Dror's Program

```
In[∘]:= M_k_[ℰ_] := Expand[ℰ /. {e_k → (1 - t) e_k + e_{k+1}, e_{k+1} → t e_k}]

In[∘]:= Table[Coefficient[M_3[e_j], e_i], {i, 7}, {j, 7}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1-t & t & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

```
In[∘]:= Table[e_i // M_2 // M_5, {i, 7}] // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} e_1 \\ e_2 - t\, e_2 + e_3 \\ t\, e_2 \\ e_4 \\ e_5 - t\, e_5 + e_6 \\ t\, e_5 \\ e_7 \end{pmatrix}$$

```
In[∘]:= Table[e_i // M_5 // M_2, {i, 7}]
```

Out[∘]= $\{e_1, e_2 - t\, e_2 + e_3, t\, e_2, e_4, e_5 - t\, e_5 + e_6, t\, e_5, e_7\}$

```
In[∘]:= lhs = Table[e_i // M_3 // M_4 // M_3, {i, 7}]
```

Out[∘]= $\{e_1, e_2, e_3 - t\, e_3 + e_4 - t\, e_4 + e_5, t\, e_3 - t^2\, e_3 + t\, e_4, t^2\, e_3, e_6, e_7\}$

```
In[∘]:= rhs = Table[e_i // M_4 // M_3 // M_4, {i, 7}]
```

Out[∘]= $\{e_1, e_2, e_3 - t\, e_3 + e_4 - t\, e_4 + e_5, t\, e_3 - t^2\, e_3 + t\, e_4, t^2\, e_3, e_6, e_7\}$

```
In[∘]:= lhs == rhs
```

Out[∘]= True

Leonard's program

```
In[∘]:= δ_{i_,j_} := If[i == j, 1, 0]    (* Delta function*)

In[∘]:= δ_{7,7}
```

Out[∘]= 1

(Alt) In[ ]:=
```
(*
α_{i,j} is the i,j entries of the n x n  matrix corresponding to the generator A_{r,s},
1 ≤ r < s ≤ n, of the pure braid group.
*)
α_{i_,j_}[A_{r_,s_}] := Which[
  s < i || i < r, δ_{i,j},
  s == i, (1 - t_i) δ_{i,r} + t_r δ_{i,j},
  r == i, (1 - t_i) (δ_{i,j} + t_i δ_{s,j}) + t_i t_s δ_{i,j},
  r < i < s, (1 - t_i) (1 - t_s) δ_{r,j} - (1 - t_r) δ_{s,j} + δ_{i,j}
 ]
```

(Alt) In[ ]:= `Table[α_{i,j}[A_{1,2}], {i, 3}, {j, 3}] // MatrixForm`

(Alt) Out[ ]//MatrixForm=

$$\begin{pmatrix} 1-t_1+t_1 t_2 & (1-t_1)\, t_1 & 0 \\ 0 & t_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(Alt) In[ ]:= `Table[α_{i,j}[A_{2,4}], {i, 5}, {j, 5}] // MatrixForm`

(Alt) Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1-t_2+t_2 t_4 & 0 & (1-t_2)\, t_2 & 0 \\ 0 & (1-t_3)\,(1-t_4) & 1 & -1+t_2 & 0 \\ 0 & 0 & 0 & t_2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(Alt) In[ ]:= `Table[α_{i,j}[A_{2,7}], {i, 8}, {j, 8}] // MatrixForm`

(Alt) Out[ ]//MatrixForm=

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1-t_2+t_2 t_7 & 0 & 0 & 0 & 0 & (1-t_2)\, t_2 & 0 \\ 0 & (1-t_3)\,(1-t_7) & 1 & 0 & 0 & 0 & -1+t_2 & 0 \\ 0 & (1-t_4)\,(1-t_7) & 0 & 1 & 0 & 0 & -1+t_2 & 0 \\ 0 & (1-t_5)\,(1-t_7) & 0 & 0 & 1 & 0 & -1+t_2 & 0 \\ 0 & (1-t_6)\,(1-t_7) & 0 & 0 & 0 & 1 & -1+t_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Checking the relations of the pure braid group

The relation $A_{r,s} A_{m,n} A_{r,s}{}^{-1} = A_{m,n}$ if s<m or m < r < s < n.

In[ ]:= `A23 = Table[α_{i,j}[A_{2,3}], {i, 7}, {j, 7}];`

In[ ]:= `A45 = Table[α_{i,j}[A_{4,5}], {i, 7}, {j, 7}];`

In[ ]:= `res1 = A23.A45.Inverse[A23];`

In[∘]:=
```
Simplify[res1] == A45
```

Out[∘]=
$$\{\{1, 0, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0, 0\}, \{0, 0, 0, 1 + t_4 (-1 + t_5), -(-1 + t_4) t_4, 0, 0\}, \{0, 0, 0, 0, t_4, 0, 0\}, \{0, 0, 0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0, 0, 1\}\} == \{\{1, 0, 0, 0, 0, 0, 0\}, \{0, 1, 0, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0, 0\}, \{0, 0, 0, 1 - t_4 + t_4 t_5, (1 - t_4) t_4, 0, 0\}, \{0, 0, 0, 0, t_4, 0, 0\}, \{0, 0, 0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0, 0, 1\}\}$$

In[∘]:=

When m < r < s < n

In[∘]:=
```
A17 = Table[α_{i,j}[A_{1,7}], {i, 9}, {j, 9}];
```

In[∘]:=
```
A35 = Table[α_{i,j}[A_{3,5}], {i, 9}, {j, 9}];
```

In[∘]:=
```
res2 = Simplify[A35.A17.Inverse[A35]]
```

Out[∘]=
$$\{\{1 + t_1 (-1 + t_7), 0, 0, 0, 0, 0, -(-1 + t_1) t_1, 0, 0\}, \{(-1 + t_2) (-1 + t_7), 1, 0, 0, 0, 0, -1 + t_1, 0, 0\}, \{(-1 + t_3) (-1 + t_7), 0, 1, 0, 0, 0, -(-1 + t_1) (-1 + t_3^2 - t_3 t_5), 0, 0\}, \{(-1 + t_4 (2 + t_3 (-1 + t_5) - t_5)) (-1 + t_7), 0, 0, 1, 0, 0, (-1 + t_1) (t_3 + (-1 + t_4) (-1 + t_5)), 0, 0\}, \{t_3 (-1 + t_5) (-1 + t_7), 0, 0, 0, 1, 0, (-1 + t_1) t_3, 0, 0\}, \{(-1 + t_6) (-1 + t_7), 0, 0, 0, 0, 1, -1 + t_1, 0, 0\}, \{0, 0, 0, 0, 0, 0, t_1, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 1\}\}$$

In[∘]:=
```
A17 == res2 (* This is not giving me true or false? *)
```

Out[∘]=
$$\{\{1 - t_1 + t_1 t_7, 0, 0, 0, 0, 0, (1 - t_1) t_1, 0, 0\}, \{(1 - t_2) (1 - t_7), 1, 0, 0, 0, 0, -1 + t_1, 0, 0\}, \{(1 - t_3) (1 - t_7), 0, 1, 0, 0, 0, -1 + t_1, 0, 0\}, \{(1 - t_4) (1 - t_7), 0, 0, 1, 0, 0, -1 + t_1, 0, 0\}, \{(1 - t_5) (1 - t_7), 0, 0, 0, 1, 0, -1 + t_1, 0, 0\}, \{(1 - t_6) (1 - t_7), 0, 0, 0, 0, 1, -1 + t_1, 0, 0\}, \{0, 0, 0, 0, 0, 0, t_1, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 1\}\} == \{\{1 + t_1 (-1 + t_7), 0, 0, 0, 0, 0, -(-1 + t_1) t_1, 0, 0\}, \{(-1 + t_2) (-1 + t_7), 1, 0, 0, 0, 0, -1 + t_1, 0, 0\}, \{(-1 + t_3) (-1 + t_7), 0, 1, 0, 0, 0, -(-1 + t_1) (-1 + t_3^2 - t_3 t_5), 0, 0\}, \{(-1 + t_4 (2 + t_3 (-1 + t_5) - t_5)) (-1 + t_7), 0, 0, 1, 0, 0, (-1 + t_1) (t_3 + (-1 + t_4) (-1 + t_5)), 0, 0\}, \{t_3 (-1 + t_5) (-1 + t_7), 0, 0, 0, 1, 0, (-1 + t_1) t_3, 0, 0\}, \{(-1 + t_6) (-1 + t_7), 0, 0, 0, 0, 1, -1 + t_1, 0, 0\}, \{0, 0, 0, 0, 0, 0, t_1, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 1\}\}$$

When s<m

In[∘]:=
```
A12 = Table[α_{i,j}[A_{1,2}], {i, 9}, {j, 9}]
```

Out[∘]=
$$\{\{1 - t_1 + t_1 t_2, (1 - t_1) t_1, 0, 0, 0, 0, 0, 0, 0\}, \{0, t_1, 0, 0, 0, 0, 0, 0, 0\}, \{0, 0, 1, 0, 0, 0, 0, 0, 0\}, \{0, 0, 0, 1, 0, 0, 0, 0, 0\}, \{0, 0, 0, 0, 1, 0, 0, 0, 0\}, \{0, 0, 0, 0, 0, 1, 0, 0, 0\}, \{0, 0, 0, 0, 0, 0, 1, 0, 0\}, \{0, 0, 0, 0, 0, 0, 0, 1, 0\}, \{0, 0, 0, 0, 0, 0, 0, 0, 1\}\}$$

In[ ]:= **A34 = Simplify@Table[$\alpha_{i,j}$[$A_{3,4}$], {i, 9}, {j, 9}]**

Out[ ]= {{1, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0},
{0, 0, $1 + t_3 (-1 + t_4)$, $-(-1 + t_3) t_3$, 0, 0, 0, 0, 0}, {0, 0, 0, t_3, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 1}}

In[ ]:= **res3 = Simplify[A12.A34.Inverse[A12]]**

Out[ ]= {{1, 0, 0, 0, 0, 0, 0, 0, 0}, {0, 1, 0, 0, 0, 0, 0, 0, 0},
{0, 0, $1 + t_3 (-1 + t_4)$, $-(-1 + t_3) t_3$, 0, 0, 0, 0, 0}, {0, 0, 0, t_3, 0, 0, 0, 0, 0},
{0, 0, 0, 0, 1, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 1, 0, 0, 0},
{0, 0, 0, 0, 0, 0, 1, 0, 0}, {0, 0, 0, 0, 0, 0, 0, 1, 0}, {0, 0, 0, 0, 0, 0, 0, 0, 1}}

In[ ]:= **res3 == A34**

Out[ ]= True