July 24, 2020

$$0 \to \mathbb{Z} \xrightarrow[g]{\cdot 2} \mathbb{Z} \xrightarrow[f]{} \mathbb{Z}/2 \to 0$$

$$\operatorname{im} f = \mathbb{Z}/2 \qquad \ker f = 2\mathbb{Z}$$

$$\operatorname{im} f \oplus \ker f = \mathbb{Z}/2 \oplus 2\mathbb{Z} \neq \mathbb{Z}$$

$(1,0) = t$

$2t = 0$

$G = \pi_1 = F_n$

$\tilde{J} = G \cdot d_0$

$$\frac{\mathbb{Z}\langle \pi_1 \rangle}{\mathbb{Z} \cdot e} \overset{?}{\cong} \mathbb{Z}\pi_1 \langle \alpha_1, \dots, \alpha_n \rangle$$

$\tilde{H} = H_1(\tilde{\Sigma}, \tilde{J})$ $\boxed{\otimes\ x}$

$\alpha_j - 1 \longleftarrow \alpha_j$ $\qquad [g\alpha_j - g] \longleftarrow g\alpha_j$

$[g\alpha_j] \not\longleftarrow g \cdot \alpha_j \qquad g \in F_n$

?

$\hat{\pi}_1$

$$0 \to H_1(\tilde{\Sigma}) \to \tilde{H} \to \frac{\mathbb{Z}[t^{\pm 1}]}{\mathbb{Z}} \to 0$$

1 $\qquad \Longleftrightarrow \qquad$ 2

$2 \Rightarrow 1$ ✓ $\qquad 1 \Rightarrow 2$?

$$\mathbb{Z}F_n \langle \alpha_1 \dots \alpha_n \rangle \longrightarrow \frac{\overset{\ker ?}{\mathbb{Z}F_n}}{\cancel{\mathbb{Z}\langle \iota \rangle}}$$

$$g\alpha_i \longmapsto [g\alpha_i - g]$$ isomorphism?

$$A \to B \xrightarrow{\varphi} \mathbb{Z} \to 0$$

$$\Updownarrow \, ?$$

$$A \to B/\mathbb{Z} \to 0$$

$$A \to \ker\varphi \to 0$$

$$0 \to \mathbb{Z} \to A \to B$$

$$\Updownarrow$$

$$0 \to A/\mathbb{Z} \to B$$

$$a : \mathbb{Z}F_n \longrightarrow \mathbb{Z}$$

$$g_i \longmapsto 1$$

$$\sum \alpha_i g_i \longmapsto \sum \alpha_i$$

$$K := \ker(a) = \left\{ \sum \alpha_i g_i : \sum \alpha_i = 0 \right\}$$ "augmentation ideal"

$$\mathbb{Z}F_n\langle x_1, \ldots, x_n\rangle \longrightarrow K$$

$$gx_i \longmapsto gx_i - g$$

isomorphism?

$$n \cdot \infty = \infty - 1 \quad !!$$

Aug 5, 2020

$$\tilde{J} = \pi^{-1}(\{d\})$$

$\langle \tilde{J} \rangle$ → ~~The free group generated by $\tilde{J}$?~~

$\langle \tilde{J} \rangle$ → The $\mathbb{Z}$-module generated by $\tilde{J}$?

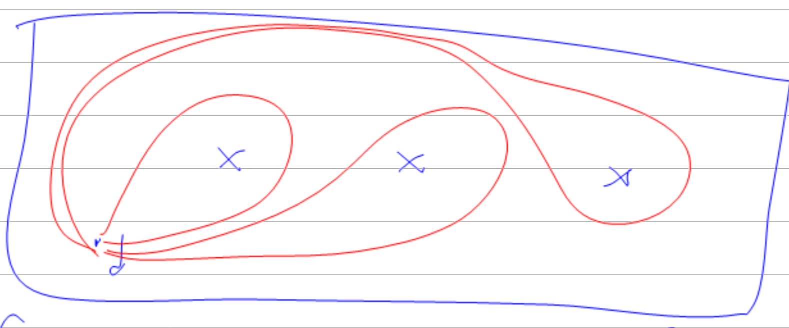
!

~~$g \in \langle \tilde{J} \rangle$~~ Fix $d_0 \in \tilde{J}$

$g \in \tilde{J}$ ~~$g - 1$~~ $g - d_0 \in \langle \tilde{J} \rangle$

$\ker \varphi = \langle g - d_0 : g \in \tilde{J} \rangle$

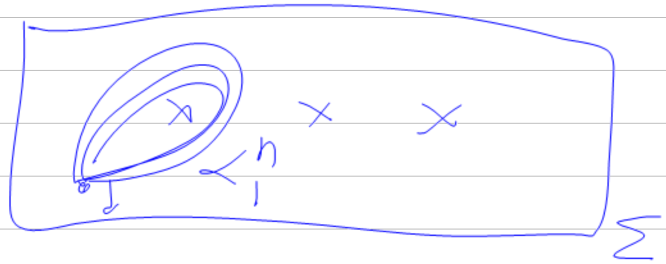
$$\begin{array}{ccc} \tilde{J} & \longleftrightarrow & \mathbb{Z} \\ \Downarrow & & \Downarrow \\ d_0 & \longleftrightarrow & 0 \end{array}$$



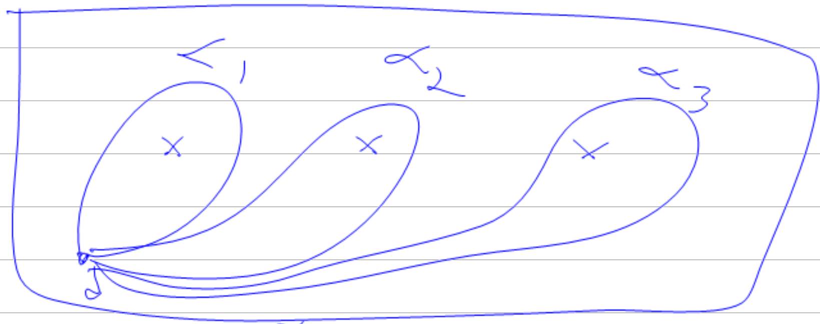
$$\{\ldots d_{-1}, d_0, d_1, d_2 \ldots\} = \tilde{J}$$

$d_1 - d_0 = \partial \gamma \qquad \gamma : [0,1] \to \Sigma$

$\boxed{\tilde{\alpha}^n_i}$ ✓ $\qquad \tilde{\alpha}^n_i$ ✗ $\qquad$ ~~$\underset{d_0 \to d_1}{\tilde{\alpha}_i} \quad \underset{d_0 \to d_1}{\tilde{\alpha}_i}$~~



Aug 11, 2020



$$h'([\underbrace{g}_{=\, d_0 * m} - d_0]) := [\underbrace{\alpha_1^m}_{\gamma}]$$

$$h'\left(\sum c_i [g_i - d_0]\right) = \sum c_i h'([g_i - c_0])$$

Is this a $\mathbb{Z}[\mathbb{Z}] = \mathbb{Z}[t, t^{-1}]$-module homomorphism?

$$\boxed{h'(t^k[g_i - d_0]) \overset{?}{=} t^k h'([g_i - d_0])}$$

Ex. First do this for $\mathbb{R} \xrightarrow{p} S^1$

$$h'([g + g' - 2d_0]) = h'([g - d_0] + [g' - d_0])$$

$$h'([g - g']) = ?$$

Aug 16, 2020 Comment:

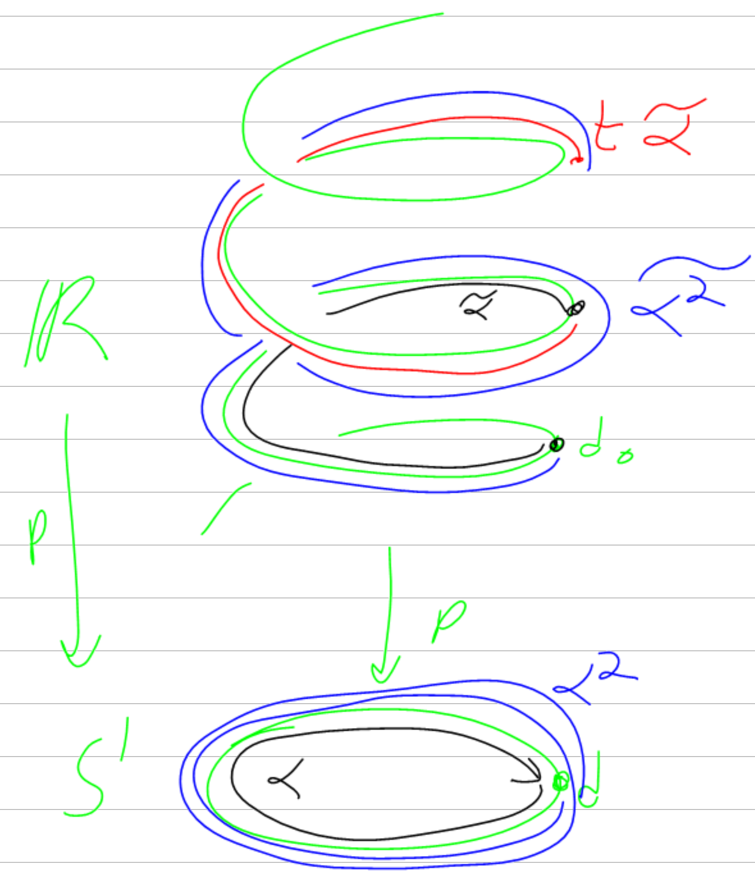
$$\underbrace{\{\text{Non-empty words}\}}_{\infty - 1} = \underbrace{\{\text{letters}\}}_{26} \times \underbrace{\{\text{all words}\}}_{\infty}$$

$$\infty - 1 = 26 \times \infty$$

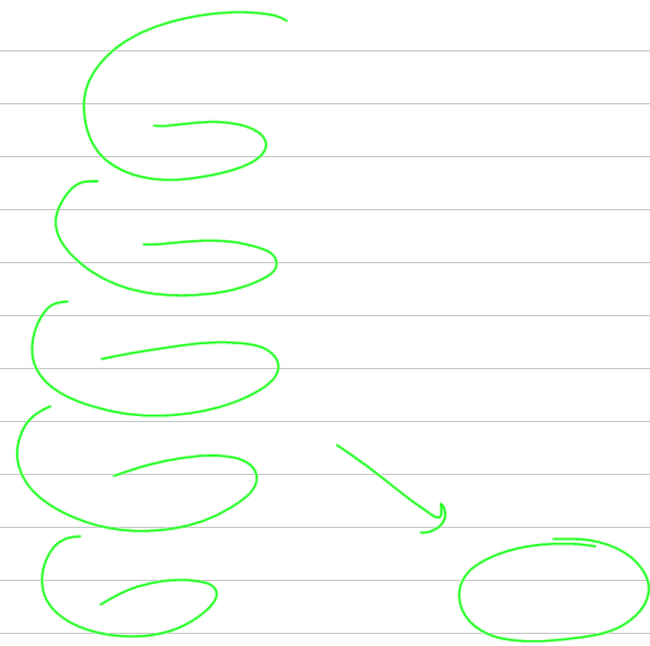
Aug 19, 2020

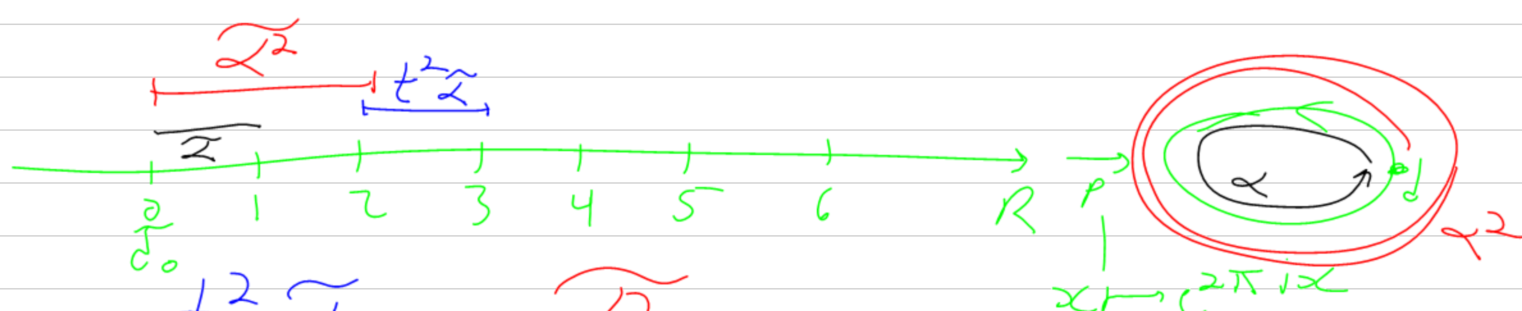
$\widetilde{\alpha_k^i}$ makes sense.

$\widetilde{\alpha_k}^i$ makes no sense b/c $\widetilde{\alpha_k}$ isn't closed.

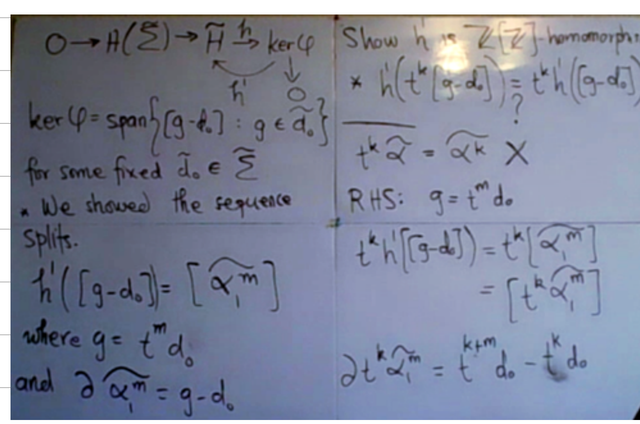


$$t^2\tilde{\alpha} \neq \widetilde{\alpha^2}$$





$t^2\tilde{\alpha}$ $\qquad$ $\widetilde{\alpha^2}$



$0 \to H(\tilde{\Sigma}) \to \tilde{H} \xrightarrow{h} \ker\varphi$, h', 0

$\ker\varphi = \operatorname{span}\{[g - \tilde{d}_0] : g \in \tilde{d}_0\}$

for some fixed $\tilde{d}_0 \in \tilde{\Sigma}$

* We showed the sequence splits.

$h'([g-d_0]) = [\widetilde{\alpha_1^m}]$

where $g = t^m d_0$

and $\partial \widetilde{\alpha_1^m} = g - d_0$

Show h' is $\mathbb{Z}[\mathbb{Z}]$-homomorphism

* $h'(t^k[g-d_0]) \overset{?}{=} t^k h'([g-d_0])$

$t^k\tilde{\alpha} = \widetilde{\alpha^k}$ X

RHS: $g = t^m d_0$

$t^k h'([g-d_0]) = t^k[\widetilde{\alpha_1^m}]$

$= [t^k \widetilde{\alpha_1^m}]$

$\partial t^k \widetilde{\alpha_1^m} = t^{k+m} d_0 - t^k d_0$

$$x_{i_1}^{\pm 1} x_{i_2}^{\pm 1} \cdots x_{i_k}^{\pm 1} - 1$$ ← For simplicity: all powers are +1

What if not?

$$= x_{i_1} x_{i_2} \cdots x_{i_k} - 1 = \sum \alpha_a (g_a x_{i_a} - g_a)$$

$$= \sum \alpha_a g_a (x_{i_a} - 1) \, ?$$

$$xy - 1 = (\ldots)(x-1) + (\ldots)(y-1)$$

$$\underline{xy - x} + \underline{x - 1} = x(y-1) + (x-1)$$

Sep 11, 2020 $\quad w \in FG(x_1, \ldots x_n)$

$$w - 1 = \sum \frac{\partial w}{\partial x_i}(x_i - 1) \quad ? \; ?$$

$w = 1 \quad x_1$

$$x_1 - 1 = 1 \cdot (x_1 - 1) + 0 \ldots \checkmark$$

$$x_i^{-1} - 1 = -x_i^{-1}(x_i - 1) + \ldots$$

$$0 = \frac{\partial}{\partial x_i}(x_i x_i^{-1}) = 1 + x_i \frac{\partial x_i^{-1}}{\partial x_i}$$

$$\frac{\partial}{\partial x_i}(x_i^{-1}) = -x_i^{-1}$$

$$W_k = \underline{W_{k-1}} \, x_{j_k}^{\epsilon_k}$$

Assume

$$W_{k-1} - 1 = \sum_{i=1}^{n} \frac{\partial W_{k-1}}{\partial x_i}(x_i - 1)$$

$$w_k - 1 = w_{k-1} x_{i_k}^{\epsilon_k}$$

$$= (w_{k-1} - 1) x_{i_k}^{\epsilon_k} + (x_{i_k}^{\epsilon_k} - 1)$$

$$= \left(\sum_{i=1}^{n} \frac{\partial w_{k-1}}{\partial x_i} (x_i - 1) \right) x_{i_k}^{\epsilon_k} + (x_{i_k}^{\epsilon_k} - 1)$$

$$\overset{?}{=} \sum_{i=1}^{k} \frac{\partial w_k}{\partial x_i} (x_i - 1)$$

$w_{k-1} x_{i_k}^{\epsilon_k}$

$$F \longrightarrow \tilde{\Sigma} \longrightarrow \Sigma$$

$$F \longrightarrow E \longrightarrow B$$

$$H_n(F) \longrightarrow H_n(\tilde{\Sigma}) \longrightarrow H_n(\Sigma) \longrightarrow H_{n-1}(F) \longrightarrow \cdots$$

"Long exact seq for a cover"

"Long exact seq of a fibration"

$$H_1(\tilde{\Sigma}, F) \qquad H_1(\tilde{\Sigma}) \qquad H_0(F)$$

$\forall W \quad W-1=\sum_{i=1}^{n}\frac{\partial W}{\partial x_i}(x_i-1)$

$x_k^{\epsilon}W'=W \qquad W'-1=\sum_{i=1}^{n}\frac{\partial W'}{\partial x_i}(x_i-1)$

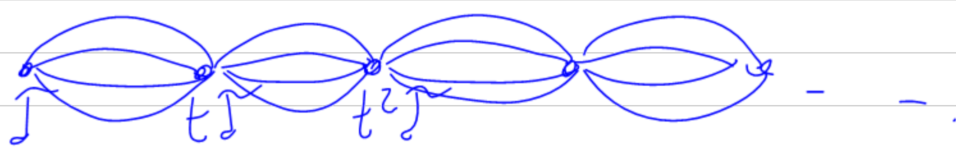
$$W-1=x_k^{\epsilon}W'-1=x_k^{\epsilon}(W'-1)+(x_k^{\epsilon}-1)$$

$$=x_k^{\epsilon}\sum_{i=1}^{n}\frac{\partial W'}{\partial x_i}(x_i-1)+(x_k^{\epsilon}-1)=\#_1$$

$W=x_k^{\epsilon}W' \qquad \frac{\partial W}{\partial x_i}=\begin{cases} x_k^{\epsilon}\frac{\partial W'}{\partial x_i} & i\neq k \\ \frac{\partial x_k^{\epsilon}}{\partial x_k}+x_k^{\epsilon}\frac{\partial W'}{\partial x_k} & i=k \end{cases}$

$$\sum_{i=1}^{n}\frac{\partial W}{\partial x_i}(x_i-1)=x_k^{\epsilon}\sum\frac{\partial W'}{\partial x_i}(x_i-1)+\frac{\partial x_k^{\epsilon}}{\partial x_k}(x_k-1)=\#_2$$

$n=4$

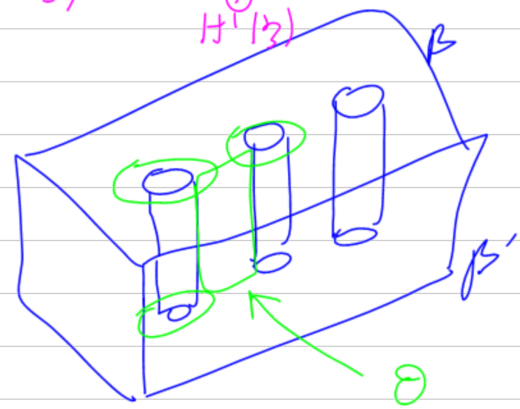


Back to Kassel-Turaev!

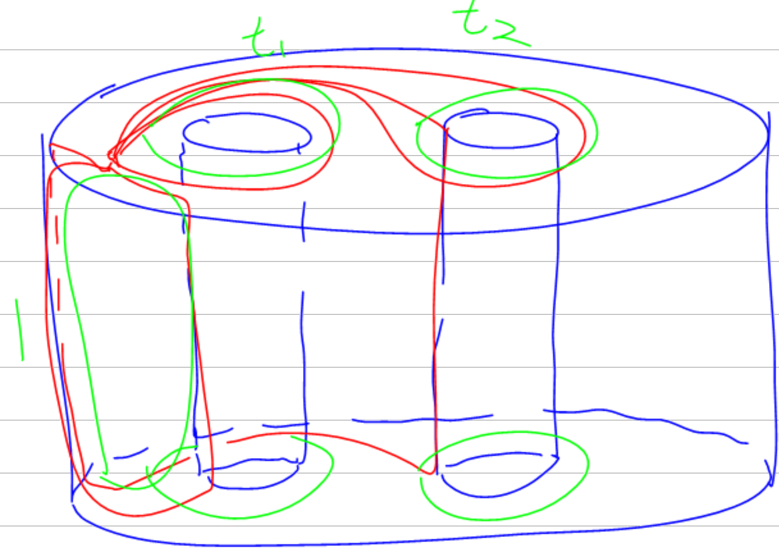
$(d\alpha)(\gamma) = 1-t \qquad \beta(\gamma) = 1$

$$d\alpha=(1-t)\beta$$

$$H^1(A\cup B) \to H^1(A)\oplus H^1(B) \longrightarrow H^1(A\cap B)$$



$$H^1(\partial X) = \left\langle \begin{matrix} \beta_1 \ldots \beta_n \\ \beta'_1 \ldots \beta'_n \end{matrix} \right\rangle \Big/ \begin{matrix} \sum (t_i - 1)\beta_i = 0 \\ \sum (t_i - 1)\beta'_i = 0 \end{matrix}$$



$$\pi_1(\partial X) \longrightarrow \pi_1(X) \xrightarrow{\phi} Q(t_1, t_2)$$

$$U \cup V = \partial X, \quad U, \quad V, \quad U\cap V = \sqcup S^1$$

$$F: \pi_1(Y) \longrightarrow G \xrightarrow{\psi} K$$

$$C_n(Y, F) = C_n(\tilde{Y}) \otimes_{\pi_1(Y)} K$$

$$\cong$$

$$C_n(\hat{Y}) \otimes_G R$$

$$G \to \hat{Y} = (\tilde{Y} \times G)/\pi_1 \to Y$$

If $n = 2$

$$\underline{\underline{2(n-1)}} = \underline{\underline{n}} = 2$$

$n=3$: $2(n-1) \downarrow 4$, $\quad n \downarrow 3$

$A_1, A_2, A_3,$

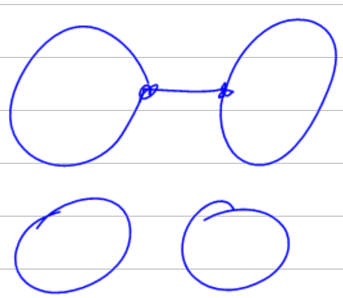
$+(t_2-1)B_1 - (t_1-1)B_2, \quad \cdot(t_3-1)$

$-(t_3-1)B_1 - (t_1-1)B_3, \quad \cdot(t_2-1)$

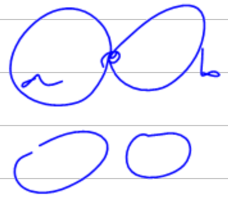
$+(t_3-1)B_2 - (t_2-1)B_3 \quad \cdot(t_1-1)$

$$0 \to H^1(\partial X) \xrightarrow{i} \begin{matrix} H^1(X_0) \\ \oplus \\ H^1(X_1) \end{matrix} \to H^1(\cancel{S^1})^{\,0}$$

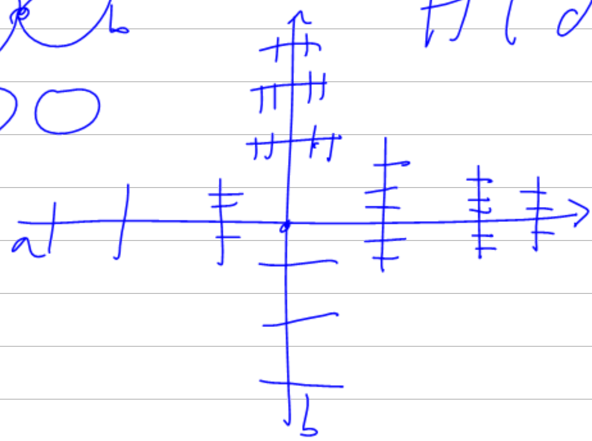
$n = 2$



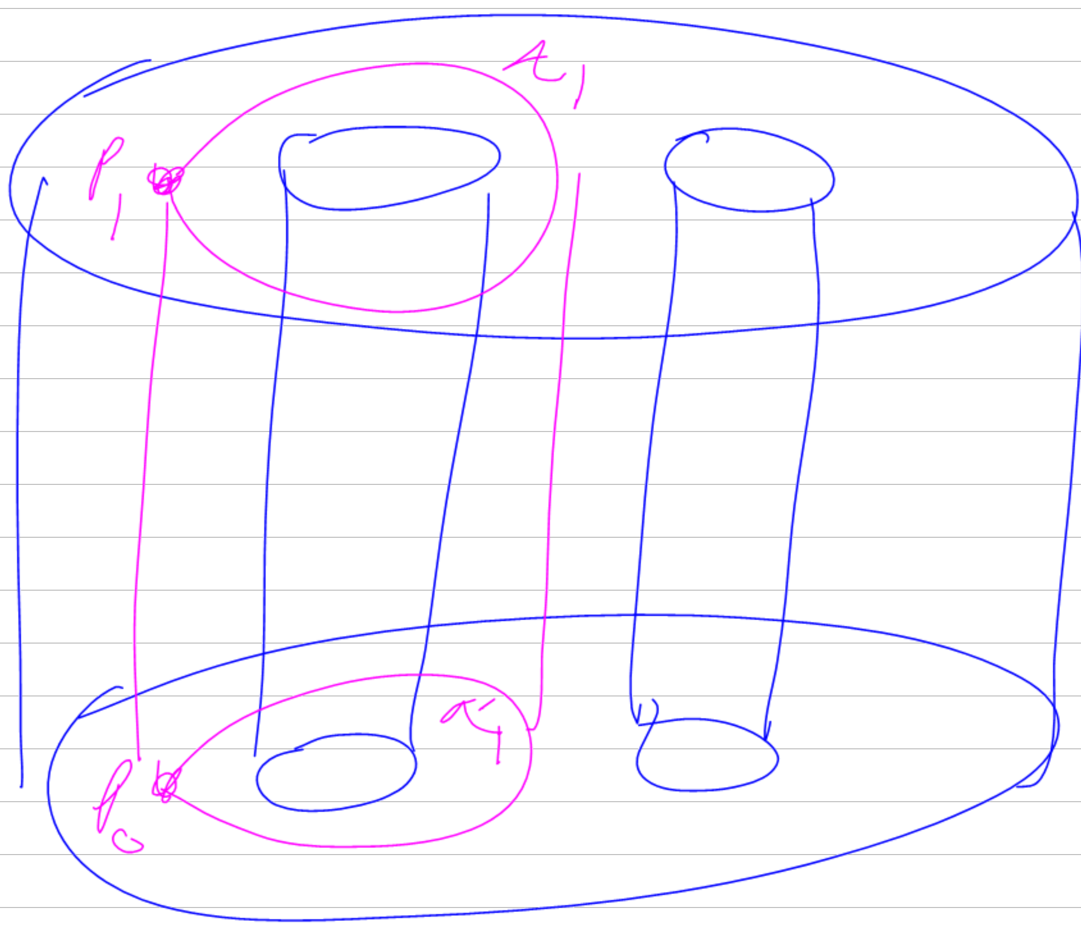
X_0

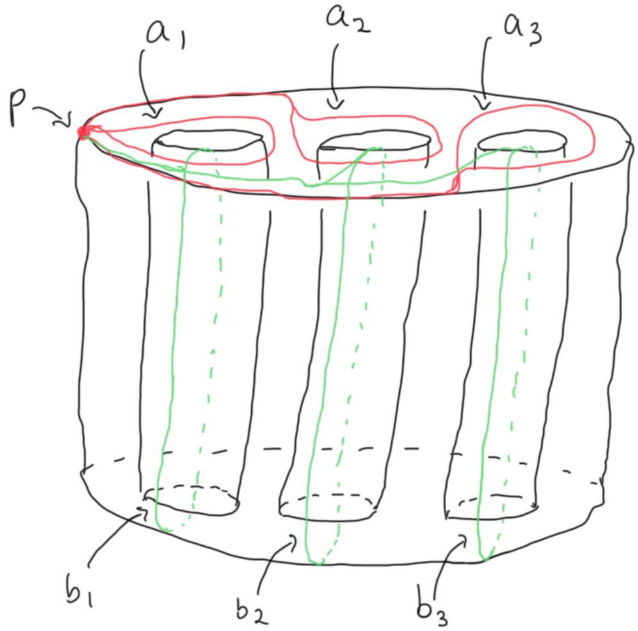


$H^1(\partial X)$



$A_j \neq A_j'$





Cell structure

0-cell : p

1-cells : a_i, b_i, $i = 1, 2, 3$

2-cell : e

$$\partial e = a_1 b_1 a_1^{-1} b_1^{-1}$$

$$\left[\begin{array}{l} C_0 = \langle \tilde{p} \rangle, \quad C_1 = \langle \tilde{a}_i, \tilde{b}_i \rangle_{i=1,2,3}, \quad C_2 = \langle \tilde{e} \rangle \\ 0 \longrightarrow C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \end{array}\right]$$

$$\left[\begin{array}{l} C^0 = \langle P \rangle, \quad C^1 = \langle A_i, B_i \rangle_{i=1,2,3}, \quad C^2 = \langle E \rangle \\ 0 \longrightarrow C^0 \xrightarrow{d^0} C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} 0 \end{array}\right]$$

We compute $\mathcal{H}^1(\partial X, F)$, $F = \mathbb{Q}(t_1^{\pm 1}, t_2^{\pm 1}, t_3^{\pm 1})$

The twisting

$$\phi : \pi_1(X, p) \longrightarrow F$$

$a_i \longmapsto t_i$ (taking meridians to t_i) $i = 1, 2, 3$

$b_i \longmapsto 1$, for $i = 1, 2, 3$.

$$\mathcal{H}^1(\partial X, F) = \ker d^1 / \operatorname{im} d^0$$

① $\ker d^1$

Let $S \in C^1$. Then $d^1(S) \in C^2$, so

$$d^1(S)(\tilde{e}) = S(\partial_2(\tilde{e})).$$

What is $\partial_2(\tilde{e})$ (find $\partial_2(\tilde{e})$ using ∂e)

$$\partial e = \underbrace{a_1 b_1 a_1^{-1} b_1^{-1}}_{①} \underbrace{a_2 b_2 a_2^{-1} b_2^{-1}}_{②} \underbrace{a_3 b_3 a_3^{-1} b_3^{-1}}_{③}.$$

Boundary of $\tilde{e}$:

$$a_1 + b_1\phi(a_1) - a_1\phi(a_1 b_1 a_1^{-1}) - b_1\phi(a_1 b_1 a_1^{-1} b_1^{-1}) \quad ①$$

$$+ a_2\phi(a_1 b_1 a_1^{-1} b_1^{-1}) + b_2\phi(a_1 b_1 a_1^{-1} b_1^{-1} a_2) - a_2\phi(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1})$$
$$- b_2\phi(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}) \quad ②$$

$$+ a_3\phi(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}) + b_3\phi(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} a_3)$$
$$- a_3\phi(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} a_3 b_3 a_3^{-1}) - b_3\phi(a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} a_3 b_3 a_3^{-1} b_3^{-1}) \quad ③$$

$$= \cancel{a_1} + t_1 b_1 - \cancel{a_1} - b_1$$
$$+ \cancel{a_2} + t_2 b_2 - \cancel{a_2} - b_2$$
$$+ \cancel{a_3} + t_3 b_3 - \cancel{a_3} - b_3$$

$$\phi : \begin{cases} a_i \longmapsto t_i \\ b_i \longmapsto 1 \end{cases}$$

$$= (t_1 - 1) b_1 + (t_2 - 1) b_2 + (t_3 - 1) b_3.$$ Thus,

$$\partial_2(\tilde{e}) = (t_1 - 1)\tilde{b}_1 + (t_2 - 1)\tilde{b}_2 + (t_3 - 1)\tilde{b}_3$$

$d'(S)(\tilde{e}) = S(\partial_2(\tilde{e})) = 0$ for what $S \in C'$?

$S(\partial_2(\tilde{e})) = 0$ if

$$S \in \operatorname{Span} \left\{ \begin{array}{l} A_1, A_2, A_3, \\ (t_2 - 1)B_1 - (t_1 - 1)B_2, \\ (t_3 - 1)B_1 - (t_1 - 1)B_3, \\ \cancel{(t_3 - 1)B_2 - (t_2 - 1)B_3} \end{array} \right\} = \ker d'$$

Image of d^0

$d^0(\mathcal{P}) \in C^1$.

$d^0(\mathcal{P})(a_i) = \mathcal{P}(\partial_1(\tilde{a}_i)) = \mathcal{P}((t_i - 1)\tilde{p}) = t_i - 1, \quad i = 1, 2, 3$

$d^0(\mathcal{P})(b_i) = \mathcal{P}(\partial_1(\tilde{b}_i)) = \mathcal{P}(0) = 0$

So $d^0(\mathcal{P}) = \sum_{i=1}^{3} (t_i - 1) A_i$

$\operatorname{Im} d^0 = \left\langle \sum_{i=1}^{3} (t_i - 1) A_i \right\rangle$

$\ker d^1 = \{ A_1, A_2, \cancel{A_3}, (t_2 - 1)B_1 - (t_1 - 1)B_2, (t_3 - 1)B_1 - (t_1 - 1)B_3, \cancel{(t_3 - 1)B_2 - (t_2 - 1)B_3} \}$

$\operatorname{Im} d^0 = \left\langle \sum_{i=1}^{3} (t_i - 1) A_i \right\rangle$

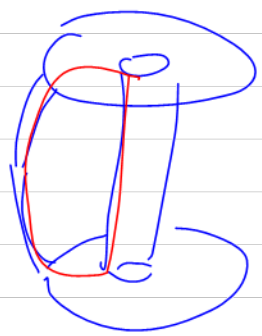
$H^1(X_{0,1})$

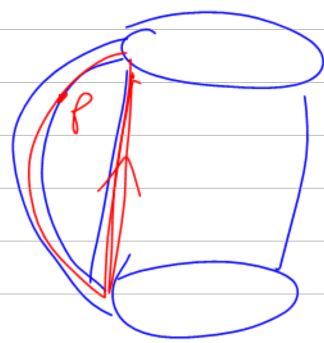
This implies

$\mathcal{H}^1(\partial X, F) \cong F^5$

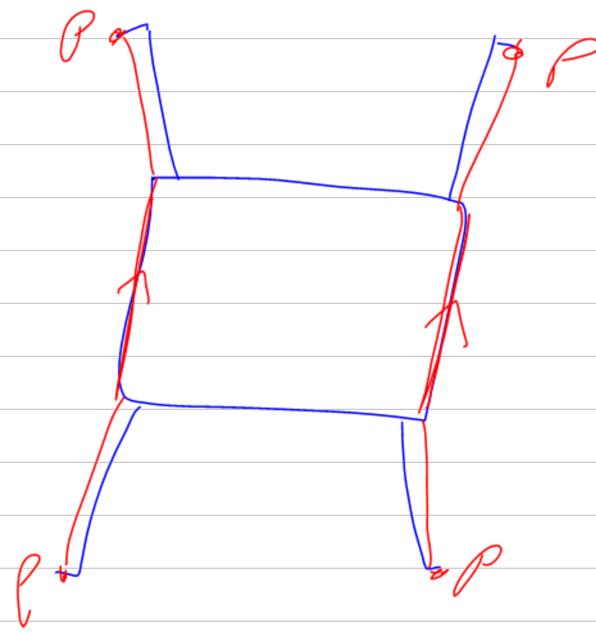
My computation still shows that the B_i's are involved for 3 holes and instead of dimension 4, I get 5.

I am not sure if I am doing something wrong.

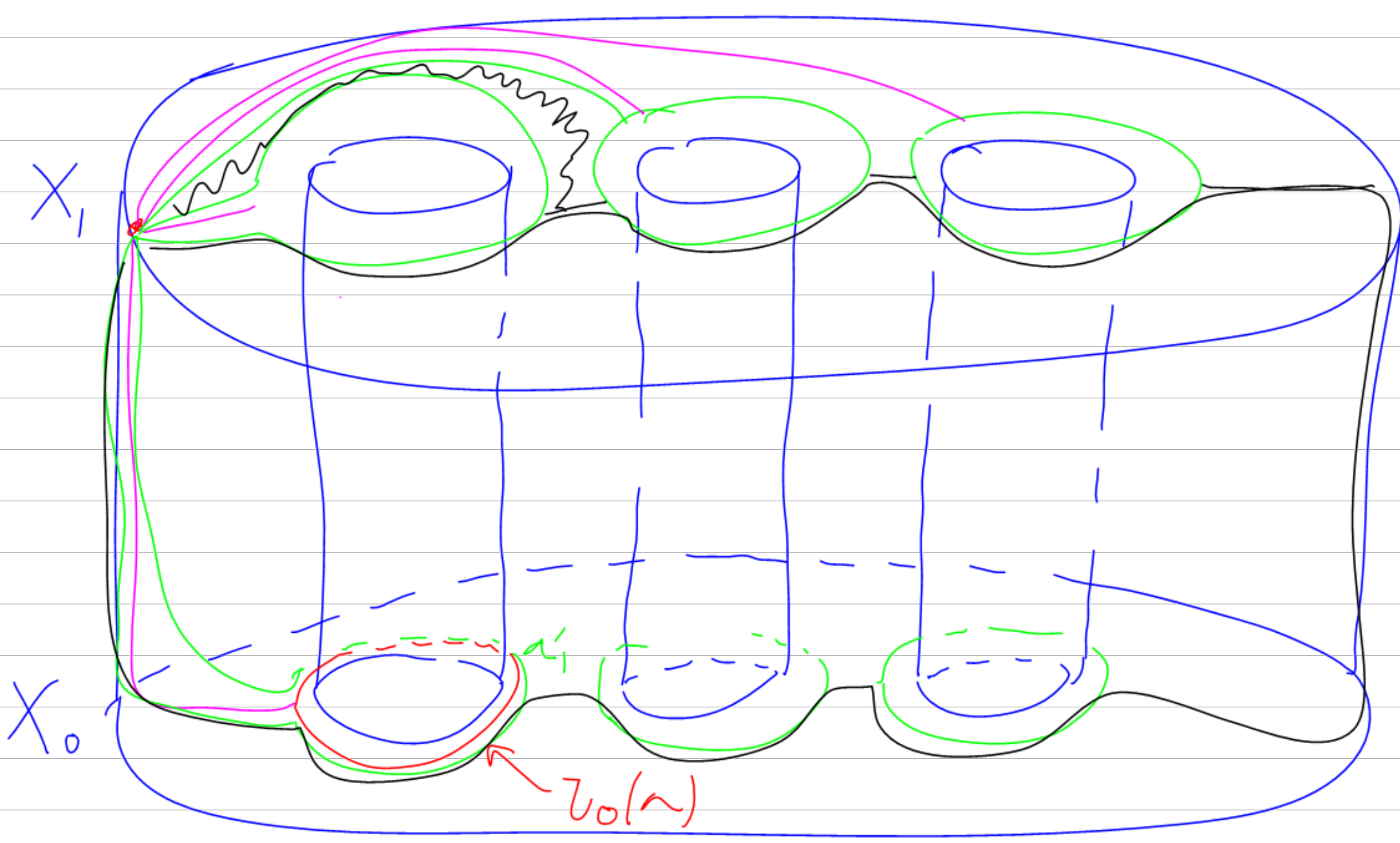




p

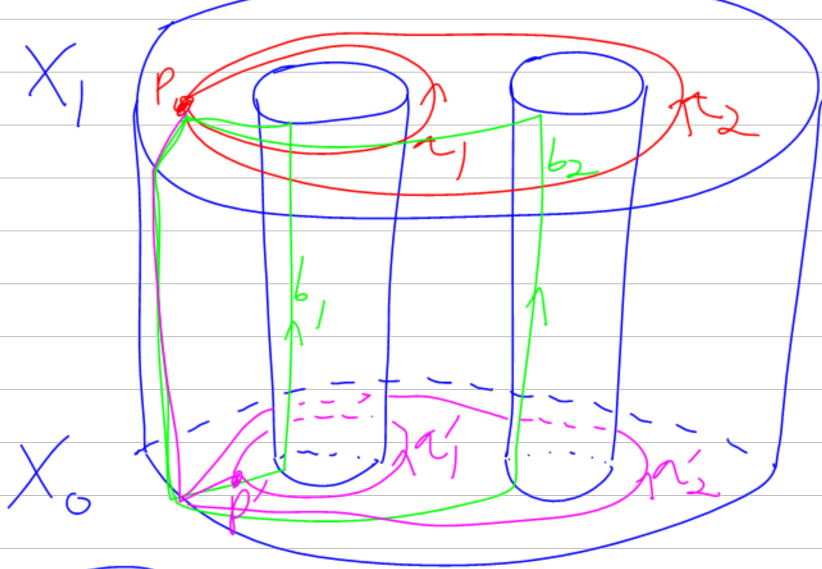


p
p
p
p



X_1
X_0
a_1
$\tau_0(\alpha)$

$$\partial \tilde{a}_i = (t_i - 1)\tilde{p}$$

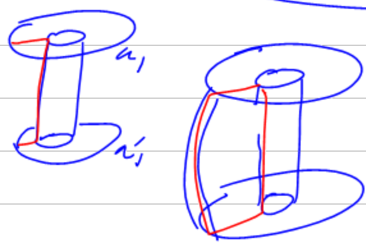


$$C = [(t_2-1)\tilde{a}_1 - (t_1-1)\tilde{a}_2]$$

$$C' = [(t_2-1)\tilde{a}_1' - (t_1-1)\tilde{a}_2']$$

$$\partial D \simeq a_1^{-1} b_1^{-1} a_1' b_1 \sim 1$$

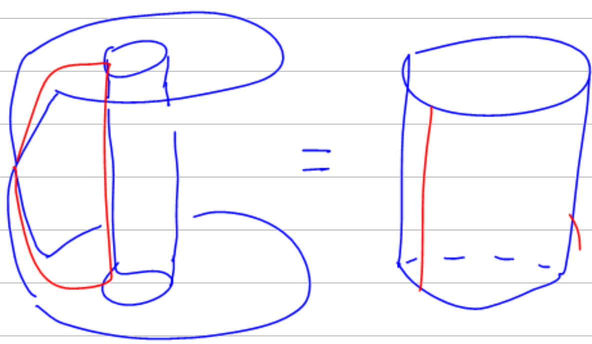
$$\partial \tilde{D} \simeq -[\tilde{a}_1] + [\tilde{a}_1'] - t_1\tilde{b}_1 + \tilde{b}_1$$



$$C - C' = (t_2-1)(t_1-1)(b_1 - b_2)$$

In $\tilde{C}_1$: $a_i' = a_i + (t_i - 1) b_i$

$A_i(a_i') = 1 + 0 = 1$



$$H_1(X_0)\langle a_{0,i}\rangle \longrightarrow a_i' = b_i^{-1} a_i b_i$$

$$\oplus \xrightarrow{\sim} H_1(\partial X)$$

$$H_1(X_1)\langle a_{1,i}\rangle \longrightarrow a_i$$

$$V = \langle x, y, z\rangle / x+y+z=0$$

$$V^* = \langle X, Y, Z\rangle / X+Y+Z=0$$

$$X(x) = 1 \quad X(y) = 0 \quad X(z) = 0$$

$$0 = X(x+y+z) = 1$$

$\langle a \ \ a' \ \ b \rangle / a - a' = (t-1)b$

$X \quad A \quad A' \quad B$

A

$V = \langle x, y, z \rangle / x + y + z = 0$

$\beta_1 = (x, y) \qquad \beta_1^* = (\varphi_1, \psi_1)$

$\varphi_1(x) = 1$

$\varphi_1(y) = 0$

$\psi_1(x) = 0$

$\psi_1(y) = 1$

$=$ $\qquad$ $\times$

$\beta_2 = (x, z) \qquad \beta_2^* = (\varphi_2, \psi_2)$

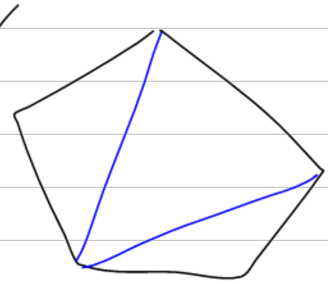
$\varphi_2(x) = 1$

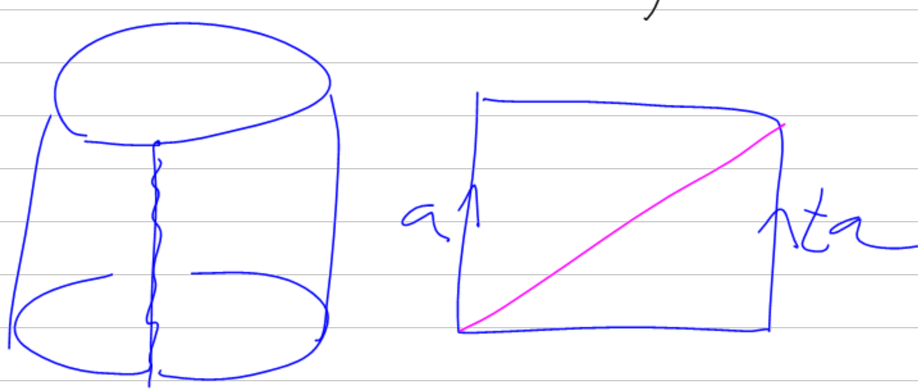
$\varphi_2(z) = 0$

$\psi_2(x) = 0$

$\psi_2(z) = 1$

$\varphi_1(y) = 0 \qquad \varphi_2(y) = \varphi_2(-x-z) = -1$

$(\alpha \cup \beta)\Bigg($  $\Bigg)$



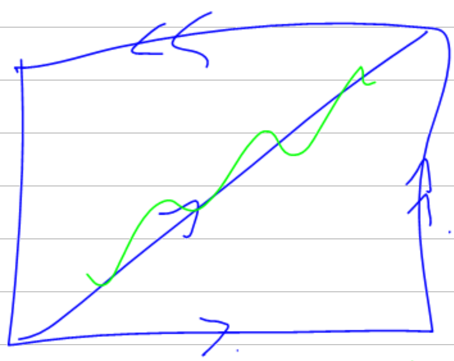
$$C^*(X;F) = \mathrm{Hom}_\pi(C_*(\tilde{X}) \to F)$$

$$\varphi \in C^k(X;F) \qquad \psi \in C^\ell(X;F)$$

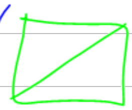
$$\sigma \in C_k(\tilde{X}) \qquad \varphi \cup \psi \in H^*(X)$$

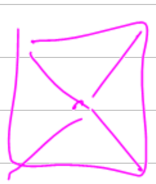
$$(\varphi \cup \psi)(\sigma) = \varphi(\sigma|_{[0 \dots k]})\,\psi(\sigma|_{[k, \dots, k+\ell]})$$

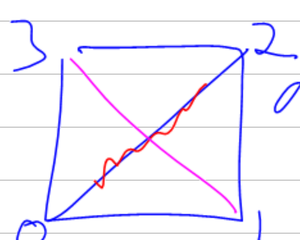
$g\sigma \qquad g\cdot \qquad \sigma \qquad g\cdot \qquad \sigma$



$\delta\phi = 0$

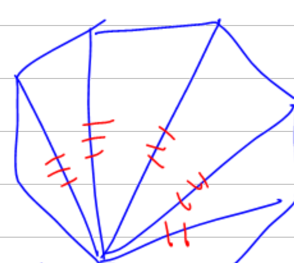
$\delta\psi = 0$ $\qquad \phi \cup \psi($  $)$





$a_1 \vee b_1$

$H^1 = \langle a_1, b_1 \rangle$

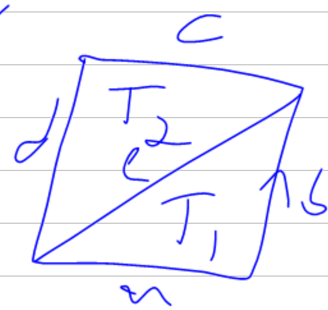


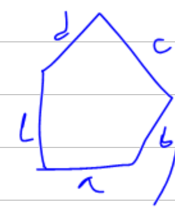
$H^1 = \langle a_1, b_1, a_2, b_2 \rangle$

$$(a_1 \vee b_1)(0123) = (a_1 \vee b_1)(012) + (a_1 \vee b_1)(023)$$

$$= -\ a_1(01)\ \cancel{a_1(02)}\ \ldots$$

$$b_1(23)\ \ldots$$

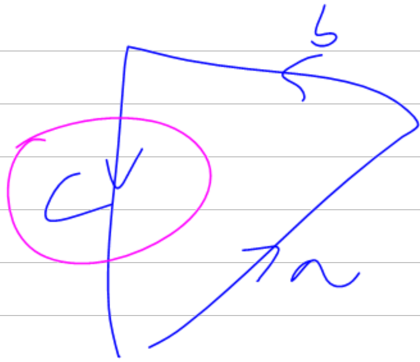
$(\alpha \vee \beta)$ () $= \ldots$

$(\alpha \vee \beta)$ () $= ?$

$$d\alpha = 0$$

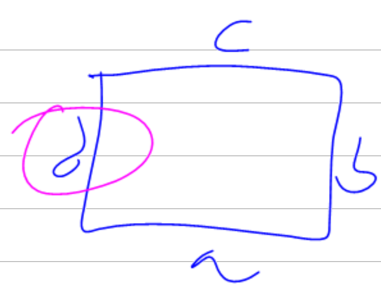
$$\alpha(\partial T_1) = 0$$

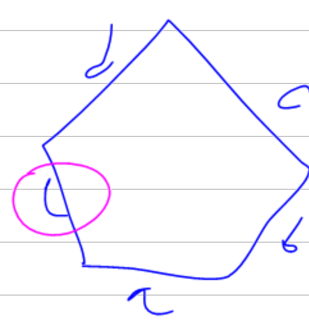
$$\alpha(e) - \alpha(a) - \alpha(b) = 0$$



$$\phi(c) = \phi(a) + \phi(b)$$

$$\phi(a)\alpha(b)$$





$$\sum_{1\le i<j\le n-1} \alpha(a_i)\beta(a_j)$$

$$\underline{\underline{A_i}} \overset{\varphi}{\longmapsto} (A_i', \underline{\underline{A_i}})$$

$$\underset{\langle v_i\rangle}{V} \oplus \underset{\langle w_j\rangle}{W} = \left\langle \begin{matrix} (v_i, 0) \\ (0, w_j) \end{matrix} \right\rangle$$

$$(v_1, w_1) + (v_2, w_2) = (v_1, w_2) + (v_2, w_1)$$

$$(v_1 + v_2, w_1 + w_2) =$$

$$A, B, \in C' \longrightarrow A, B \in C'(\,, \bar{F})$$

$$\phi = aA + bB \longrightarrow \bar{a}A + \bar{b}B$$

$$\eta : H'(\partial X, F) \xrightarrow[\text{lin}]{\text{Conj}} H'(\partial X, \bar{F})$$

$$\phi \longmapsto \phi$$

$$a\psi \longmapsto \bar{a}\psi$$

$$\phi \cup \phi \longrightarrow \phi \cup \eta(\phi)$$

$$\phi \cup \phi$$

$$\alpha \cup \beta$$

$$\alpha \in H_1(\partial X, F)$$

$$\beta \in H_1(\partial X, \bar{F})$$

$$\phi = B_2 - \frac{t_1(t_2 - 1)}{t_1 - 1} B_1 - \ldots$$

$$\downarrow$$

$$\overline{B_2 - \left(\frac{t_1(t_2 - 1)}{t_1 - 1}\right) B_1 - \ldots}$$

$V=\mathbb{C}^n \qquad \langle x,y\rangle$

$$\langle\ ,\ \rangle : \begin{cases} \text{sesqui-linear map } V\times V \to \mathbb{C} & (A) \\ \text{linear map } V\times \bar{V} \longrightarrow \mathbb{C} & (B) \end{cases}$$

$\bar{V} = V$ as a set $\qquad \underbrace{\alpha x}_{\text{in } \bar{V}} = \underbrace{\bar{\alpha}\cdot x}_{\text{in } V}$

$\eta : V \to \bar{V}$ is the identity map, yet $\eta(\alpha x) = \bar{\alpha}\,\eta(x)$

$$\langle \alpha x, \alpha x\rangle : \begin{cases} A: \ \alpha\bar{\alpha}\langle x,x\rangle = |\alpha|^2\cdot \|x\|^2 \\ B: \ \langle \alpha x, \underbrace{\eta(\alpha x)}_{\bar{V}}\rangle = \langle \alpha x, \bar{\alpha}\underbrace{\eta(x)}_{\bar{V}}\rangle \end{cases}$$

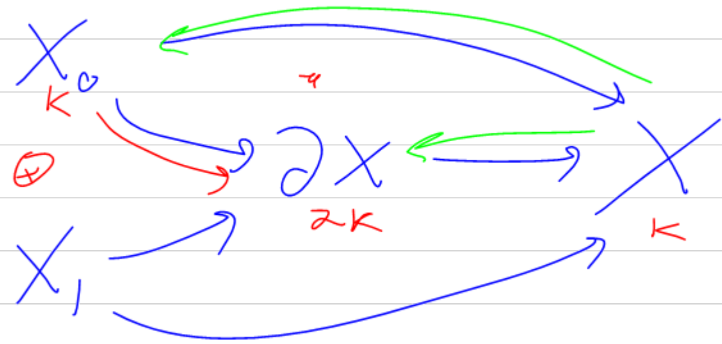
$$= \alpha\bar{\alpha}\langle x, \eta(x)\rangle = |\alpha|^2\|x\|^2$$

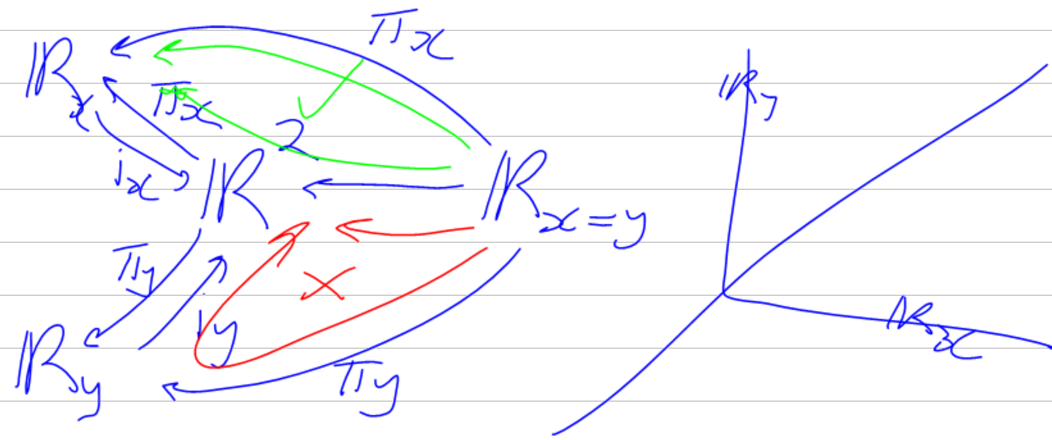
$$x, y \in \boxed{H^1(\partial X; F)} = \langle \emptyset, \psi = A_{1,0} + A_{1,1}\rangle$$

$t_1 \to t_1^{-1}$

$$x \cup \eta(y) = -\overline{y \cup \eta(x)}$$

$$\begin{array}{ccc} H^1(\partial X, F) \otimes H^1(\partial X, \bar{F}) & \xrightarrow{\ \cup\ } & H^2(\partial X, \mathbb{C}) \\ \downarrow & & \downarrow \\ H^1(X_0, F) \otimes H^1(X_0, \bar{F}) & \xrightarrow{\ \cup\ } & H^2(X_0, \mathbb{C}) = 0 \end{array}$$





$$\varphi : \mathbb{R}^n \longrightarrow (\mathbb{R}^n)^*$$

$$\mathbb{R}^n \cong (\mathbb{R}^n)^*$$

$$\eta(B_i) \neq B_i \notin C^1(\partial X, \bar{F})$$

$$\begin{tikzcd}
C_i(X;F) = C_i(\tilde{X}) \otimes_{\pi_1(X)} F \arrow[d, "\eta"] & \\
C_i(X;\bar{F}) = C_i(\tilde{X}) \otimes_{\pi_1(X)} \bar{F} &
\end{tikzcd}$$

$$\begin{tikzcd}
C_i(X;F) \arrow[d, "\eta"] & C_i(\tilde{X}) \arrow[d, "I"] \otimes_{\pi_1(X)} & F \arrow[d, "-"] \\
C_i(X;\bar{F}) & C_i(\tilde{X}) \otimes_{\pi_1(X)} & \bar{F}
\end{tikzcd}$$

$$\eta(B_i) = B_i \circ \eta$$

$$\eta(B_i)(t^k b_i) = B_i(\eta(t^k b_i))$$

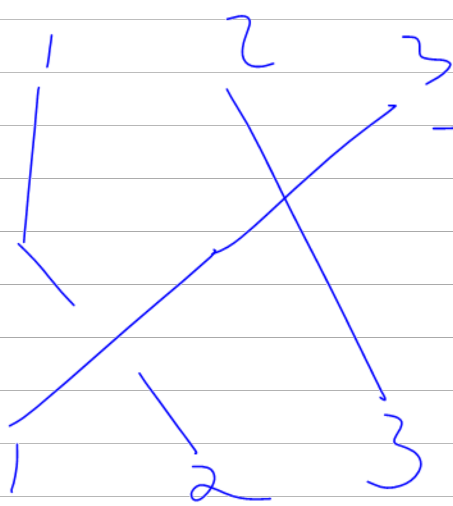
$$= B_i(t^{-k} b_i) = t^{-k}$$

$$B_i(t^k b_i) = t^k$$

$$\begin{array}{ccccc} PwB_n & \overset{\tau}{\hookrightarrow} & wB_n & \xrightarrow[\pi]{\substack{s_i \mapsto 1 \\ \sigma_i \mapsto \sigma_{i,i+1}}} & PwB_n \\ \| & & \| & & \\ \langle \sigma_{ij} \rangle & & \langle \sigma_i, s_i \rangle & & \end{array}$$

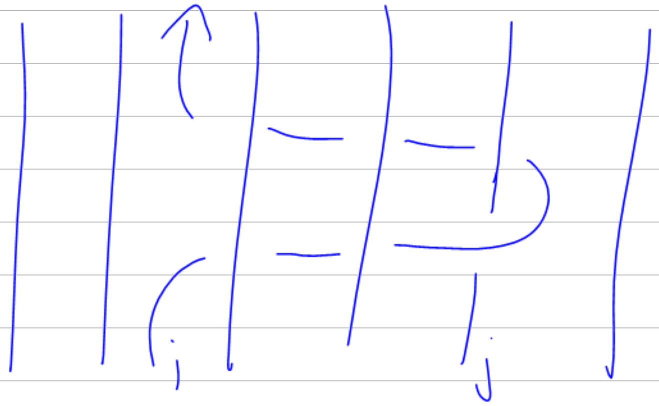
$$\tau(\sigma_{i,i+1}) = \sigma_i s_i \text{ or } s_i \sigma_i$$

$$wB_n = PwB_n \rtimes S_n = S_n \ltimes PwB_n$$



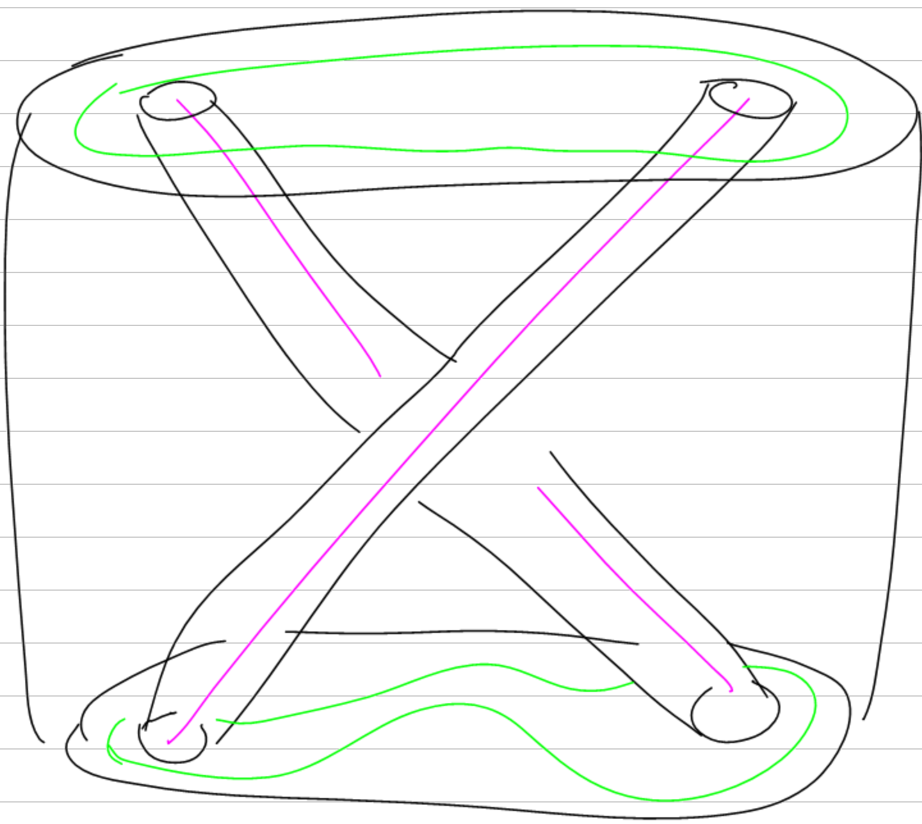
$\to \sigma_{32}$

$\to \sigma_{12}$

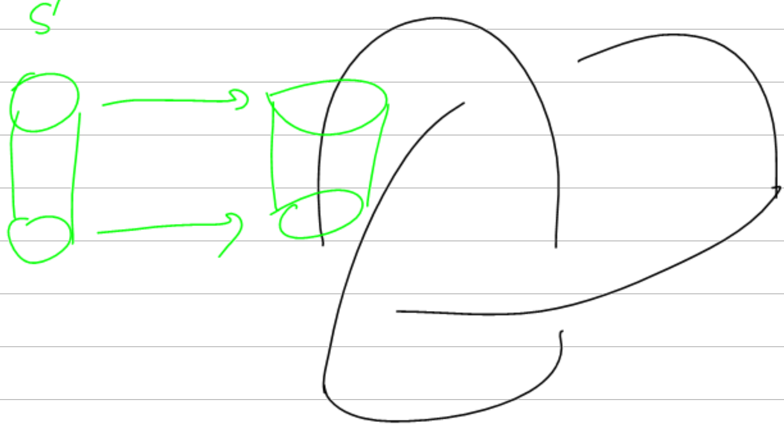
$\gamma_{ij} =$ 

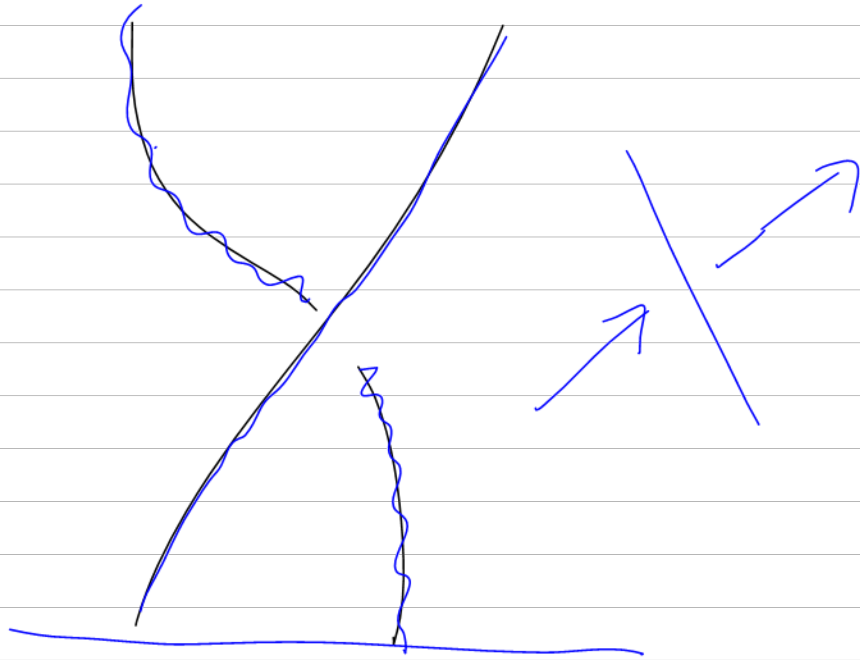
$\in PB_n$

$$\gamma_{ij} = \sigma_{i+1,i}^{-1}\,\sigma_{i+2,i}^{-1}\cdots\sigma_{j-1,i}^{-1}\,\sigma_{ji}\,\sigma_{ji}\cdots$$

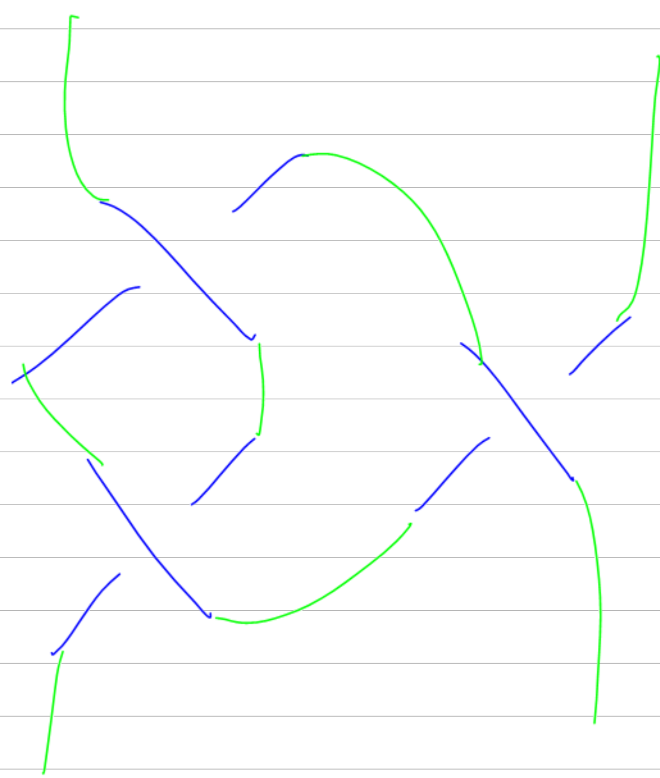


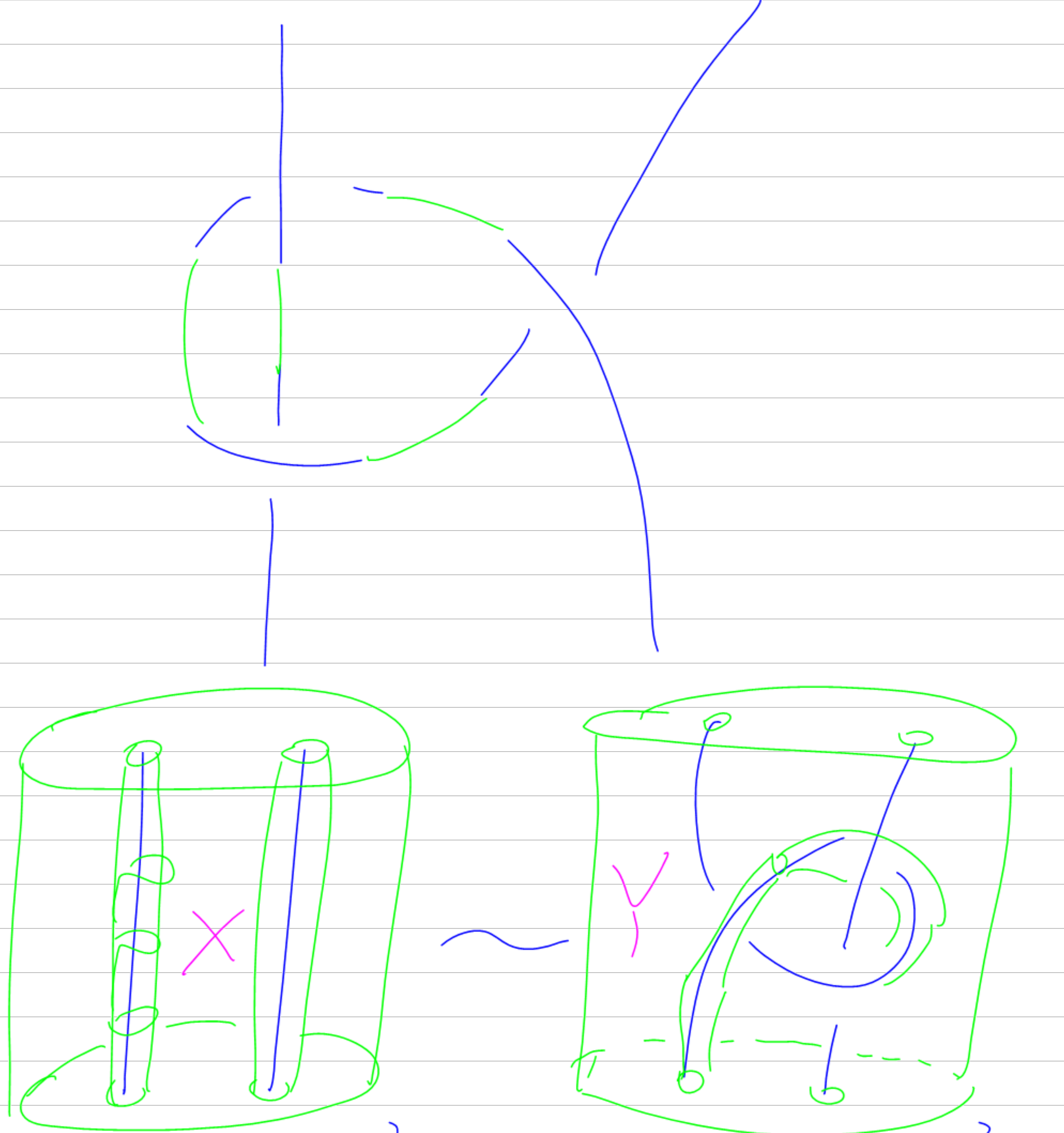
S^1



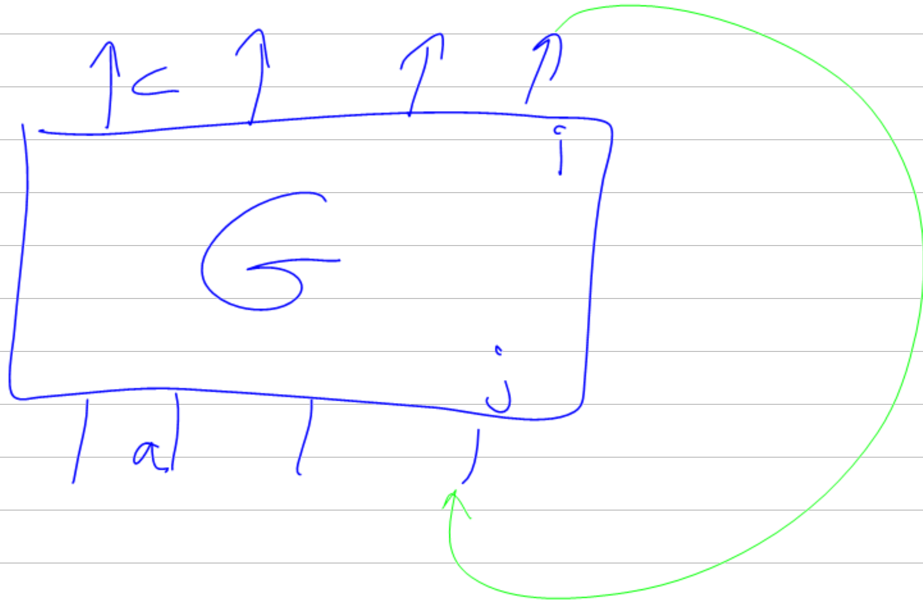


$$\begin{tikzcd} A & B \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, phantom, "\cong" rotate=90] & C \arrow[d, phantom, "\cong" rotate=90] \\ A' & B' \arrow[l, "\sim"'] \arrow[r, "\sim"] & C' \end{tikzcd}$$

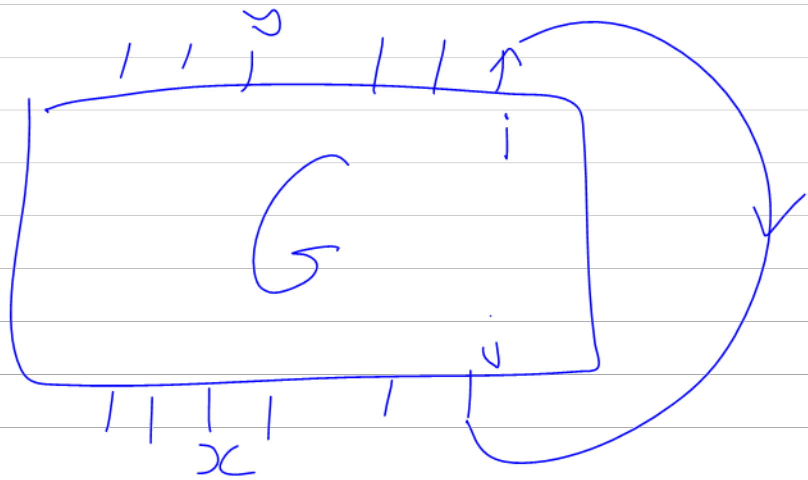


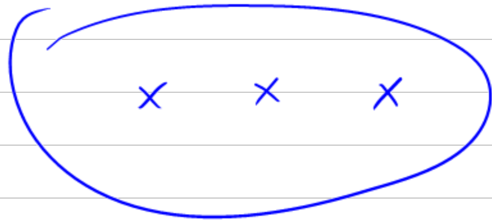
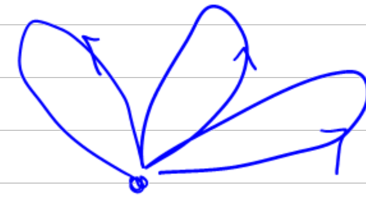


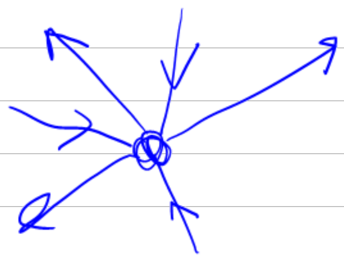
$I \sqcup I \hookrightarrow D^2 \times I \qquad\qquad I \sqcup I \hookrightarrow D^2 \times I$

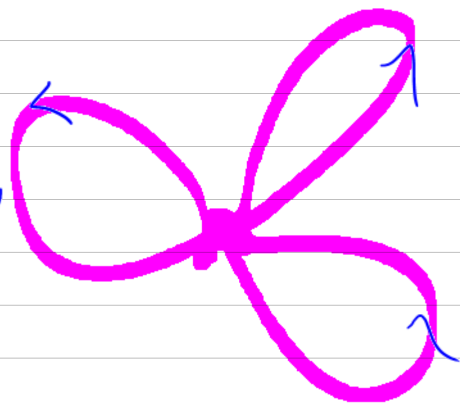


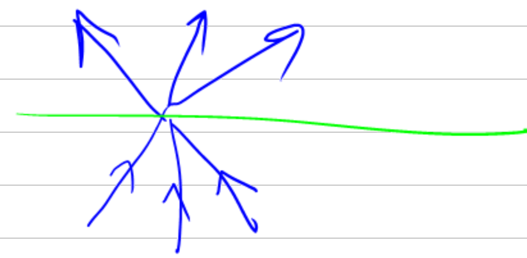
$$\left(G/m^{ij}_{k}\right)_{xy} = G_{xy} + G_{xi}G_{jy} + \dots$$

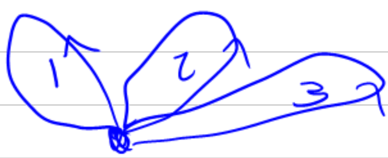


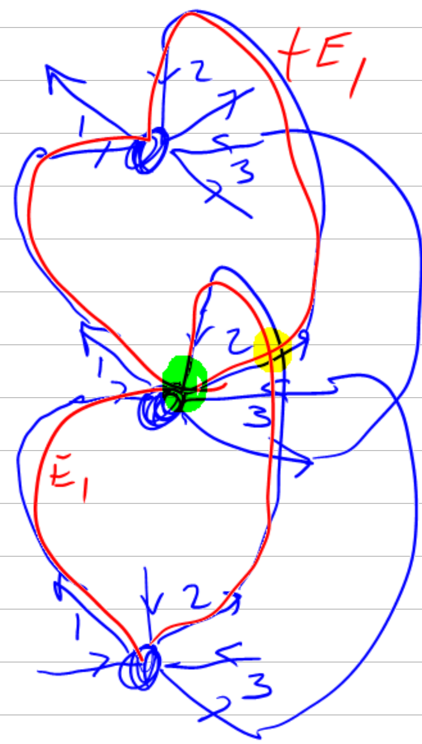
$D_3 =$  $=$ 

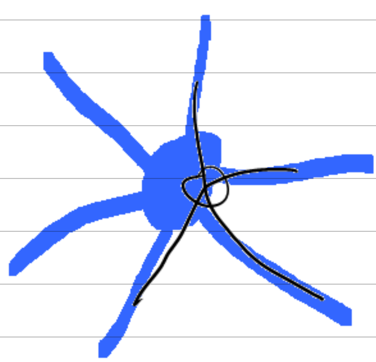


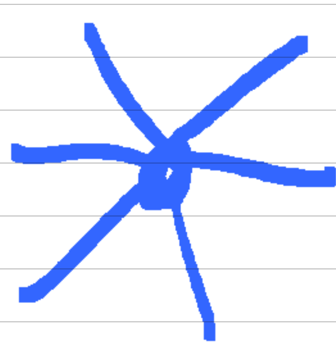




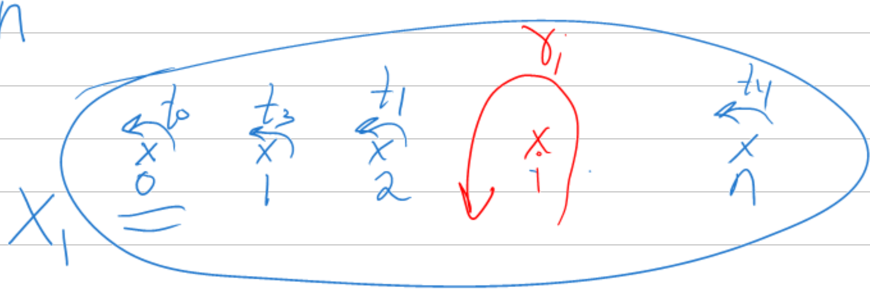




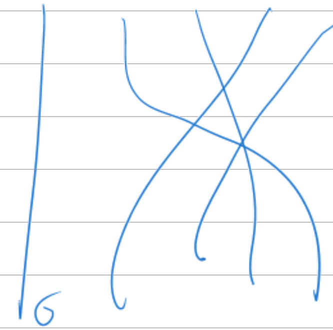




n

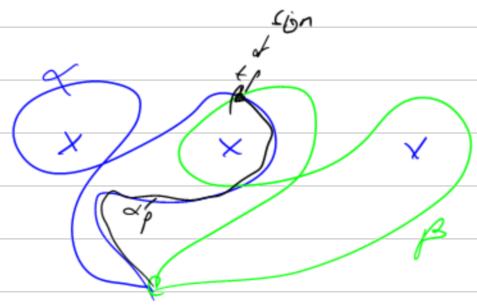


$F = \{t_3, t_1, \ldots, t_4\}$



$$H_1(X_1, F) = h[F, \textstyle\sum \alpha_i \beta_i]$$

$$\beta_i = (t_0 - 1)\gamma_i - (F_i - 1)\gamma_0$$



$$\langle \alpha, \beta \rangle = \sum_{p \in \alpha \cap \beta} \epsilon_p W(\alpha'_p \beta'_p)$$

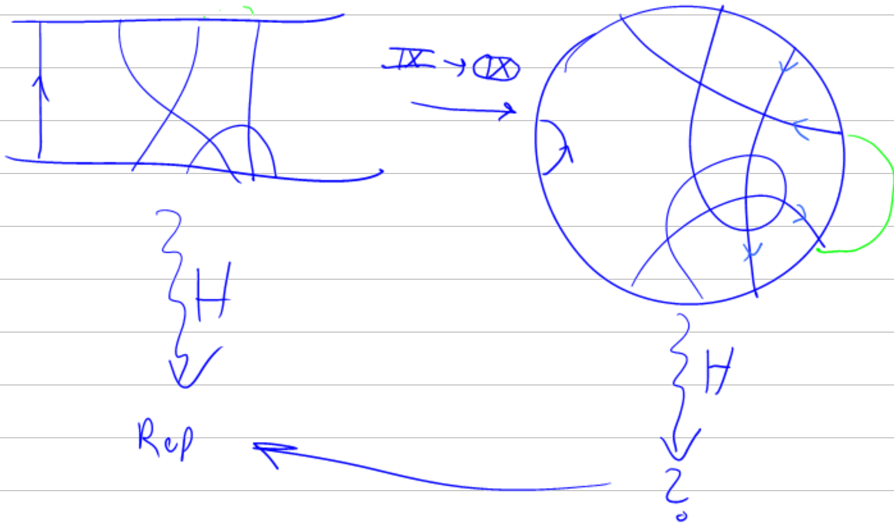
$$\alpha = \alpha'_p \alpha''_p \qquad \beta = \beta'_p \beta''_p$$

$W(\gamma)$ is such that

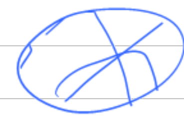
curve

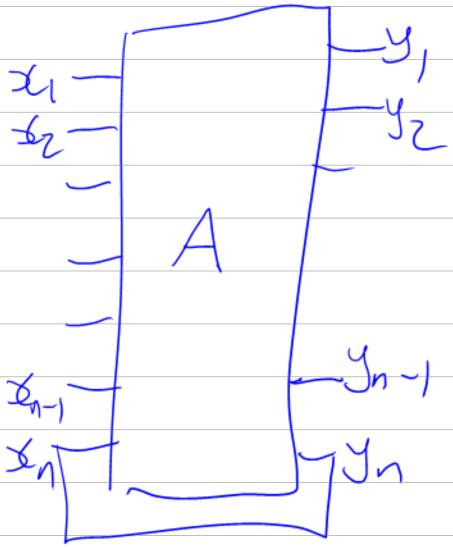
$$W(\gamma) \cdot \tilde{\gamma}(0) = \tilde{\gamma}(1)$$

power of t lift



$\Longleftrightarrow$ Cars

Q. Can you find a homological context for the cars invariant in the  context.



$$\tilde{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} = \begin{pmatrix} \vdots \\ \bar{y} \\ \vdots \\ y_n \end{pmatrix} \qquad \tilde{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \vdots \\ \bar{x} \\ \vdots \\ x_n \end{pmatrix}$$

$$\bar{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_{n-1} \end{pmatrix} \qquad \bar{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \end{pmatrix}$$

$$\tilde{y} = A\tilde{x} \qquad A = \left(\begin{array}{c|c} B & \phi \\ \hline \theta & \beta \end{array}\right)$$

$$y_n = x_n$$

$$\Downarrow$$

$$\begin{pmatrix} \bar{y} \\ y_n \end{pmatrix} = \begin{pmatrix} B & \phi \\ \theta & \beta \end{pmatrix} \begin{pmatrix} \bar{x} \\ x_n \end{pmatrix} \iff \begin{aligned} \bar{y} &= B\bar{x} + \phi x_n \\ y_n &= \theta\bar{x} + \beta x_n \end{aligned}$$

$$y_n = x_n$$

diff $y_n = x_n$

$$0 = \theta\bar{x} + (\beta - 1)x_n$$

$$\Updownarrow$$

$$x_n = \frac{\theta x}{1-\beta}$$

$$\bar{y} = B\bar{x} + \phi\frac{\theta\bar{x}}{1-\beta} = \underbrace{\left(B + \frac{\phi\theta}{1-\beta}\right)}_{\bar{A}}\bar{x}$$

$$\bar{y} = \bar{A}\bar{x}$$

<u>Q</u>. If $A \in M_{n\times n}(\mathbb{C})$ is unitary w.r.t some $\overbrace{\text{inner product}}^{\langle\ \rangle}$ on $\mathbb{C}^n$, is $\bar{A}$ also unitary wrt to some $\overbrace{\text{inner product}}^{\langle\ \rangle'}$ on $\mathbb{C}^{n-1}$?

<u>conj</u>

$$\langle v, w\rangle' = \langle \iota v, \iota w\rangle + \gamma\langle \iota v, e_n\rangle\langle e_n, \iota w\rangle$$

where $\iota : \mathbb{C}^{n-1} \to \mathbb{C}^n$ $\quad \bar{v} \mapsto \begin{pmatrix} \bar{v} \\ \vdots \\ 0 \end{pmatrix}$ for some $\gamma \in \mathbb{C}$ probably $\gamma = \pm 1$.
or $\pm\langle e_n, e_n\rangle^{\pm 1}$.