Gassner representation for 

$$\begin{pmatrix} \frac{t_j - 1}{t_i} & \frac{1}{t_i} \\ 1 & 0 \end{pmatrix}$$

```
In[•]:= σ_i_,j_[ℰ_] := Expand[ℰ /. {x_i → x_i + (t_j - 1)/t_i x_j, x_j → 1/t_i x_j}]

In[•]:= {x_1, x_2, x_3} // σ_1,2 // σ_1,3

Out[•]=
```

$$\left\{x_1 - \frac{x_2}{t_1} + \frac{t_2\, x_2}{t_1} - \frac{x_3}{t_1} + \frac{t_3\, x_3}{t_1}, \frac{x_2}{t_1}, \frac{x_3}{t_1}\right\}$$

```
In[•]:= lhs = {x_1, x_2, x_3} // σ_1,2 // σ_1,3 // σ_2,3

Out[•]=
```

$$\left\{x_1 - \frac{x_2}{t_1} + \frac{t_2\, x_2}{t_1} - \frac{x_3}{t_1} + \frac{t_3\, x_3}{t_1}, \frac{x_2}{t_1} - \frac{x_3}{t_1\, t_2} + \frac{t_3\, x_3}{t_1\, t_2}, \frac{x_3}{t_1\, t_2}\right\}$$

```
In[•]:= {x_1, x_2, x_3} // σ_2,3 // σ_1,3

Out[•]=
```

$$\left\{x_1 - \frac{x_3}{t_1} + \frac{t_3\, x_3}{t_1}, x_2 - \frac{x_3}{t_1\, t_2} + \frac{t_3\, x_3}{t_1\, t_2}, \frac{x_3}{t_1\, t_2}\right\}$$

```
In[•]:= rhs = {x_1, x_2, x_3} // σ_2,3 // σ_1,3 // σ_1,2

Out[•]=
```

$$\left\{x_1 - \frac{x_2}{t_1} + \frac{t_2\, x_2}{t_1} - \frac{x_3}{t_1} + \frac{t_3\, x_3}{t_1}, \frac{x_2}{t_1} - \frac{x_3}{t_1\, t_2} + \frac{t_3\, x_3}{t_1\, t_2}, \frac{x_3}{t_1\, t_2}\right\}$$

```
In[•]:= lhs == rhs

Out[•]=
True

In[•]:= {{t_i, 0}, {0, t_j}} . {{(t_j - 1)/t_i, 1/t_i}, {1, 0}} . {{t_j^-1, 0}, {0, t_i^-1}}

Out[•]=
```

$$\left\{\left\{\frac{-1 + t_j}{t_j}, \frac{1}{t_i}\right\}, \{1, 0\}\right\}$$

In[•]:=
```
Simplify[{{(-1 + t_j)/t_j, 1/t_i}, {1, 0}} /. {t_i_ :> t_i^-1}] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 1-t_j & t_i \\ 1 & 0 \end{pmatrix}$$

In[•]:=
```
n = 5
```

Out[•]=

5

In[•]:=
```
σ_i_[ℰ_] := Expand[ℰ /. {t_i → t_(i+1), t_(i+1) → t_i, x_i → x_(i+1) + (t_i - 1)/t_(i+1) x_i, x_(i+1) → 1/t_(i+1) x_i}]
```

In[•]:=
```
{x_1, x_2, x_3, x_4} // σ_1
```

Out[•]=

$$\left\{-\frac{x_1}{t_2}+\frac{t_1\,x_1}{t_2}+x_2,\ \frac{x_1}{t_2},\ x_3,\ x_4\right\}$$

In[•]:=
```
{x_1, x_2, x_3, x_4} // σ_1 // σ_2 // σ_1
```

Out[•]=

$$\left\{-\frac{x_1}{t_3}+\frac{t_1\,x_1}{t_3}-\frac{x_2}{t_3}+\frac{t_2\,x_2}{t_3}+x_3,\ -\frac{x_1}{t_2\,t_3}+\frac{t_1\,x_1}{t_2\,t_3}+\frac{x_2}{t_3},\ \frac{x_1}{t_2\,t_3},\ x_4\right\}$$

In[•]:=
```
{x_1, x_2, x_3, x_4} // σ_2 // σ_1 // σ_2
```

Out[•]=

$$\left\{-\frac{x_1}{t_3}+\frac{t_1\,x_1}{t_3}-\frac{x_2}{t_3}+\frac{t_2\,x_2}{t_3}+x_3,\ -\frac{x_1}{t_2\,t_3}+\frac{t_1\,x_1}{t_2\,t_3}+\frac{x_2}{t_3},\ \frac{x_1}{t_2\,t_3},\ x_4\right\}$$

In[•]:=
```
f_i_ := Table[Coefficient[σ_i[x_j], x_k], {j, n}, {k, n}]
```

In[•]:=
```
f_1 // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} -\frac{1}{t_2}+\frac{t_1}{t_2} & 1 & 0 & 0 & 0 \\ \frac{1}{t_2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[•]:=
```
Simplify[f_1.f_2.f_1] // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} \frac{-t_2+t_2^2+(-1+t_1)^2\,t_3}{t_2^2\,t_3} & \frac{-1+t_1}{t_2} & 1 & 0 & 0 \\ \frac{-1+t_1}{t_2^2} & \frac{1}{t_2} & 0 & 0 & 0 \\ \frac{1}{t_2\,t_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
Simplify[f2.f1.f2] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{-1+t_1}{t_2} & \frac{-1+t_2}{t_3} & 1 & 0 & 0 \\ \frac{-1+t_2}{t_2\,t_3} & \frac{1}{t_3} & 0 & 0 & 0 \\ \frac{1}{t_2\,t_3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
Simplify[f1.f2.f1 /. t_ → t] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{-1+t}{t} & \frac{-1+t}{t} & 1 & 0 & 0 \\ \frac{-1+t}{t^2} & \frac{1}{t} & 0 & 0 & 0 \\ \frac{1}{t^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
Simplify[f2.f1.f2 /. t_ → t] // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} \frac{-1+t}{t} & \frac{-1+t}{t} & 1 & 0 & 0 \\ \frac{-1+t}{t^2} & \frac{1}{t} & 0 & 0 & 0 \\ \frac{1}{t^2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

In[ ]:=
```
β_i_[ℰ_] := Expand[ℰ /. {x_i → x_{i+1} + (t - 1)/t x_i, x_{i+1} → 1/t x_i}]
```

In[ ]:=
```
{x1, x2, x3, x4} // β1 // β2 // β1
```

Out[ ]=

$$\left\{x_1 - \frac{x_1}{t} + x_2 - \frac{x_2}{t} + x_3,\ -\frac{x_1}{t^2} + \frac{x_1}{t} + \frac{x_2}{t},\ \frac{x_1}{t^2},\ x_4\right\}$$

In[ ]:=
```
{x1, x2, x3, x4} // β2 // β1 // β2
```

Out[ ]=

$$\left\{x_1 - \frac{x_1}{t} + x_2 - \frac{x_2}{t} + x_3,\ -\frac{x_1}{t^2} + \frac{x_1}{t} + \frac{x_2}{t},\ \frac{x_1}{t^2},\ x_4\right\}$$

In[ ]:=
```
x1 + x2 + x3 + x4 // β1
```

Out[ ]=

$$x_1 + x_2 + x_3 + x_4$$

In[ ]:=
```
x1 + x2 + x3 + x4 // β2
```

Out[ ]=

$$x_1 + x_2 + x_3 + x_4$$

In[ ]:= $x_1 + x_2 + x_3 + x_4 \;//\; \beta_3$

Out[ ]=

$$x_1 + x_2 + x_3 + x_4$$