w_2 + f_0\, t_1\, u_1\, u_2\, w_2 + f_0\, u_1\, u_3\, w_2 - f_0\, t_1\, u_1\, u_3\, w_2$, 0, 0, 0, 0, 0, 0}

In[•]:= **ff = f$_0$ + f$_1$ η[1] + f$_2$ η[2] + f$_3$ η[3] + f$_4$ η[4];**
FTGchecks4 = Expand@{ff v$_1$, ff v$_2$, ff v$_3$, ff v$_4$, u$_1$, u$_2$, u$_3$, u$_4$, w$_1$, w$_2$, w$_3$, w$_4$};

In[•]:= **Short[FOC1 = FTGchecks4 // FTG$_{1,2}$ // FTG$_{1,3}$ // FTG$_{1,4}$]**

Out[•]//Short= $\left\{\text{<<30>>} + f_3\, v_1\, \eta[3] + f_4\, v_1\, \eta[4],\ \text{<<1>>},\ \text{<<1>>},\ \text{<<6>>},\ \frac{\text{<<1>>}}{\text{<<1>>}},\ \frac{w_3}{t_1},\ \frac{w_4}{t_1}\right\}$

In[•]:= **Short[FOC2 = FTGchecks4 // FTG$_{1,3}$ // FTG$_{1,2}$ // FTG$_{1,4}$]**

Out[•]//Short= $\left\{\text{<<30>>} + f_3\, v_1\, \eta[3] + f_4\, v_1\, \eta[4],\ \text{<<1>>},\ \text{<<1>>},\ \text{<<6>>},\ \frac{\text{<<1>>}}{\text{<<1>>}},\ \frac{w_3}{t_1},\ \frac{w_4}{t_1}\right\}$

In[•]:= **Short[FOC3 = FTGchecks4 // FTG$_{1,4}$ // FTG$_{1,2}$ // FTG$_{1,3}$]**

Out[•]//Short= $\left\{\text{<<30>>} + f_3\, v_1\, \eta[3] + f_4\, v_1\, \eta[4],\ \text{<<1>>},\ \text{<<1>>},\ \text{<<6>>},\ \frac{\text{<<1>>}}{\text{<<1>>}},\ \frac{w_3}{t_1},\ \frac{w_4}{t_1}\right\}$

In[•]:= **Short[FOC4 = FTGchecks4 // FTG$_{1,4}$ // FTG$_{1,3}$ // FTG$_{1,2}$]**

Out[•]//Short= $\left\{\text{<<30>>} + f_3\, v_1\, \eta[3] + f_4\, v_1\, \eta[4],\ \text{<<10>>},\ \frac{w_4}{t_1}\right\}$

In[•]:= **FOC1 - FOC2 - FOC3 + FOC4**

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[•]:= **FOC1 - FOC2**

Out[•]= $\{0, -f_0 u_1 u_2 w_3 + f_0 t_1 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 - f_0 t_1 u_1 u_3 w_3,$
$-f_0 u_1 u_2 w_2 + f_0 t_1 u_1 u_2 w_2 + f_0 u_1 u_3 w_2 - f_0 t_1 u_1 u_3 w_2, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$

In[•]:= **FOC2 - FOC3**

Out[•]= $\{0, f_0 u_1 u_2 w_3 - f_0 t_1 u_1 u_2 w_3 - f_0 u_1 u_3 w_3 + f_0 t_1 u_1 u_3 w_3 -$
$f_0 u_1 u_2 w_4 + f_0 t_1 u_1 u_2 w_4 + f_0 u_1 u_4 w_4 - f_0 t_1 u_1 u_4 w_4, f_0 u_1 u_2 w_2 - f_0 t_1 u_1 u_2 w_2 -$
$f_0 u_1 u_3 w_2 + f_0 t_1 u_1 u_3 w_2 - f_0 u_1 u_3 w_4 + f_0 t_1 u_1 u_3 w_4 + f_0 u_1 u_4 w_4 - f_0 t_1 u_1 u_4 w_4,$
$-f_0 u_1 u_2 w_2 + f_0 t_1 u_1 u_2 w_2 + f_0 u_1 u_4 w_2 - f_0 t_1 u_1 u_4 w_2 - f_0 u_1 u_3 w_3 +$
$f_0 t_1 u_1 u_3 w_3 + f_0 u_1 u_4 w_3 - f_0 t_1 u_1 u_4 w_3, 0, 0, 0, 0, 0, 0, 0, 0\}$

In[•]:= **FOC1 - FOC3**

Out[•]= $\{0, -f_0 u_1 u_2 w_4 + f_0 t_1 u_1 u_2 w_4 + f_0 u_1 u_4 w_4 - f_0 t_1 u_1 u_4 w_4,$
$-f_0 u_1 u_3 w_4 + f_0 t_1 u_1 u_3 w_4 + f_0 u_1 u_4 w_4 - f_0 t_1 u_1 u_4 w_4, -f_0 u_1 u_2 w_2 + f_0 t_1 u_1 u_2 w_2 + f_0 u_1 u_4 w_2 -$
$f_0 t_1 u_1 u_4 w_2 - f_0 u_1 u_3 w_3 + f_0 t_1 u_1 u_3 w_3 + f_0 u_1 u_4 w_3 - f_0 t_1 u_1 u_4 w_3, 0, 0, 0, 0, 0, 0, 0, 0\}$

In[•]:= **(FTGchecks4 // $FTG_{1,2}$ // $FTG_{1,3}$ // $\overline{FTG}_{1,2}$ // $\overline{FTG}_{1,3}$ // $FTG_{1,4}$ // $FTG_{1,3}$ // $FTG_{1,2}$ // $\overline{FTG}_{1,3}$ // $\overline{FTG}_{1,2}$ // $\overline{FTG}_{1,4}$) - FTGchecks4**

Out[•]= $\{0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$

In[•]:= **(FTGchecks4 // $FTG_{1,2}$ // $FTG_{1,3}$ // $\overline{FTG}_{1,2}$ // $\overline{FTG}_{1,3}$ // $FTG_{1,3}$ // $FTG_{1,3}$ // $FTG_{1,2}$ // $\overline{FTG}_{1,3}$ // $\overline{FTG}_{1,2}$ // $\overline{FTG}_{1,3}$) - FTGchecks4**

Out[•]= $\{0, f_0 u_1^2 w_3 - 2 f_0 t_1 u_1^2 w_3 + f_0 t_1^2 u_1^2 w_3 - f_0 u_1 u_2 w_3 + 2 f_0 t_1 u_1 u_2 w_3 - f_0 t_1^2 u_1 u_2 w_3,$
$f_0 u_1^2 w_2 - 2 f_0 t_1 u_1^2 w_2 + f_0 t_1^2 u_1^2 w_2 - f_0 u_1 u_2 w_2 + 2 f_0 t_1 u_1 u_2 w_2 - f_0 t_1^2 u_1 u_2 w_2,$
$0, 0, 0, 0, 0, 0, 0, 0, 0\}$

In[•]:= **ff = $f_0 + f_1 \eta[1] + f_2 \eta[2] + f_3 \eta[3] + f_4 \eta[4] + f_5 \eta[5]$;**
FTGchecks5 = Expand@{ff v_1, ff v_2, ff v_3, ff v_4, ff v_5 u_1, u_2, u_3, u_4, u_5, w_1, w_2, w_3, w_4, w_5};
(FTGchecks5 // $FTG_{1,2}$ // $FTG_{1,3}$ // $\overline{FTG}_{1,2}$ // $\overline{FTG}_{1,3}$ // $FTG_{1,4}$ // $FTG_{1,5}$ // $\overline{FTG}_{1,4}$ // $\overline{FTG}_{1,5}$) - FTGchecks5

Out[•]= $\{0, -f_0 u_1 u_2 w_3 + f_0 t_1 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 - f_0 t_1 u_1 u_3 w_3,$
$-f_0 u_1 u_2 w_2 + f_0 t_1 u_1 u_2 w_2 + f_0 u_1 u_3 w_2 - f_0 t_1 u_1 u_3 w_2,$
$-f_0 u_1 u_4 w_5 + f_0 t_1 u_1 u_4 w_5 + f_0 u_1 u_5 w_5 - f_0 t_1 u_1 u_5 w_5,$
$-f_0 u_1^2 u_4 w_4 + f_0 t_1 u_1^2 u_4 w_4 + f_0 u_1^2 u_5 w_4 - f_0 t_1 u_1^2 u_5 w_4, 0, 0, 0, 0, 0, 0, 0, 0, 0\}$

The Turbo-Burau Representation

```
In[•]:= η /: η[i_]^2 = 0; η /: η[i_] η[j_] = 0;
        TB_i_,j_[ξ_] :=
          Expand[ξ /. {
              f_. v_k_ :> Plus[f v_k /. v_j → (1 - t - η[i]) v_i + (t + η[i]) v_j,
                 (t - 1) (∂_η[i] f - ∂_η[j] f) (u_k /. u_j → (1 - t) u_i + t u_j) u_i w_j,
                 K δ_k,i (f /. _η → 0) (u_j - u_i) u_i w_j],
              u_j → (1 - t) u_i + t u_j,
              w_i → w_i + (1 - t^-1) w_j, w_j → t^-1 w_j}];
        ff = f_0 + f_1 η[1] + f_2 η[2] + f_3 η[3];
        TBchecks = {ff v_1, ff v_2, ff v_3, u_1^2 w_1, u_1^2 w_2, u_1, u_2, u_3, w_1, w_2, w_3};

In[•]:= Short[R3l = TBchecks // TB_1,2 // TB_1,3 // TB_2,3, 10]

Out[•]//Short= {f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + t f_1 u_1^2 w_2 + f_2 u_1^2 w_2 - t f_2 u_1^2 w_2 + f_0 u_1 u_2 w_2 - f_0 u_1^2 w_3 - f_1 u_1^2 w_3 +
  t f_1 u_1^2 w_3 + 2 f_2 u_1^2 w_3 - f_2 u_1^2 w_3 / t - t f_2 u_1^2 w_3 - f_3 u_1^2 w_3 + f_3 u_1^2 w_3 / t - f_2 u_1 u_2 w_3 + t f_2 u_1 u_2 w_3 +
  f_3 u_1 u_2 w_3 - t f_3 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 + f_1 v_1 η[1] + f_2 v_1 η[2] + f_3 v_1 η[3], <<9>>, w_3 / t^2}

In[•]:= R3r = TBchecks // TB_2,3 // TB_1,3 // TB_1,2; R3l - R3r

Out[•]= {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0}

In[•]:= Short[OCl = TBchecks // TB_1,2 // TB_1,3]

Out[•]//Short= {f_0 v_1 - f_0 u_1^2 w_2 - f_1 u_1^2 w_2 + t f_1 u_1^2 w_2 + <<12>> + f_0 u_1 u_3 w_3 + f_1 v_1 η[1] + f_2 v_1 η[2] + f_3 v_1 η[3],
  <<1>>, <<1>>, <<5>>, w_1 + <<6>>, w_2 / t, w_3 / t}

In[•]:= OCr = TBchecks // TB_1,3 // TB_1,2; OCl - OCr

Out[•]= {0, -f_0 u_1 u_2 w_3 + t f_0 u_1 u_2 w_3 + f_0 u_1 u_3 w_3 - t f_0 u_1 u_3 w_3,
  -f_0 u_1 u_2 w_2 + t f_0 u_1 u_2 w_2 + f_0 u_1 u_3 w_2 - t f_0 u_1 u_3 w_2, 0, 0, 0, 0, 0, 0, 0, 0}
```