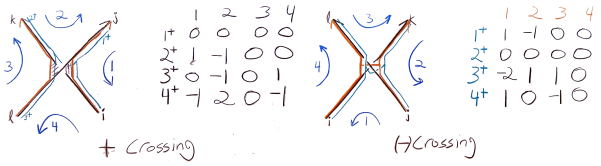
Pensieve header: Knot Signatures as in KnotTheory` and as in Kashaev. Continues pensieve://Projects/Signatures/AsInKT.nb.

In[•]:=
```
SetDirectory["C:\\drorbn\\AcademicPensieve\\Projects\\Signatures"];
<< Common.m
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

Tristram-Levine

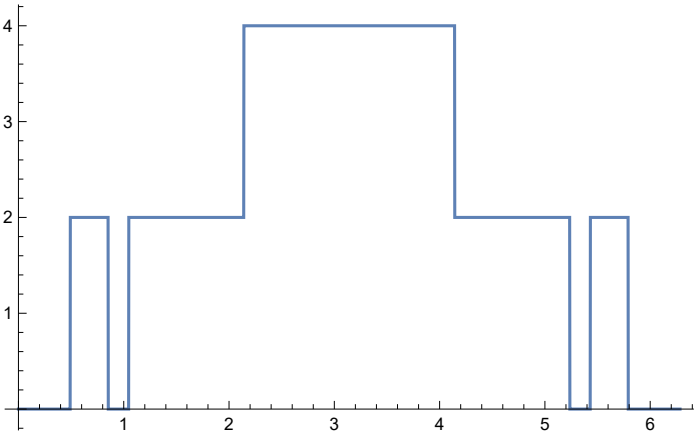
{\def\nbpdfText#1{\[\includegraphics[width=\linewidth]{#1}\]}



}

In[•]:=
```
TLS1[K_, ω_] := Module[{spd, a, s = 0, c, cs, A, is},
  spd = Times @@ PD[K] /. x_X :> If[PositiveQ@x, Xp, Xm] @@ x;
  cs = spd /. {
       Xp[i_, j_, k_, l_] :> a_j,i[++s] a_k,-j[++s] a_-l,-k[++s] a_-i,l[++s],
       Xm[i_, j_, k_, l_] :> a_-j,i[++s] a_k,j[++s] a_l,-k[++s] a_-i,-l[++s]
      } //. a_i_,j_[x__] a_j_,k_[y__] :> a_i,k[x, y] /. a__[x__] :> a[x];
  A = Table[0, Length@cs, Length@cs];
  Do[is = Position[cs, 4 i - #][[1, 1]] & /@ {3, 2, 1, 0};
   A[[is, is]] += If[spd[[i, 0]] === Xp, {{0, 0, 0, 0}, {1, -1, 0, 0}, {0, -1, 0, 1}, {-1, 2, 0, -1}}, {{1, -1, 0, 0}, {0, 0, 0, 0}, {-2, 1, 1, 0}, {1, 0, -1, 0}}],
   {i, Length[spd]}];
  Total[Sign[Select[
     Eigenvalues[(1 - ω) A + (1 - ω*) A^T], Abs[#] > 10^-6 &]]];
```

```
In[•]:= Plot[TLS1[Knot@"K12a422", e^(i t)], {t, 0, 2 π}]
```




```
Out[•]=
```

```
In[•]:= Total[(TLS1[#, -1] == -KnotSignature[#]) & /@ AllKnots[{3, 10}]]
```

KnotTheory: Loading precomputed data in PD4Knots`.

```
Out[•]= 249 True
```

```
In[•]:= K = Knot[4, 1];
        ω = E^(2 π i RandomReal[])
```

```
Out[•]= 0.592988 - 0.805211 i
```

```
In[•]:= {spd, a, s = 0, c, cs, A, is}
```

```
Out[•]= {spd, a, 0, c, cs, A, is}
```

```
In[•]:= PD[K]
```

```
Out[•]= PD[X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4], X[2, 7, 3, 8]]
```

```
In[•]:= spd = Times @@ PD[K] /. x_X :> If[PositiveQ@x, Xp, Xm] @@ x
```

```
Out[•]= Xm[2, 7, 3, 8] Xm[6, 3, 7, 4] Xp[4, 2, 5, 1] Xp[8, 6, 1, 5]
```

```
In[•]:= spd /. {
          Xp[i_, j_, k_, l_] :> a_{j,i}[++s] a_{k,-j}[++s] a_{-l,-k}[++s] a_{-i,l}[++s],
          Xm[i_, j_, k_, l_] :> a_{-j,i}[++s] a_{k,j}[++s] a_{l,-k}[++s] a_{-i,-l}[++s]
         }
```

```
Out[•]= a_{-8,5}[16] a_{-7,2}[1] a_{-6,-4}[8] a_{-5,-1}[15] a_{-4,1}[12] a_{-3,6}[5] a_{-2,-8}[4]
        a_{-1,-5}[11] a_{1,-6}[14] a_{2,4}[9] a_{3,7}[2] a_{4,-7}[7] a_{5,-2}[10] a_{6,8}[13] a_{7,3}[6] a_{8,-3}[3]
```

```
In[•]:= s = 0; spd /. {
            Xp[i_, j_, k_, l_] :> a_{j,i}[++s] a_{k,-j}[++s] a_{-l,-k}[++s] a_{-i,l}[++s],
            Xm[i_, j_, k_, l_] :> a_{-j,i}[++s] a_{k,j}[++s] a_{l,-k}[++s] a_{-i,-l}[++s]
           } //. a_{i_,j_}[x__] a_{j_,k_}[y__] :> a_{i,k}[x, y]
```

```
Out[•]= a_{-7,-7}[1, 9, 7] a_{-6,-6}[8, 12, 14] a_{-5,-5}[15, 11] a_{-3,-3}[5, 13, 3] a_{-2,-2}[4, 16, 10] a_{3,3}[2, 6]
```

```
In[•]:= s = 0; cs = spd /. {
          Xp[i_, j_, k_, l_] :> a_j,i[++s] a_k,-j[++s] a_-l,-k[++s] a_-i,l[++s],
          Xm[i_, j_, k_, l_] :> a_-j,i[++s] a_k,j[++s] a_l,-k[++s] a_-i,-l[++s]
        } //. a_i_,j_[x__] a_j_,k_[y__] :> a_i,k[x, y] /. a__[x__] :> a[x]
Out[•]= a[2, 6] a[15, 11] a[1, 9, 7] a[4, 16, 10] a[5, 13, 3] a[8, 12, 14]

In[•]:= A = Table[0, Length@cs, Length@cs]
Out[•]= {{0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0},
         {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}, {0, 0, 0, 0, 0, 0}}

In[•]:= A // MatrixForm
Out[•]//MatrixForm=
        ⎛0 0 0 0 0 0⎞
        ⎜0 0 0 0 0 0⎟
        ⎜0 0 0 0 0 0⎟
        ⎜0 0 0 0 0 0⎟
        ⎜0 0 0 0 0 0⎟
        ⎝0 0 0 0 0 0⎠

In[•]:= i = 1; is = Position[cs, 4 i - #][[1, 1]] & /@ {3, 2, 1, 0}
Out[•]= {3, 1, 5, 4}

In[•]:= A[[is, is]]
Out[•]= {{0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}, {0, 0, 0, 0}}

In[•]:= A[[is, is]] += If[spd[[i, 0]] === Xp, ⎛ 0  0  0  0⎞, ⎛ 1 -1  0  0⎞]
                                              ⎜ 1 -1  0  0⎟  ⎜ 0  0  0  0⎟
                                              ⎜ 0 -1  0  1⎟  ⎜-2  1  1  0⎟
                                              ⎝-1  2  0 -1⎠  ⎝ 1  0 -1  0⎠
Out[•]= {{1, -1, 0, 0}, {0, 0, 0, 0}, {-2, 1, 1, 0}, {1, 0, -1, 0}}

In[•]:= A // MatrixForm
Out[•]//MatrixForm=
        ⎛ 0  0  0  0  0  0⎞
        ⎜ 0  0  0  0  0  0⎟
        ⎜-1  0  1  0  0  0⎟
        ⎜ 0  0  1  0 -1  0⎟
        ⎜ 1  0 -2  0  1  0⎟
        ⎝ 0  0  0  0  0  0⎠

In[•]:= i = 2; is = Position[cs, 4 i - #][[1, 1]] & /@ {3, 2, 1, 0}
Out[•]= {5, 1, 3, 6}
```

```
In[•]:= A[[is, is]] += If[spd[[i, 0]] === Xp, {{0, 0, 0, 0}, {1, -1, 0, 0}, {0, -1, 0, 1}, {-1, 2, 0, -1}}, {{1, -1, 0, 0}, {0, 0, 0, 0}, {-2, 1, 1, 0}, {1, 0, -1, 0}}]
Out[•]= {{2, 0, -2, 0}, {0, 0, 0, 0}, {-2, 0, 2, 0}, {1, 0, -1, 0}}

In[•]:= A // MatrixForm
Out[•]//MatrixForm=
⎛0  0  0  0  0  0⎞
⎜0  0  0  0  0  0⎟
⎜0  0  2  0 -2  0⎟
⎜0  0  1  0 -1  0⎟
⎜0  0 -2  0  2  0⎟
⎝0  0 -1  0  1  0⎠

In[•]:= i = 3; is = Position[cs, 4 i - #][[1, 1]] & /@ {3, 2, 1, 0}
Out[•]= {3, 4, 2, 6}

In[•]:= A[[is, is]] += If[spd[[i, 0]] === Xp, {{0, 0, 0, 0}, {1, -1, 0, 0}, {0, -1, 0, 1}, {-1, 2, 0, -1}}, {{1, -1, 0, 0}, {0, 0, 0, 0}, {-2, 1, 1, 0}, {1, 0, -1, 0}}]
Out[•]= {{2, 0, 0, 0}, {2, -1, 0, 0}, {0, -1, 0, 1}, {-2, 2, 0, -1}}

In[•]:= A // MatrixForm
Out[•]//MatrixForm=
⎛0  0  0   0   0   0⎞
⎜0  0  0  -1   0   1⎟
⎜0  0  2   0  -2   0⎟
⎜0  0  2  -1  -1   0⎟
⎜0  0 -2   0   2   0⎟
⎝0  0 -2   2   1  -1⎠

In[•]:= i = 4; is = Position[cs, 4 i - #][[1, 1]] & /@ {3, 2, 1, 0}
Out[•]= {5, 6, 2, 4}

In[•]:= A[[is, is]] += If[spd[[i, 0]] === Xp, {{0, 0, 0, 0}, {1, -1, 0, 0}, {0, -1, 0, 1}, {-1, 2, 0, -1}}, {{1, -1, 0, 0}, {0, 0, 0, 0}, {-2, 1, 1, 0}, {1, 0, -1, 0}}]
Out[•]= {{2, 0, 0, 0}, {2, -2, 0, 2}, {0, 0, 0, 0}, {-2, 2, 0, -2}}
```

```
In[•]:= A // MatrixForm
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & -2 & 0 \\ 0 & 0 & 2 & -2 & -2 & 2 \\ 0 & 0 & -2 & 0 & 2 & 0 \\ 0 & 0 & -2 & 2 & 2 & -2 \end{pmatrix}$$

```
In[•]:= MatrixForm[(1 - ω) A + (1 - ω*) A^T]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0.+0.\,i & 0.+0.\,i & 0.+0.\,i & 0.+0.\,i & 0.+0.\,i & 0. \\ 0.+0.\,i & 0.+0.\,i & 0.+0.\,i & 0.+0.\,i & 0.+0.\,i & 0. \\ 0.+0.\,i & 0.+0.\,i & 1.62805+0.\,i & 0.814024-1.61042\,i & -1.62805+0.\,i & -0.81402 \\ 0.+0.\,i & 0.+0.\,i & 0.814024+1.61042\,i & -1.62805+0.\,i & -0.814024-1.61042\,i & 1.628 \\ 0.+0.\,i & 0.+0.\,i & -1.62805+0.\,i & -0.814024+1.61042\,i & 1.62805+0.\,i & 0.814024 \\ 0.+0.\,i & 0.+0.\,i & -0.814024-1.61042\,i & 1.62805+0.\,i & 0.814024+1.61042\,i & -1.62 \end{pmatrix}$$

```
In[•]:= Eigenvalues[(1 - ω) A + (1 - ω*) A^T]
```

```
Out[•]= {-4.86072, 4.86072, -1.11468 × 10^-15, 4.4537 × 10^-18, 0., 0.}
```

```
In[•]:= Select[
         Eigenvalues[(1 - ω) A + (1 - ω*) A^T], Abs[#] > 10^-6 &]
```

```
Out[•]= {-4.86072, 4.86072}
```

```
In[•]:= Sign[Select[
           Eigenvalues[(1 - ω) A + (1 - ω*) A^T], Abs[#] > 10^-6 &]]
```

```
Out[•]= {-1, 1}
```

```
        Total[Sign[Select[
             Eigenvalues[(1 - ω) A + (1 - ω*) A^T], Abs[#] > 10^-6 &]]]
```

```
In[•]:= Total[Sign[Select[
             Eigenvalues[(1 - ω) A + (1 - ω*) A^T], Abs[#] > 10^-6 &]]]
```

```
Out[•]= 0
```

A new TLS program

```
In[•]:= Clear[ω]
```

```
In[•]:= A = {{0, 0, 0, 0}, {1, -1, 0, 0}, {0, -1, 0, 1}, {-1, 2, 0, -1}}; MatrixForm[(1 - ω) A + (1 - ω*) A^T]
```

Out[•]//MatrixForm=

$$\begin{pmatrix} 0 & 1-\text{Conjugate}[\omega] & 0 & -1+\text{Conjugate}[\omega] \\ 1-\omega & -2+\omega+\text{Conjugate}[\omega] & -1+\text{Conjugate}[\omega] & 2\times(1-\text{Conjugate}[\omega]) \\ 0 & -1+\omega & 0 & 1-\omega \\ -1+\omega & 2\times(1-\omega) & 1-\text{Conjugate}[\omega] & -2+\omega+\text{Conjugate}[\omega] \end{pmatrix}$$

```
In[•]:= A = {{1, -1, 0, 0}, {0, 0, 0, 0}, {-2, 1, 1, 0}, {1, 0, -1, 0}}; MatrixForm[(1 - ω) A + (1 - ω*) A^T]
```

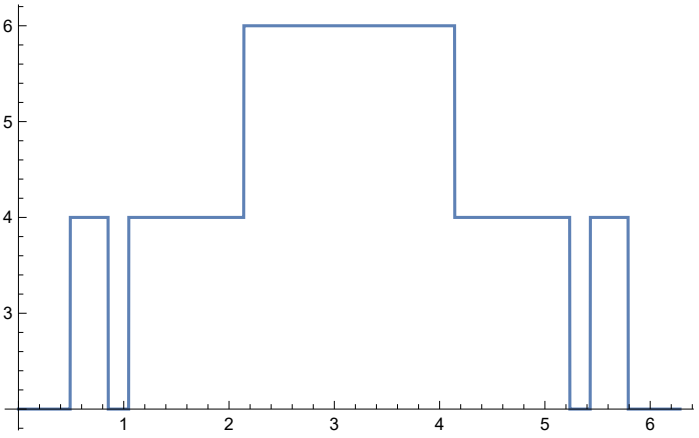
Out[•]//MatrixForm=

$$\begin{pmatrix} 2-\omega-\text{Conjugate}[\omega] & -1+\omega & -2\times(1-\text{Conjugate}[\omega]) & 1-\text{Conjugate}[\omega] \\ -1+\text{Conjugate}[\omega] & 0 & 1-\text{Conjugate}[\omega] & 0 \\ -2\times(1-\omega) & 1-\omega & 2-\omega-\text{Conjugate}[\omega] & -1+\text{Conjugate}[\omega] \\ 1-\omega & 0 & -1+\omega & 0 \end{pmatrix}$$

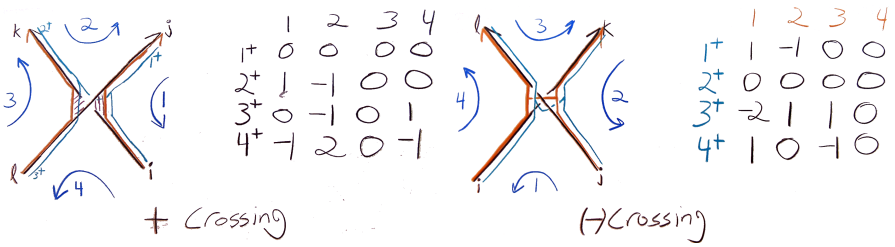
```
In[•]:= TLS2[K_, ω_] := Module[{spd, a, s = 0, c, cs, A, is},
  spd = Times @@ PD[K] /. x_X :> If[PositiveQ@x, Xp, Xm] @@ x;
  cs = spd /. {
       Xp[i_, j_, k_, l_] :> a_{j,i}[++s] a_{k,-j}[++s] a_{-l,-k}[++s] a_{-i,l}[++s],
       Xm[i_, j_, k_, l_] :> a_{-j,i}[++s] a_{k,j}[++s] a_{l,-k}[++s] a_{-i,-l}[++s]
      } //. a_{i_,j_}[x__] a_{j_,k_}[y__] :> a_{i,k}[x, y] /. a__[x__] :> a[x];
  A = Table[0, Length@cs, Length@cs];
  Do[is = Position[cs, 4 i - #][[1, 1]] & /@ {3, 2, 1, 0};
   A[[is, is]] += If[spd[[i, 0]] === Xp,
     {{0, 1-ω*, 0, -1+ω*}, {1-ω, -2+ω+ω*, -1+ω*, 2×(1-ω*)}, {0, -1+ω, 0, 1-ω}, {-1+ω, 2×(1-ω), 1-ω*, -2+ω+ω*}},
     {{2-ω-ω*, -1+ω, -2×(1-ω*), 1-ω*}, {-1+ω*, 0, 1-ω*, 0}, {-2×(1-ω), 1-ω, 2-ω-ω*, -1+ω*}, {1-ω, 0, -1+ω, 0}}
    ],
   {i, Length[spd]}];
  Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-6 &]]]
 ];
```

```
In[•]:= Plot[TLS2[Knot@"K12a422", e^(i t)], {t, 0, 2 π}]
```

Out[•]=



Kashaev



KC for “Kashaev’s Conjecture”.

```
In[•]:= KC[K_, x1_, x2_] := Module[{spd, a, s = 0, c, cs, A, is, w = 0},
  spd = Times @@ PD[K] /. X_X :> If[PositiveQ@x, Xp, Xm] @@ x;
  cs = spd /. {
       Xp[i_, j_, k_, l_] :> a_{j,i}[++s] a_{k,-j}[++s] a_{-l,-k}[++s] a_{-i,l}[++s],
       Xm[i_, j_, k_, l_] :> a_{-j,i}[++s] a_{k,j}[++s] a_{l,-k}[++s] a_{-i,-l}[++s]
      } //. a_{i_,j_}[x__] a_{j_,k_}[y__] :> a_{i,k}[x, y] /. a_[x__] :> a[x];
  A = Table[0, Length@cs, Length@cs];
  Do[is = Position[cs, 4 i - #][[1, 1]] & /@ {3, 2, 1, 0};
   A[[is, is]] += If[spd[[i, 0]] === Xp,
     ++w; {{1, x1, 1, x1}, {x1, x2, x1, 1}, {1, x1, 1, x1}, {x1, 1, x1, x2}},
     --w; {{-x2, -x1, -1, -x1}, {-x1, -1, -x1, -1}, {-1, -x1, -x2, -x1}, {-x1, -1, -x1, -1}}
    ],
   {i, Length[spd]}];
  (Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-6 &]]] - w) / 2
 ];
```

```
In[•]:= t = RandomReal[{0, 2 * π}]
        ω = e^(i * t);
        x1 = (1 / 2) * (e^((1 / 2) * i * t) + e^(- (1 / 2) * i * t));
        x2 = (1 / 2) * (e^(i * t) + e^(-i * t));
        Total[(KC[#, x1, x2] == TLS2[#, ω]) & /@ AllKnots[{3, 10}]]
```

Out[•]= 2.22305

Out[•]= 249 True

```
In[•]:= KC2[K_, x1_, x2_] := Module[{spd, a, s = 0, c, cs, A, is, w = 0},
          spd = Times @@ PD[K] /. x_X :> If[PositiveQ@x, Xp, Xm] @@ x;
          cs = spd /. {
                Xp[i_, j_, k_, l_] :> a_{j,i}[++s] a_{k,-j}[++s] a_{-l,-k}[++s] a_{-i,l}[++s],
                Xm[i_, j_, k_, l_] :> a_{-j,i}[++s] a_{k,j}[++s] a_{l,-k}[++s] a_{-i,-l}[++s]
               } //. a_{i_,j_}[x__] a_{j_,k_}[y__] :> a_{i,k}[x, y] /. a__[x__] :> a[x];
          A = Table[0, Length@cs, Length@cs];
          Do[is = Position[cs, 4 i - #][[1, 1]] & /@ {3, 2, 1, 0};
           A[[is, is]] += If[spd[[i, 0]] === Xp,
             ++w;
              {{1, x1, 1, x1}, {x1, x2, x1, 1}, {1, x1, 1, x1}, {x1, 1, x1, x2}} - {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}, --w;
              {{-x2, -x1, -1, -x1}, {-x1, -1, -x1, -1}, {-1, -x1, -x2, -x1}, {-x1, -1, -x1, -1}} + {{1, 0, 0, 0}, {0, 1, 0, 0}, {0, 0, 1, 0}, {0, 0, 0, 1}}
            ],
           {i, Length[spd]}];
          (Total[Sign[Select[Eigenvalues[A], Abs[#] > 10^-6 &]]]) / 2
         ];
```

```
In[•]:= t = RandomReal[{0, 2 * π}]
        ω = e^(i * t);
        x1 = (1 / 2) * (e^((1 / 2) * i * t) + e^(- (1 / 2) * i * t));
        x2 = (1 / 2) * (e^(i * t) + e^(-i * t));
        Total[(KC2[#, x1, x2] == TLS2[#, ω]) & /@ AllKnots[{3, 10}]]
```

Out[•]= 5.13387

Out[•]= 167 False + 82 True