Math 117 Course Description

Classical and Statistical Mechanics, Spring 1994

- *Time and place*: MWF 1PM, Science Center 507.
- *Instructor*: Dror Bar-Natan, Science Center 426G, 5-8797, dror@math.
- *Office hours*: Wednesday 2-3PM and Friday 12-1PM.
- *Teaching fellow*: David Nowakowski, 3-6932.
- *Review sessions*: To be determined.
- *Textbooks*: Gelfand-Fomin's *Calculus of Variations* for the first few weeks; about the rest we'll see later.
- *Goal*: Discuss some of the physics that every mathematician should know, from the point of view of a mathematician. (I wish I could give a course on *all* the physics that every mathematician should know, but I don't know enough physics for that.) There will be some over-emphasis of subjects that I want to understand better by the end of the course.
- *Intended for*: Math majors and beginning math graduate students.
- *Tentative course plan:* (it is *conceivable* that we will actually follow it)

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      M  W  F Topics
Feb      2  4 Introduction: path integrals, F=ma, the Fourier semigroup,
      7  9 11 the EPR paradox, Maxwell's equations.
     14 16 18 The first few chapters of Gelfand-Fomin: Euler's equation,
     XX 23 25 constrains, Nother's theorem, Hamilton's equations, second
Mar  28  2  4 variations.
      7  9 11
     14 16 18 The last chapter of Bamberg-Sternberg - entropy, temperature,
     21 23 25 thermodynamics, statistical mechanics.
Apr  XX XX XX                    (Spring recess - no classes)
      4  6  8
     11 13 15 A little on quantum mechanics - the uncertainty principle,
     18 20 22 quantum probability, the hydrogen atom.
     25 27 29 Very little on quantum field theory - an introduction
May   2  4  6 to perturbation theory via the Chern-Simons example.
      9 11 13 Reading period - I plan to finish everything before that,
     16 18    but plans are there only so that they can be changed later.
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- *Prerequisites:* Differential forms and Stokes' theorem (math 22, 25-55 or math 134 should do), really understanding diagonalization of matrices, having heard of Hilbert spaces and linear operators on them, and no fear of ODEs.
- *Homework* will be assigned weekly and be due the following week.
- *Grading*: Two midterms, homework, and a final. Dates and weights will be announced later on.

INFORMATION SHEET FOR MATH 117

Name:

Class:

Dorm phone number:

Dorm address:

Electronic mail address:

I want to major in:

I'm taking this class because:

I've taken the following math courses before:

I've taken the following science courses before:

The other math/science courses that I'm taking this term are:

In short, what did you think of class today? (too fast, too slow, too high, too low, ...)

What happens to a particle in a quantum harmonic oscillator $\frac{\pi}{2}$ seconds after it was thrown in?

Of course, the above question is not particularily interesting. Luckily, while deriving the answer to that question we will pass by few of the most fundamental ideas in physics, the key one being the idea of integration over infinite dimensional spaces, which is central to quantum field theory. To my understanding, quantum field theory might as well be considered as a part of mathematics exceptional in not being completly rigorous, but yet deep, elegant and powerful. So our real purpose here is to see a very simple but yet essential use of the basic ideas of quantum field theory.

Not everything along the way will be accurate and rigorous although the discussion below can be made completely so. The reasons for that are lack of time, and as the weaker part of QFT is nonrigorous anyway, also lack of motivation. And last comment - few of the expressions further down are going to look pretty horrible, but the end result will be neat, familiar, and maybe a bit unexpected.

The question: Let the complex valued function $\psi=\psi(t,x)$ be a solution of the schrodinger equation

$$\frac{\partial\psi}{\partial t}=-i\left(-\frac{1}{2}\Delta_x+\frac{1}{2}x^2\right)\psi \quad \text{with } \psi\big|_{t=0}=\psi_0$$

what is $\psi\big|_{t=T=\pi/2}$?

In fact, big part of our discussion will work just as well for the general schrodinger equation --

$$\frac{\partial\Psi}{\partial t}=-iH\Psi\ ,\quad H=-\tfrac{1}{2}\Delta_x+V(x)\ ,\quad \Psi|_{t=0}=\Psi_0\ ,\quad T \text{ arbitrary.}$$

Ψ – "the wave function", $|\Psi(t,x)|^2$ is the probability of finding our particle at time t in position x.
H – "the Hamiltonian", "the evolution operator".
$-\frac{1}{2}\Delta_x$ – "kinetic energy term".
$V(x)$ – "the potential at a point x"

Solution:

$\frac{\partial\Psi}{\partial t}=-iH\Psi$, $\Psi|_{t=0}=\Psi_0$ implies formally:

$$\Psi(T,x)=(e^{-iTH}\Psi_0)(x)=(e^{i\frac{T}{2}\Delta-iTV}\Psi_0)(x)=$$

by aside 1, with $n=10^{58}+17$, and for convenience set $x_n=x$

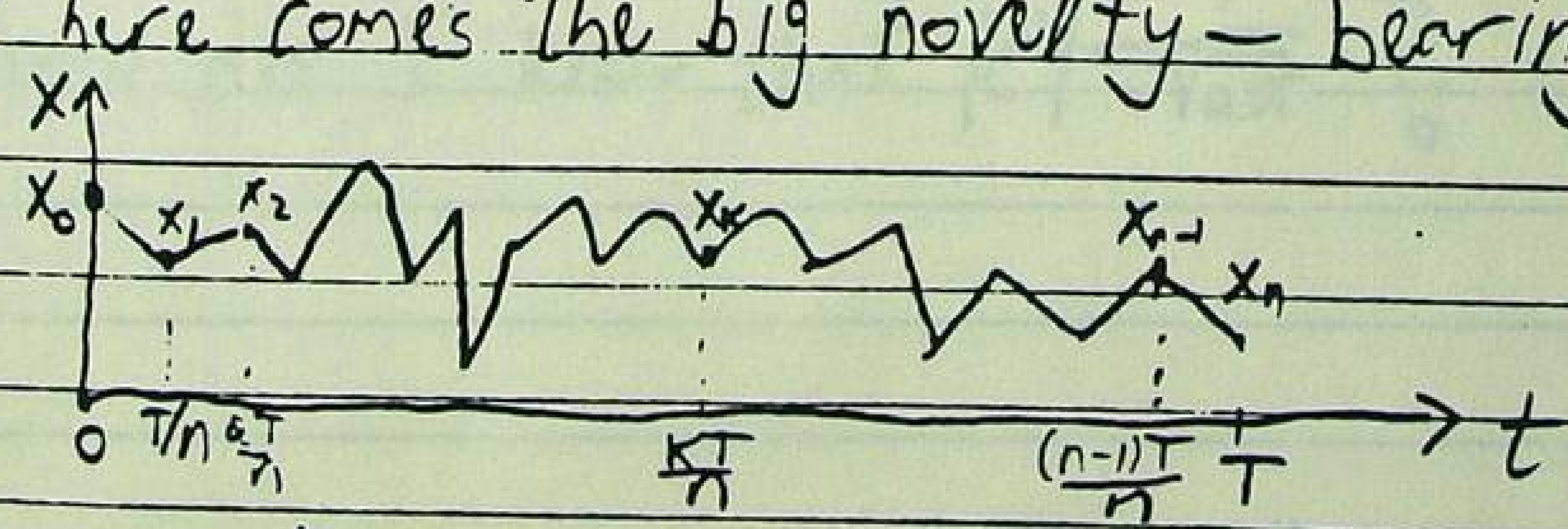
$$=(e^{i\frac{T}{2n}\Delta}\cdot e^{-i\frac{T}{n}V}\cdot e^{i\frac{T}{2n}\Delta}e^{-i\frac{T}{n}V}\cdot\ldots\cdot e^{i\frac{T}{2n}\Delta}e^{-i\frac{T}{n}V}\Psi_0)(x_n)=$$

$$=C\cdot\int dx_{n-1}e^{i\frac{(x_n-x_{n-1})^2}{2T/n}}e^{-i\frac{T}{n}V(x_{n-1})}\int dx_{n-2}e^{i\frac{(x_{n-1}-x_{n-2})^2}{2T/n}}e^{-i\frac{T}{n}V(x_{n-2})}\ldots$$

$$\ldots\int dx_0\,e^{i\frac{(x_1-x_0)^2}{2T/n}}e^{-i\frac{T}{n}V(x_0)}\Psi_0(x_0)=$$

$$=C\cdot\int dx_0\ldots dx_{n-1}\exp\left(i\frac{T}{2n}\sum_{k=1}^{n}\left(\frac{x_n-x_{n-1}}{T/n}\right)^2-i\frac{T}{n}\sum_{k=0}^{n-1}V(x_k)\right)\cdot\Psi(x_0)=$$

Now here comes the big novelty – bearing in mind the picture



we can write

$$\hat{=}\,C\int dx_0\int_{W_{x_0x_n}}\mathcal{D}x\,\exp\left(i\int_0^T dt\left(\tfrac{1}{2}\dot{x}^2(t)-V(x(t))\right)\right)\Psi(x_0)=$$

$$W_{x_0x_n}=\left\{\begin{array}{l}x:[0,T]\to\mathbb{R}\\ x(0)=x_0,\ x(T)=x_n\end{array}\right\}$$

$$=c\int dx_0\,\psi(x_0)\int_{W_{x_0x_n}}\mathcal{D}x\,\exp(i\mathcal{L}(x))=$$

Let x_c be the minimum point of $\mathcal{L}(x)$, write $x=x_c+x_q$ and get

$$=c\int dx_0\,\psi(x_0)\int_{W_{00}}\mathcal{D}x_q\,\exp(i\mathcal{L}(x_c+x_q))=$$

In our particular case, using aside 4, we get

$$=c\int dx_0\,\psi(x_0)\int_{W_{00}}\mathcal{D}x_q\,\exp(i\mathcal{L}(x_c)+i\mathcal{L}(x_q))=$$

The path integral is now independent of x_0, and so it factors out & Therfore

$$=c'\int dx_0\,\psi(x_0)\,e^{i\mathcal{L}(x_c)}=$$

in our case, with $t=\frac{\pi}{2}$

$$=c'\int dx_0\,\psi(x_0)\exp\left(i\int_0^{\pi/2}\tfrac{1}{2}[(x_n\sin t+x_0\cos t)']^2-\tfrac{1}{2}(x_n\sin t+x_0\cos t)^2dt\right)=$$

$$=c'\int dx_0\,\psi(x_0)\exp(-ix_0x_n)$$

So how do I know that $|c'|=\frac{1}{\sqrt{2\pi}}$?

Aside 1: If A and B are matrices, then

$$e^{A+B} = \lim_{n\to\infty} (e^{A/n} e^{B/n})^n$$

Proof Just expand both sides as power series, and use some combinatorics to compare the coefficients. For a smoother proof see Glimm-Jaffe page 47. Slightly cheating what they say is that

$$e^{\frac{A+B}{n}} = e^{\frac{A}{n}} e^{\frac{B}{n}} \text{ to } \frac{\text{const}}{n^2} \quad \text{(trivial)}$$

and so

$$(e^{\frac{A+B}{n}})^n = (e^{\frac{A}{n}} e^{\frac{B}{n}})^n \text{ to } \frac{\text{const}}{n}$$

Aside 2: $(e^{itV}\psi_0)(x) = e^{itV(x)}\psi_0(x)$ — Trivial

Aside 3: $(e^{i\frac{t}{2}\Delta}\psi_0)(x) = c \cdot \int dx' e^{i\frac{(x-x')^2}{2t}} \psi_0(x')$

Proof: In fact, the left hand side is just the solution $\psi(t,x)$ of Schrödinger's equation with $V\equiv 0$:

$$\frac{\partial\psi}{\partial t} = i\frac{1}{2}\Delta_x \psi \qquad \psi|_{t=0} = \psi_0 .$$

Taking Fourier transform $\tilde{\psi}(t,p) = \frac{1}{\sqrt{2\pi}}\int e^{-ixp}\psi(t,x)dx$:

$$\frac{\partial\tilde{\psi}}{\partial t} = -i\frac{p^2}{2}\tilde{\psi} \qquad \tilde{\psi}|_{t=0} = \tilde{\psi}_0$$

For a fixed p, this is just a trivial ordinary differential equation with respect to t, and thus:

$$\tilde{\psi}(t,p) = e^{-i\frac{tp^2}{2}}\tilde{\psi}_0(p)$$

Taking inverse Fourier transform, which takes products to convolutions and Gaussians to Gaussians, we get Q.E.D.

Aside 4 Determining the minimum point of $\mathcal{L}(X)$ on $W_{x_0 x_n}$:

If x_c is the minimum point in $W_{x_0 x_n}$, then for arbitrary $x_q \in W_{00}$ there will be no term in

$$\mathcal{L}(X_c + \epsilon X_q)$$

which is linear in ϵ. Now

$$\mathcal{L}(X) = \int_0^T dt \left(\tfrac{1}{2}\dot{x}^2(t) - V(x(t))\right)$$

So using $V(X_c + \epsilon X_q) \cong V(X_c) + \epsilon X_q V'(X_c)$, we get that the linear term in ϵ in $\mathcal{L}(X_c + \epsilon X_q)$ is

$$\int_0^T dt \left(\dot{x}_c \cdot \dot{x}_q - V'(X_c) \cdot X_q\right) =$$

integrating by parts and using $X_q(0) = X_q(T) = 0$:

$$= \int_0^T dt \left[-\ddot{x}_c - V'(x_c)\right] \cdot x_q .$$

For this integral to vanish independently of x_q, we must have $-\ddot{x}_c - V'(X_c) \equiv 0$, or

$$\ddot{x}_c = -V'(X_c).$$ (The famous $F = ma$ of Newton! We have just rediscovered the Principle of least action!)

In our very particular case $V(X) = \frac{1}{2}X^2$ we get:

$$\ddot{x}_c = -X_c, \quad X_c(0) = X_0, \quad X_c\left(\tfrac{\pi}{2}\right) = X_n$$

and therfore:

$$x_c(t) = X_n \sin t + X_0 \cos t$$

Dror Bar-Natan

Math 117, 2/2/94.

Go through course description.

Distribute [illegible]

First Snapshot:

"what happens to a particle in a 1-D quantum harmonic oscillator $\frac{\pi}{2}$ seconds after it was thrown in"

$\Psi_0(x)$ complex valued "wave function"

$\|\Psi_0(x)\|^2$ - prob. of finding P at x.

$$\int |\Psi(x)|^2 dx = 1$$

$$T = \frac{\pi}{2}, \quad \Psi_T(x_T) = \int dx_0 \int_{\substack{x:[0,T]\to\mathbb{R} \\ x(0)=x_0;\ x(T)=x_T}} \mathcal{D}x\, e^{i\mathcal{L}(x)} =$$

$W(x_0, x)$

$$V(x) = \frac{1}{2}x^2$$

$$\mathcal{L}(x) = \text{"The lagrangian"} = \int_0^T \underbrace{\frac{1}{2}\dot{x}^2(t)}_{\text{kin}} - \underbrace{V(x(t))}_{\text{pot.}}\, dt$$

$= \ldots$

BULLETIN OF THE
ADVANCED MATHEMATICAL SOCIETY
Volume 23, Number 6, November 2001

Copied From:
"The Joy of TeX"
by M. Spivak.

WHAT EVERY YOUNG MATHEMATICIAN SHOULD KNOW

BY LORD K. ELVIN

ABSTRACT. We evaluate an interesting definite integral.

The purpose of this paper is to call attention to a result of which many mathematicians seem to be ignorant.

THEOREM. *The value of* $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ *is*

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

PROOF: We have

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy \qquad \text{by Fubini}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy\\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}r\,dr\,d\theta \qquad \text{using polar coordinates}\\
&= \int_{0}^{2\pi}\left[\int_{0}^{\infty} e^{-r^2}r\,dr\right]d\theta\\
&= \int_{0}^{2\pi}\left[-\frac{e^{-r^2}}{2}\Bigg|_{r=0}^{r=\infty}\right]d\theta\\
&= \int_{0}^{2\pi}\left[\frac{1}{2}\right]d\theta\\
&= \pi.
\end{aligned}$$

Remark: A mathematician is one to whom *that* is as obvious as that twice two makes four is to you.

INSTITUTE FOR HAUGHTY ATTITUDES

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A little About the Fourier Transform

Math 117, February 4, 1994

Dror Bar-Natan

Definition 1 *Let f be an integrable function on $\mathbf{R}$. Define its Fourier transform $\tilde{f}$ by:*

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(x)dx.$$

Theorem 1 *(The Fourier inversion theorem) One can reconstruct f from $\tilde{f}$ using:*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \tilde{f}(p)dp.$$

Remark Notice that it follows that $\tilde{\tilde{f}}(x) = f(-x)$ and that $\tilde{\tilde{\tilde{\tilde{f}}}} = f$.

Fact Let $f_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}$. Then $\tilde{f}_\sigma(p) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma p^2}{2}}$.

Claim 1 *If $f(x) = g(x - x_0)$ then $\tilde{f}(p) = e^{-ipx_0} \tilde{g}(p)$.*

Claim 2 *If $f(x) = e^{ip_0 x} g(x)$, then $\tilde{f}(p) = \tilde{g}(p - p_0)$.*

Definition 2 *The convolution of two functions f and g is defined by:*

$$(f * g)(x) = \int_{-\infty}^{\infty} dy\, f(x-y)g(y) = \int_{-\infty}^{\infty} dy\, f(y)g(x-y).$$

Claim 3 $\widetilde{f * g} = \sqrt{2\pi}\tilde{f}\tilde{g}$ *and* $\widetilde{fg} = \frac{1}{\sqrt{2\pi}} \tilde{f} * \tilde{g}$.

Remark Pick $g = f_\sigma$ and you can prove the Fourier inversion theorem!!

Claim 4 $\widetilde{f'}(p) = ip\tilde{f}(p)$ *and* $\widetilde{xf(x)} = i\frac{d}{dp}\tilde{f}$.

Problems:

1. Compute the Fourier transform of the function χ defined by $\chi(x) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{otherwise} \end{cases}$.
2. Compute the convolution $\chi * \chi$.
3. Check that indeed $\widetilde{\chi * \chi} = \sqrt{2\pi}\tilde{\chi} \cdot \tilde{\chi}$.
4. Prove the *Plancherel identity*: Let f be a complex-valued function on $\mathbf{R}$.
   (a) Let $g(x) = \overline{f(-x)}$. Prove that $\tilde{g}(p) = \overline{\tilde{f}(p)}$.
   (b) Evaluate $(f * g)(0)$ and $(\widetilde{\tilde{f}\tilde{g}})(0)$ and deduce that
   $$\int |f(x)|^2 dx = \int |\tilde{f}(p)|^2 dp.$$
   (c) in what sense is the Plancherel identity a variation of the famous theorem of Pythagoras?

More information can be found in any standard analysis book, such as Rudin's *Funcational Analysis* or *Real and Complex Analysis.*

Feb 4 1994:

How was class today? (Fast, slow, clear, messy, boring, amazing, ... ?)

Feb 4 1994:

How was class today? (Fast, slow, clear, messy, boring, amazing, ... ?)

Feb 4 1994:

How was class today? (Fast, slow, clear, messy, boring, amazing, ... ?)

Math 117, February 4 1994

Re-distribute information sheets etc.

Introduce David Nowakowski; let him distribute his handout.

Vote regarding HW.

<u>Reminder</u>:

$$\Psi_T(x_T) = \int dx_0\, \Psi(x_0) \underbrace{\int \mathcal{D}x\, e^{i\mathcal{L}(x)}}_{W(x_0,x_T)=\text{all paths } x \text{ from } x_0 \text{ to } x_T} \quad ; \quad \mathcal{L}(x) = \int_0^T \left(\tfrac{1}{2}\dot{x}^2 - V(x)\right)$$

$$V(x) = \tfrac{1}{2}x^2$$

minimized $\mathcal{L}(x)$; min x_c satisfied $\ddot{x}_c = -V'(x)$ $(F=ma)$

Finish this example. Why is $c = \frac{1}{\sqrt{2\pi}}$?

A few words on the Fourier transform:

Math 117 evaluation Name: (optional) Date:

What did you think of class today? (Fast, slow, clear, messy, boring, amazing, too rigorous, to vague ... ?)

How is it going in general?

Math 117 - HW assignment #1 Feb 7, 1994

due Feb 14, 1994

1a. Compute the Fourier transform of the function χ defined by $\chi(x) = \begin{cases} 1 & |x|<1 \\ 0 & \text{otherwise} \end{cases}$

b. Compute the convolusion $\chi * \chi$.

c. Verify that indeed $\widetilde{\chi * \chi} = \sqrt{2\pi}\, \tilde{\chi} \cdot \tilde{\chi}$

2. Prove the "plancherel identity": Let f be a complex valued function on $\mathbb{R}$.

a. Let $g(x) = \overline{f(-x)}$. Prove that $\tilde{g}(p) = \overline{\tilde{f}(p)}$

b. Evaluate $(f*g)(0)$ and $(\widetilde{\tilde{f}\tilde{g}})(0)$ and deduce that

$$\int |f(x)|^2 dx = \int |\tilde{f}(p)|^2 dp$$

c. In what sense is the Plancherel identity a variation of the Pythagoras theorem?

3a. Check (using path integrals & the tricks used in class) that

$$\psi(T, x) = C_T \int dx_0\, \psi_0(x_0) \cdot e^{\frac{i}{2\sin T}\left((x_0^2 + x^2)\cos T - 2x_0 x\right)}$$

where the constant C_T depends only on T. $(T \neq \frac{\pi}{2} k)$

b. Why is $C_T = \left(\pi(1 - e^{2iT})\right)^{1/2}$?

c. Check that $\psi(T, x)$ satisfies Schrödinger's eqn:

$$\frac{\partial \psi}{\partial T} = -i\left(-\frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + \frac{1}{2}x^2\psi\right)$$

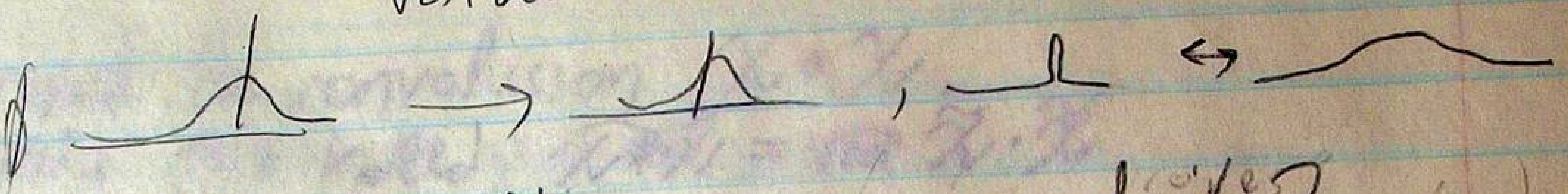
Math 117, Feb 7 1994 $\left(\begin{array}{l} Q(0)=0 \\ \Rightarrow Q(x_c+x_0)=Q(x_c)+Q(x_0) \end{array} \right)$?

Remind $\tilde{f}$; write inversion formula.

$$\frac{1}{\sqrt{2\pi}\sqrt{2\pi\sigma}}\int e^{-\frac{x^2}{2\sigma}-ipx}\,dx = \frac{1}{2\pi\sqrt{\sigma}}\int e^{-\frac{1}{2\sigma}(\underbrace{x+i\sigma p}_{y})^2-\frac{\sigma p^2}{2}}\,dx = e^{-\frac{\sigma p^2}{2}}\frac{1}{\sqrt{2\pi}}\int_{\operatorname{im} y=\sigma p} e^{-\frac{y^2}{2\sigma}} = \ldots$$

$$\sqrt{2\pi\cdot\sigma}$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{\sigma p^2}{2}}$$

i.e. 

Comment How would you measure velocity?

$\left(\underset{ma}{\ddot{x}} = \underset{F}{-x} \Rightarrow x\left(\frac{\pi}{2}\right) = \dot{x}(0)\right.$ check this!)

The Heisenberg uncertainty principle!

on same grounds, expect $\tilde{\varphi}(x) = -x$ (Follows from inversion formula)

6. Props of Fourier

1. $f(x) = g(x-x_0) \Rightarrow \tilde{f}(p) = e^{-ipx_0}\tilde{g}$
2. $f(x) = e^{ip_0x}g(x) \Rightarrow \tilde{f}(p) = \tilde{g}(p-p_0)$

$\Rightarrow$ used in proving inversion formula

3. $\widetilde{f'}(p) = ip\tilde{f}(p)$
4. $\widetilde{xf(x)} = i\frac{d}{dp}\tilde{f}$

used in diff'l Q's.

5. $\widetilde{f*g} = \sqrt{2\pi}\tilde{f}\tilde{g}$
6. $\widetilde{fg} = \frac{1}{\sqrt{2\pi}}\tilde{f}*\tilde{g}$

used below

Example: compute $\frac{\partial}{\partial T}\psi(T,x)\Big|_{T=0}$.

Math 117, Feb 9 1994

Ref: Feynman & Hibbs
Any analysis book
comment on Fourier series.

1. The semigroup.

$$(U_T\psi)(x) = \psi\left(\begin{matrix}\text{P.I. with } T \\ \text{replacing } \pi/2\end{matrix} \ \&\ \psi \text{ replacing } \psi_0\right) \quad \left(\begin{matrix}\text{computable} \\ \text{in} \\ \text{our case}\end{matrix}\right)$$

!

claim $\quad U_{T_2} U_{T_1} \psi = U_{T_1+T_2}\psi$

Topological QFT.

2. $\psi(T,x) = (U_T\psi)(x)$; what is $\frac{\partial}{\partial T}\psi(T,x)$?

First, at $T=0$

$$\psi(\epsilon,x) \sim e^{-i\epsilon V(x)}\int dy\, \psi_0(y)\, e^{\frac{i(x-y)^2}{2\epsilon}} \sim e^{-i\epsilon V(x)}\, \psi_0 * F_{i\epsilon}$$

$$\sim e^{-i\epsilon V(x)} F^{-1}\left(\tilde\psi_0 \cdot F_{\frac{1}{i\epsilon}}\right) \sim e^{-i\epsilon V(x)} F^{-1}\left(\tilde\psi_0 \cdot e^{-\frac{i\epsilon p^2}{2}}\right)$$

$$\sim (1 - i\epsilon V(x))\, F^{-1}\left(\tilde\psi_0\left(1 - \frac{i\epsilon p^2}{2}\right)\right)$$

$$\sim \tilde\psi_0(x) - i\epsilon V(x)\psi_0(x) + i\frac{\epsilon}{2}\frac{\partial^2}{\partial x^2}\psi$$

$$\Rightarrow \frac{\partial \psi}{\partial T} = -iH\psi \quad ; \quad H\psi = \left(-\frac{1}{2}\frac{\partial^2}{\partial x^2} + V\right)\psi$$

"Schrödinger's equation"

Reading purpose

Math 22b, Apr 27 1992

The principle of least action $\dot{\square} = \frac{d}{dt}\square$

minimize $S(q) = \int_a^b \left(\frac{1}{2} m \dot{q}^2 - V(q(t))\right) dt$

use $q \mapsto q + \delta q$, assume δq is such that $\delta q^2 \simeq 0$

These are Newton's equations!

The Relativistic case. Remember $\left\langle \begin{pmatrix} t_1 \\ x_1 \\ y_1 \\ z_1 \end{pmatrix}, \begin{pmatrix} t_2 \\ x_2 \\ y_2 \\ z_2 \end{pmatrix} \right\rangle = t_1 t_2 - x_1 x_2 - y_1 y_2 - z_1 z_2$

minimize $S(q) = \int_a^b \frac{1}{2} m \|q'\|^2 ds - eA$

A is the vector potential

$\square' = \frac{d}{ds}\square$

$S(q+\delta q) = S(q) + \int_a^b m \langle q', \delta q' \rangle ds - e \int_S dA =$

$= S(q) + \int_a^b -m \langle q'', \delta q \rangle - e \int_S dA$

$\Rightarrow \forall \delta q, \quad -m \langle q'', \delta q \rangle - e\, dA(\delta q, q') = 0$

might as well assume

$dA = E_x dx \wedge dt + E_y dy \wedge dt + E_z dz \wedge dt$
$+ B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy$

$q' = \begin{pmatrix} 1 \\ q'_x \\ q'_y \\ q'_z \end{pmatrix} \quad \delta q = \begin{pmatrix} 0 \\ \delta q_x \\ \delta q_y \\ \delta q_z \end{pmatrix}$

Get:

$$0 = m q''_x \delta q_x + \ldots - e \begin{pmatrix} E_x \delta q_x + E_y \delta q_y + E_z \delta q_z + \\ B_x (\delta q_y q'_z - \delta q_z q'_y) + B_y (\delta q_z q'_x - \delta q_x q'_z) + \ldots \end{pmatrix}$$

$\Rightarrow m q'' = e(E + q' \times B)$!

Math 22b, Apr 29 1992.

Remind of P.L.A, $\int_a^b \frac{1}{2} m\|q'\|^2 ds - eA$, $dA = \begin{matrix} E_x\, dx\wedge dt + \dots \\ B_x\, dy\wedge dz + \dots \end{matrix}$

$$\Rightarrow mq'' = e(E + q' \times B)$$

Two probs: 1. $\int w\wedge d\sigma = -(-1)^{\deg w}\int (dw)\wedge\sigma$ if w,σ vanish away from sight

2. $w\wedge *\sigma = \sigma \wedge * w$ if w,σ are of equal degs

E&M is $\underset{\text{1-form}}{S(A)} = \int_{\mathbb{R}^4} \frac12 (dA\wedge *dA) + \underset{\text{Three form}}{J\wedge A}$

varying

this is better

orient like $t\ x\ y\ z$

$$0 = \int d\delta A\wedge *dA + J\wedge\delta A = \int (*dA)\wedge d\delta A + J\wedge\delta A = \int -d{*}dA\wedge\delta A + J\wedge\delta A = \int (J - d{*}dA)\wedge\delta A$$

$\Rightarrow d{*}dA = J$ write $dA = F$, eqn's are

$$\left.\begin{matrix} dF = 0 \\ d{*}F = J \end{matrix}\right\}$$ Maxwell's equations!

$F = E_x\, dx\wedge dt + \dots + B_x\, dy\wedge dz + \dots$ $\qquad J = \rho\, dx\wedge dy\wedge dz - j_x\, dy\wedge dz\wedge dt - \dots$

$= \lambda_E\wedge dt + *^{(3)}\lambda_B$

$dF = \begin{cases} dx\wedge dy\wedge dz: & d^{(3)} *^{(3)} \lambda_B = 0 \Rightarrow \operatorname{div} B = 0 \\ dy\wedge dz\wedge dt: & \partial_y E_z - \partial_z E_y + \partial_t B_x = 0 \end{cases}$

$\Rightarrow$ no magnetic monopoles.

$\operatorname{curl} E = -\frac{\partial B}{\partial t} \Rightarrow$ Varying magnetic field creates a circulation of the Electric field

$$*F = -\lambda_B\wedge dt + *^{(3)}\lambda_E$$

this is where Lorentz comes in

$d{*}F = J \Rightarrow \begin{cases} \operatorname{div} E = \rho \\ \operatorname{curl} B = \frac{\partial E}{\partial t} + j \end{cases}$

The flux of E leaving ... = charge ins

The famous Maxwell term!

1. understand how Lorentz trans. act
2. know how to combine with G.R.
3. More elegant $\Rightarrow$ better apps in future

This indeed holds!

1. Alt source for QM: Sakurai Modern QM

Math 117, Feb 11 1994

Poincaré's Lemma

Exterior algebra ; diff forms ; d ; $d^2 = 0$; ~~d~~ Leibnitz's rule.

The $*$ operator by

$$\omega \wedge * \sigma = \langle \omega, \sigma \rangle \, dx^{\{1 \cdots n\}}$$

Example:

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d} \Omega^3$$

$\{f\}$ $\{\lambda_F\}$ $\{*\lambda_F\}$ $\{*f\}$

Two probs: ... continue as in Math 22b Apr 29 1992

(Continued on Feb 14 1994)

Math 117 – HW assignment #2

Feb 14 1994
due Feb 21 1994

1. Re-derive Maxwell's equations from the action principle, only this time using the correct space time metric

$$\left\|\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}\right\|^2 = t^2 - x^2 - y^2 - z^2$$

2. Figure out how a Lorentz transformation, such as

$$\begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \cosh\mu & \sinh\mu & 0 & 0 \\ \sinh\mu & \cosh\mu & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

acts on the electric and magnetic fields E & B.

3. Express the action S in terms of E & B.

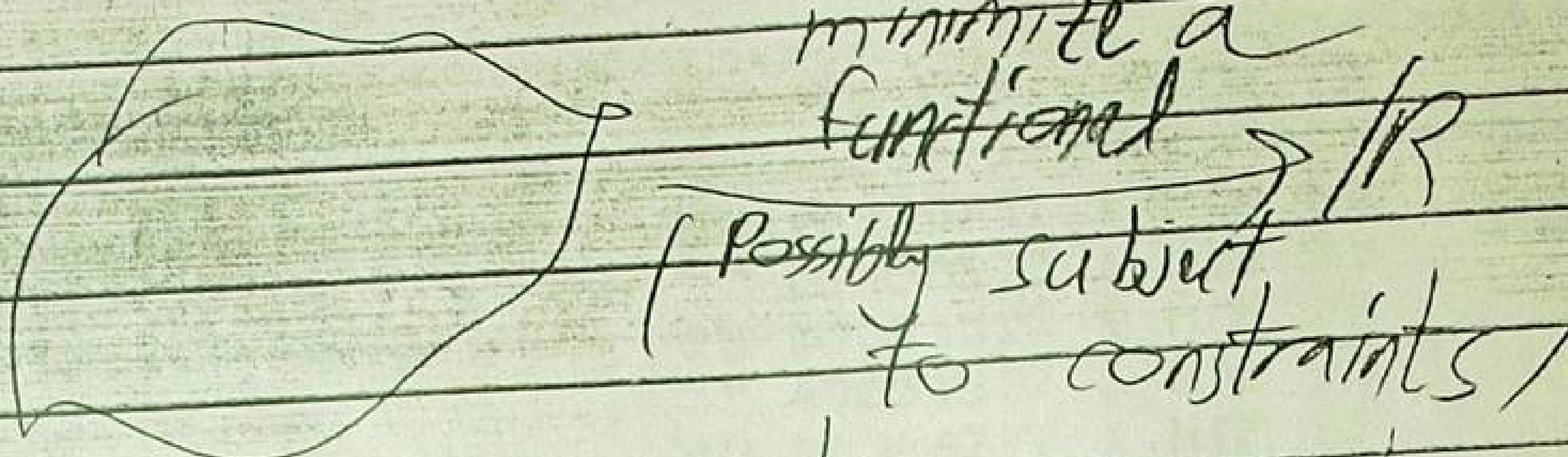
4. Solve the free Schrödinger equation

$$i\frac{\partial\psi}{\partial t} = H\psi \; ; \; H\psi = -\frac{1}{2}\frac{\partial^2\psi}{\partial x^2} \; ; \; \psi(0,x) = e^{ipx}$$

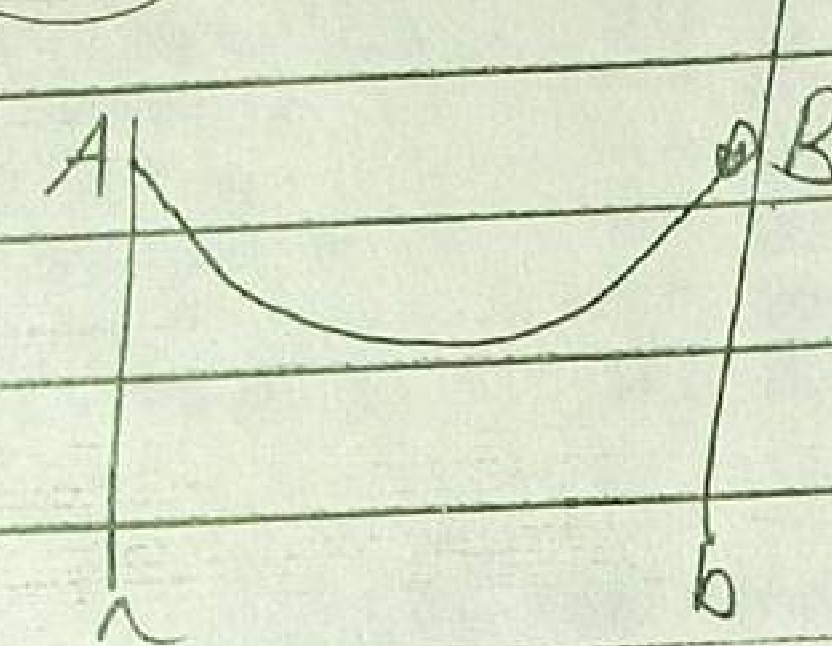
and explain your results in physical terms.

Math 115, Nov 27 1991.

I have an extra book!



Ex.



$F = \int_a^b m y \sqrt{1+y'^2}\,dx$

constraint: $l = \int dx \sqrt{1+y'^2}$

bndry: $y(a)=A \quad y(b)=B$

derive Euler, Lagrange: $F_y - \frac{d}{dx} F_{y'} = 0$
(by adding ϵh, $h(a)=h(b)=0$)

1. F indep. of y $\quad F_{y'} = \text{const}$
" " y' $\quad F_y = 0$

F indep of x:

$$\boxed{\begin{aligned} &0 = F_y - (F_{y'})' = F_y - F_{y'y}y' - F_{y'y'}y'' = 0 \quad / \cdot y' \\ &\Rightarrow \quad y'F_y - F_{y'y}y'^2 - F_{y'y'}y''y' = 0 \\ &\Rightarrow \frac{d}{dx}(F - y'F_{y'}) = 0 \Rightarrow F - y'F_{y'} = \text{Const.} \end{aligned}}$$

In our case $y\sqrt{1+y'^2} - y'\dfrac{yy'}{\sqrt{1+y'^2}} = C$

$\Rightarrow y'^2 = \dfrac{y^2 - C^2}{C^2} \Rightarrow y = C\cosh\dfrac{x-C'}{C}$

what's wrong here?

HW: read 1-4 Do $\square$ explicitly, 1.11, 15b, d

Math 117, Feb 16 1994

Reminder: min $S(A) = \frac{1}{2}\int |dA|^2 + J \wedge A$; got Maxwell's eqn's.

~~Learned~~ $dJ = 0 \Rightarrow \frac{\partial \rho}{\partial t} = \operatorname{div} j$
(charge conservation)

Advantages:
1. understand how Lorentz's trans. act
2. Know how to Combine w/ G.R.
3. More elegant ⇒ More advantages; easier to generalize, ...

Then follow math 115, Nov 27 1991

Math 115, Dec 2 1991.

Return Exams: 80^+ A 60^+ B 40^+ C

Av: 74

Review: $J(y) = \int_a^b F(x,y,y')\,dx \quad y(a)=A;\ y(b)=B$

$$F_y - \frac{d}{dx}F_{y'} = 0$$ (Euler Lagrange)

HW: $F = y'$, $F = xyy'$, $F = (y')^2/x^3$

Power lines: $F = y\sqrt{1+y'^2}$ (EL is too hard)

If F is indep of x: $0 = F_y - F_{y'y}y' - F_{y'y'}y''$ $\quad /\cdot y'$

$$y'F_y - F_{y'y}y'^2 - F_{y'y'}y''y' = 0$$

$\square$

$$(F - y'F_{y'})' = 0 \Rightarrow F - y'F_{y'} = C_1$$

$$\frac{dy}{dx} = g(y) \Rightarrow \frac{dy}{g(y)} = dx \quad \int \frac{dy}{g(y)} = x + C_2$$

Our case: $y\sqrt{1+y'^2} - y'\frac{yy'}{\sqrt{1-y'^2}} = C_1$

$$y\frac{1}{\sqrt{1-y'^2}} = C_1 \qquad \sqrt{1-y^2} = y/C_1$$

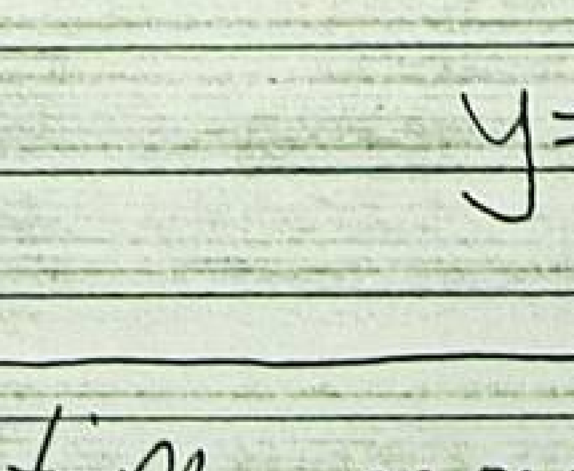
$$y' = \sqrt{1 - y^2/C_1^2}$$

$$C_1\int\frac{d\frac{y}{C_1}}{\sqrt{1-\left(\frac{y}{C_1}\right)^2}} = x + C_2 \Rightarrow C_1\cdot\cosh^{-1}\frac{y}{C_1} = x + C_2$$

$$y = C_1\cosh\frac{x+C_2}{C_1}$$

If time, generalities about gradients & Free ends.

The Brachistochrone:



$\frac{1}{2}mv^2 = gy$

$ds = v\,dt$

$ds = \sqrt{dx^2 + dy^2}$

$\int dt = \ldots$

HW: redo $\square$ for 1. $F = y\sqrt{1+y'^2}$

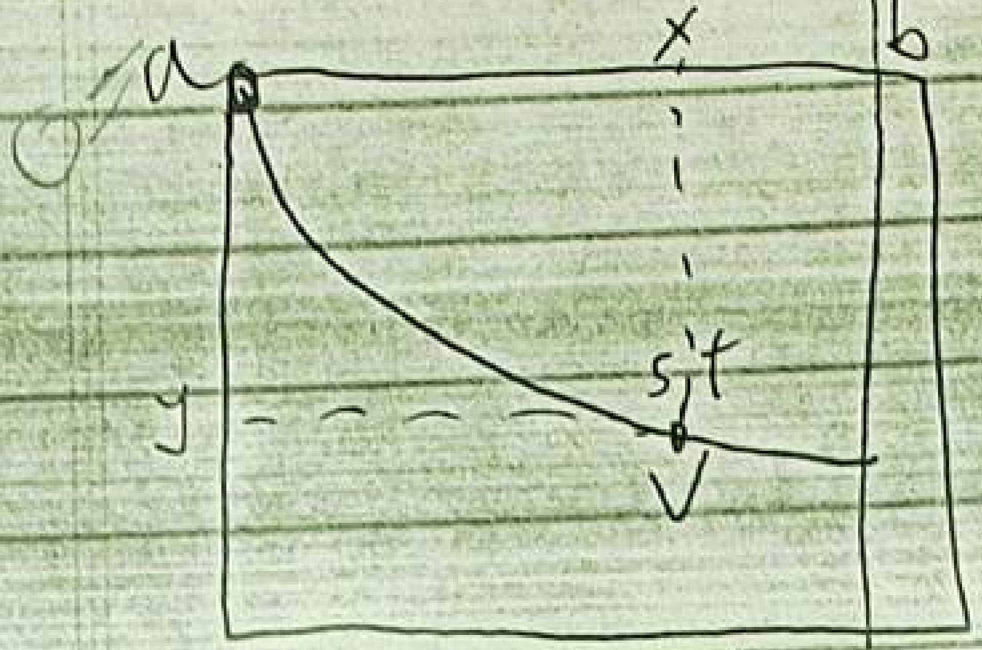
& 2. $F = \sqrt{\frac{1+y'^2}{y}}$

*Solve EL for 2.

do 15a,b,c or 20 if time permits.

Math 115, Dec 4 1991

The Brachistochrone:



$\frac{1}{2}mv^2 = gy$

$ds = v\,dt$

$ds = \sqrt{dx^2+dy^2} = \sqrt{1+y'^2}\,dx$

$\int dt \ldots$

$$J(y) = \int \sqrt{\frac{1+y'^2}{y}}\,dx$$

Conditions: $y(0)=0$

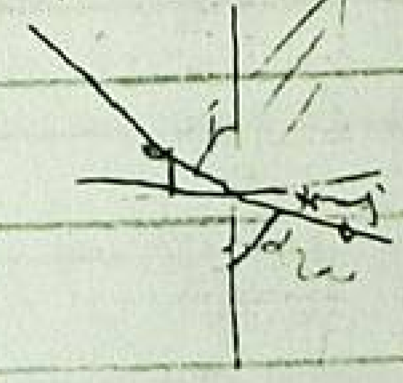
$F_{y'}(b) = 0$

$F - y'F_{y'} = C_1^{-1/2}$

derive by first doing a finite dim analog.

$$F - y'F_{y'} = \sqrt{\frac{1+y'^2}{y}} - y'\frac{y'}{\sqrt{(1+y'^2)y}} = \frac{1}{\sqrt{y(1+y'^2)}} = C_1^{-\frac{1}{2}}$$

Snell's law:



$\frac{V_1}{\sin\alpha_1} = \frac{V_2}{\sin\alpha_2}$

$\frac{V}{\sin\alpha} = \text{const}$

$V = \sqrt{y} \quad \sin\alpha = \frac{1}{\sqrt{1+y'^2}}$

$y(1+y'^2) = C_1$

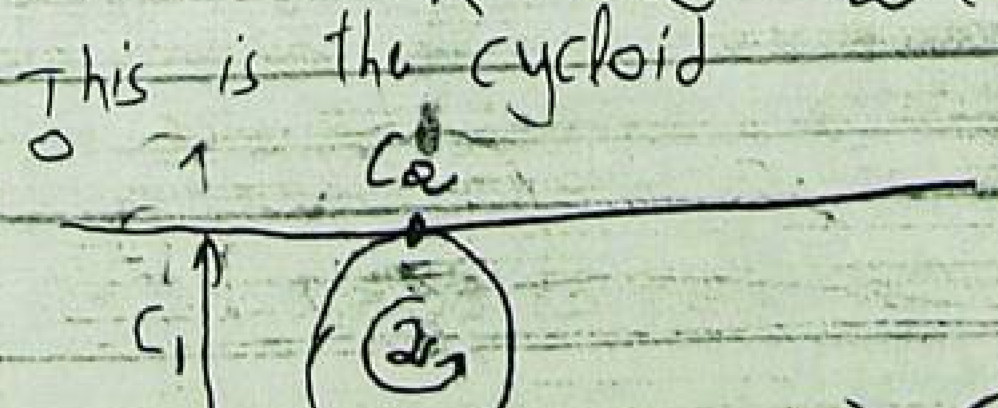
$y' = \sqrt{\frac{c_1}{y} - 1} \qquad \frac{dy}{\sqrt{\frac{c_1}{y}-1}} = dx$

$x - c_2 = \int \underbrace{\sqrt{\frac{y}{c_1 - y}}}_{tg\, t}\, dy$ ... in principle soluble, in practice hard

trick: $\Rightarrow y = c_1 \sin^2 t = \frac{c_1}{2}(1 - \cos 2t) \qquad dy = \ldots$

$x = c_2 + \frac{c_1}{2}(2t - \sin 2t)$

This is the cycloid



center $= \begin{pmatrix} c_2 + c_1/2 \\ c_1/2 \end{pmatrix}$

disp $= \frac{c_1}{2}\begin{pmatrix} -\sin 2t \\ -\cos 2t \end{pmatrix}$

$F_{y'} = 0 \Rightarrow y' = 0$!

Problem solved !

$\Rightarrow C_2 = 0$; at b $2t = \pi$; $b = \frac{c_1}{2}\pi$

HW: 1. complete details
2. read 3. Do 18, 20, ...

```
Mathematica 2.2 for SPARC
Copyright 1988-93 Wolfram Research, Inc.
 -- Open Look graphics initialized --
 -- Local version !! --

In[1]:= << Calculus'VariationalMethods'

In[2]:= << Calculus'DSolve'

In[3]:= F=Sqrt[(1+y'[x]^2)/y[x]]

                          2
               1 + y'[x]
Out[3]= Sqrt[----------]
                 y[x]

In[4]:= EulerEquations[F,y[x],x]

                      2
         -(1 + y'[x]  + 2 y[x] y''[x])
Out[4]= ------------------------------ == 0
                                2
                 3   1 + y'[x]   3/2
           2 y[x]   (----------)
                        y[x]

In[5]:= DSolve[%,y,x]

Solve::tdep: The equations appear to involve transcendental functions of the
     variables in an essentially non-algebraic way.

                             1 - y C[1]
                        Sqrt[----------] (-1 + 2 y C[1])
                                 y
                ArcTan[--------------------------------]
                          2 Sqrt[C[1]] (-1 + y C[1])
Out[5]= {Solve[---------------------------------------- +
                                 2 C[1]

                  1 - y C[1]
          y Sqrt[----------]
                      y
>         ------------------ == C[2] + #1, y],
              Sqrt[C[1]]

                          1 - y C[1]
                     Sqrt[----------] (-1 + 2 y C[1])
                              y                             1 - y C[1]
            -ArcTan[--------------------------------]  y Sqrt[----------]
                       2 Sqrt[C[1]] (-1 + y C[1])                 y
>    Solve[----------------------------------------- - ------------------ ==
                            2 C[1]                        Sqrt[C[1]]

>      C[2] + #1, y]}

In[6]:= FirstIntegrals[F,y[x],x]

                          1
Out[6]= {-(---------------------)}
                               2
                      1 + y'[x]
             y[x] Sqrt[----------]
                         y[x]

In[7]:= DSolve[First[%]==c1,y,x]
```

```
Solve::tdep: The equations appear to involve transcendental functions of the
     variables in an essentially non-algebraic way.

                                                          2
                                              2 c1 (-1 + c1  y)
                                      ArcTan[------------------------------]
                            2                              2
                     1 - c1  y                      1 - c1  y             2
              y Sqrt[---------]               Sqrt[---------] (-1 + 2 c1  y)
                         y                              y
Out[7]= {Solve[----------------- - -------------------------------------- -
                      c1                                    2
                                                       2 c1

                                        2
                                 1 - c1  y
                          y Sqrt[---------]
                                     y
>       #1 == C[1], y], Solve[-(-----------------) +
                                      c1

                                  2
                     2 c1 (-1 + c1  y)
        ArcTan[------------------------------]
                          2
                   1 - c1  y               2
              Sqrt[---------] (-1 + 2 c1  y)
                       y
>       ------------------------------------ - #1 == C[1], y]}
                          2
                      2 c1

In[8]:= Integrate[Sqrt[y/(c1-y)],y]

                                                          y
                                     (c1 - 2 y) Sqrt[-(-------)]
                                                       -c1 + y
                            c1 ArcTan[----------------------------]
                    y                            2 y
Out[8]= Sqrt[-(-------)] (-c1 + y) - ---------------------------------------
               -c1 + y                               2
```

Math 117, Feb 18 1994

1. Correct last class's example.

2. Do math 115, Dec 4 1991

MATH 117 EVALUATION name: (optional) ____________________ date: __________

What did you think of class today? (fast, slow, clear, messy, boring, amazing, too rigorous, too vague, ...)

Did you feel that you learned something new today?

How is it going in general?

MATH 117 EVALUATION name: (optional) ____________________ date: __________

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MATH 117 EVALUATION name: (optional) ____________________ date: __________

What did you think of class today? (fast, slow, clear, messy, boring, amazing, too rigorous, too vague, ...)

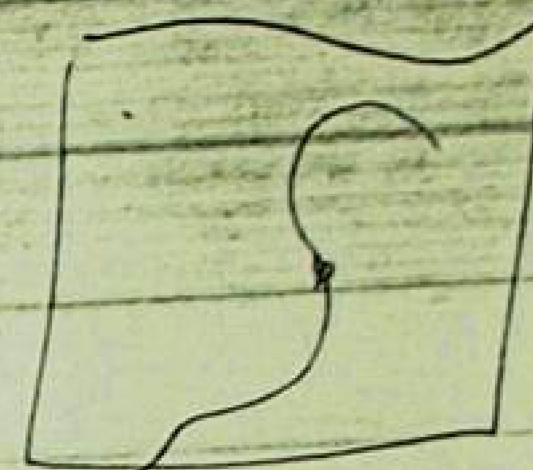
Did you feel that you learned something new today?

How is it going in general?

Math 115, Dec 6 1991

The isoperimetric ineaquallity

"Among all domains with boundary length l the disk has the most area."



Lagrange multipliers: maximize $f(x,y)$ under $g(x,y)=0$

stupid way;

smart way $h_\lambda(x,y) = f(x,y) + \lambda g(x,y)$ $\begin{cases} \nabla h_\lambda = 0 \\ g(x,y) = 0 \end{cases}$

Example: Find the point nearest to the origin on the curve $x^2+xy+y^2=1$

$h_\lambda = x^2+y^2+\lambda(x^2+xy+y^2-1)$

$\partial_x h_\lambda = 2x+2\lambda x+y=0$

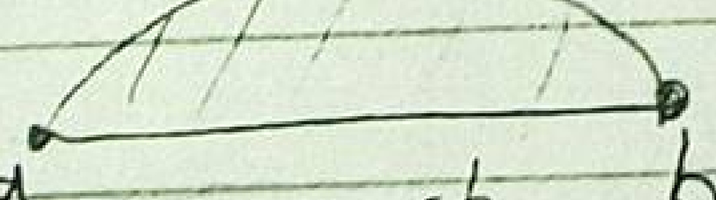
$\partial_y h_\lambda = 2y+2\lambda y+x=0$

$x^2+xy+y^2=1$

$y=-2(1+\lambda)x$ $\quad y=x$ $\quad 3x^2=1$

$x=-2(1+\lambda)y$ $\quad y=-x$ $\quad x^2=1$

Example



$y(a)=0$

$y(b)=0$

$J=\int_a^b y\,dx \qquad G=\int_a^b \sqrt{1+y'^2}\,dx = l$

$J+\lambda G = \int_a^b (y+\lambda\sqrt{1+y'^2})\,dx \qquad F_\lambda = y+\lambda\sqrt{1+y'^2}$

Rare case of Euler-Lagrange is simpler than its simplification:

$0=F_y-\frac{d}{dx}F_{y'} = 1-\lambda\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} \Rightarrow \frac{\lambda y'}{\sqrt{1+y'^2}} = x-c_1$

solve for y', get $y'=\frac{x-c_1}{\sqrt{\lambda^2-(x-c_1)^2}} \Rightarrow y-c_2 = -\sqrt{\lambda^2-(x-c_1)^2}$

$\Rightarrow (x-c_1)^2+(y-c_2)^2=\lambda^2$

! ! !

HW. 2.17–19; 22

Math 117, Feb 23 1994

More examples:

<u>Thm</u> $S(y) = \int_a^b F(x,y,y')dx \quad C(y) = \int_a^b G(x,y,y') \quad y(a) = A \quad y(b) = B$

if y is an extremum of $S(y)$ subject to $C(y) = l$, and y is not an extremal of $C(y)$, then there exists a constant λ s.t. y is an extremal of $S + \lambda C$.

<u>Proof in 2D</u> (gradients are proportional)

<u>Example</u> Isoperimetric inequality as in Math 115, Dec 6 1991

<u>Proof</u> 1. Define $\frac{\delta S}{\delta y} = F_y - \frac{d}{dx} F_{y'}$, "variational derivative", h property

2. Find $x_0 \in [a,b]$ s.t. $\frac{\delta C}{\delta y}(x_0) \neq 0$ &

st $\lambda = -\frac{\delta S}{\delta y}(x_0) \Big/ \frac{\delta C}{\delta y}(x_0)$

3. Let $x_1 \in [a,b]$, st $h = h_{x_1} - \left(\frac{\delta C}{\delta y}(x_1) \Big/ \frac{\delta C}{\delta y}(x_0)\right) \cdot h_{x_0}$

$$C(y + \epsilon h) = C(y) + o(\epsilon)$$

$$\Rightarrow S(y+\epsilon h) - S(y) = o(\epsilon)$$

$$\Rightarrow \frac{\delta S}{\delta y}(x_1) - \frac{\delta C}{\delta y}(x_1) \Big/ \frac{\delta C}{\delta y}(x_0) \, \frac{\delta S}{\delta y}(x_0) \Rightarrow \ldots$$

Details in text.

HW: (due Feb 28) 1.14, 15 b&d, 20; 2.17, 19, 22; complete isoperimetric

Requiring the action to be stationary leads to the generalized Euler-Lagrange equations

copied from Itzykson-Zuber

$$\frac{\delta I}{\delta \varphi_i(x)} \equiv \frac{\partial \mathscr{L}(x)}{\partial \varphi_i(x)} - \partial_\mu \frac{\partial \mathscr{L}(x)}{\partial[\partial_\mu \varphi_i(x)]} = 0 \tag{1-44}$$

Requiring the action to be stationary leads to the generalized Euler-Lagrange equations

copied from Itzykson-Zuber

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Requiring the action to be stationary leads to the generalized Euler-Lagrange equations

copied from Itzykson-Zuber

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Math 115, Dec 9 1991

Five more classes, Five more topics:
1. other types
2. Second derivatives
3. Hammiltonian formulation
4. Symmetries
5. "integral calculus"

The Hammiltonian formulation

motivation: $L = \int \left(\frac{1}{2}mv^2 - V(q)\right)dt$ $q = q(t)$, $V = \dot{q}$, describing a particle in a pot. field.

$L \leftrightarrow J$, $v \leftrightarrow y'$, $L \leftrightarrow F$, $q \leftrightarrow y$, $t \leftrightarrow x$

(Dict: $L \leftrightarrow J$, $y \leftrightarrow q$, $x \leftrightarrow t$)

E-L: $F_q \frac{d}{dt}(F_{\dot{q}}) = 0 \Rightarrow m\ddot{q} = -V'(q)$ ← Newton's law

maybe we should look at E-L as an initial value problem rather than a boundary value problem.

$$F_{q_i} - \frac{d}{dt}F_{\dot{q}_i} \quad \begin{cases} q_i(0) = q_{0i} \\ \dot{q}_i(0) = V_{0i} \end{cases}$$ a 2nd order ODE w/ initial cond.

in higher dims just add subscripts & natural

Theorem: In the variables q_i, $p_i \triangleq F_{\dot{q}_i}$ $(= m\dot{q} = \text{momentum})$ the E-L eqns are equivalent to the Hamilton equations:

$$\dot{q} = \frac{\partial H}{\partial p} \quad \dot{p} = -\frac{\partial H}{\partial q}$$ where the "hamiltonian" H is

$$H(q_i, p_i) \triangleq \sum \dot{q}_i p_i - F \quad \left(= \frac{1}{2}mv^2 + V(q) = \text{total energy}\right)$$

$\frac{p^2}{2m} + V(q)$

Proof: $p_i = F_{\dot{q}_i}$ $H = \sum p_i \dot{q}_i - F$

compute dH in two ways. $F_{q_i} - \frac{d}{dt}F_{\dot{q}_i} = 0$; consider $dq_i, d\dot{q}_i, dp_i, dH, dF$

$$\sum \frac{\partial H}{\partial p_i}dp_i + \sum \frac{\partial H}{\partial q_i}dq_i = dH = \cancel{\sum p_i d\dot{q}_i} + \sum \dot{q}_i dp_i - \sum \frac{\partial F}{\partial q_i}dq_i - \cancel{\sum \frac{\partial F}{\partial \dot{q}_i}d\dot{q}_i}$$

$$= \sum \dot{q}_i dp_i - \sum \dot{p}_i dq_i \qquad \text{Q.E.D}$$

Definition: P.B. (of fncts of p, q):

$$\{F, g\} = \sum \left(\frac{\partial F}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

claim:
1. $\frac{\partial F}{\partial t} = 0 \Rightarrow \frac{dF}{dt} = \{F, H\}$
2. $\{H, H\} = 0$ ("H" is an integral of motion")
3. $\frac{\partial F}{\partial t} = 0 \Rightarrow$ conservation of energy ; (we actually know that already)

HW: Read 16 | if P.B. is done: Do 4.1, 2 & Read 17.
Do everything explicitly for the harmonic oscillator $F = \frac{1}{2}m\dot{q}^2 - \frac{1}{2}kq^2$

Math 117, Feb 24 1994

Additional topics:

1. PP 16 - Sol w/ no 2nd order der.
2. PP 22 - Several var's
3. PP 24 - Minimal area surfaces
4. PP 29 - change of vars
5. PP 34 - Several unknow funtions
6. PP. 37 - Geodesics (as an example of a prob w/ several unknown functions)
7. PP 39 - Parametric form
8. PP. 41 - higher order derivatives.
9. PP. 46 - Finite subsidiary conditions.
10 PP. 49 - Geodesics on a sphere.
11. PP. 59 - End points on a curve
12. PP. 61 - Broken extremals.

plan: 1. topics

Math 115, Dec 11 1991.

Conservation laws

Review Hamilton's equations

Definition: P.B:

$$\{f,g\} = \sum \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

claim: 1. $\frac{\partial F}{\partial t} = 0 \Rightarrow \frac{dF}{dt} = \{F, H\}$

2. $\{H, H\} = 0$

3. $\frac{\partial E}{\partial t} = 0 \Rightarrow$ conservation of energy, energy is an integral of motion } we already know that

4. $\{q, p\} = 1$ This the beginning of QM!

5. anything that can be said about classical mechanics can be said using P.B. $\Rightarrow$ symplectic geometry

Noether's theorem:

1. If F is invariant under time trans E is conserved

2. If F is invariant under coordinate momentum is conserved — know but didn't prove yet

3. If F is inv. under rotations, angular momentum is conserved

Let us ~ things which are functions,

Nother's thm If $J(\tilde{q}_i) = \int_{t_0}^{t_1} F(t, q_i, \dot{q}_i)\,dt$ is invariant under

$$t^* = T(t, q_i, \dot{q}_i, \epsilon) \; ; \; q_i^* = Q(t, q_i, \dot{q}_i, \epsilon)$$ (Namely, ..

where $T_\epsilon = t$ ..

then

$$\left(\sum p_i \frac{\partial Q_i}{\partial \epsilon} - H\frac{\partial T}{\partial \epsilon} \right)\Big|_{\epsilon=0}$$

is conserved (Namely ...)

Example: 1. $T = t+\epsilon$, $Q = q \Rightarrow H$ is conserved if F is t indep.

2. $T = t$, $Q = q+\epsilon \Rightarrow p$ is conserved if F is q indep.

HW: Read 17, Noether: 13, 20 Do 4.1, 2, 3 (Noether: 5)

Math 117, Feb 28, 1994

$F = F(t, q_i, \dot q_i)$:

<u>Thm</u> In the variables $q_i, p_i \triangleq F_{\dot q_i}$ (= "the momentum conjugate to q_i") E-L are equiv to

$$\dot q = \frac{\partial H}{\partial p} \quad \dot p = -\frac{\partial H}{\partial q} \quad \text{where } H(q_i, p_i) = \sum \dot q_i p_i - F$$

<u>Proof</u> $\sum \left(\frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial q_i} dq_i\right) = dH = \sum \dot q_i dp_i + p_i d\dot q_i - \frac{\partial F}{\partial q_i} dq_i - \frac{\partial F}{\partial \dot q} d\dot q_i$

$$\begin{tikzcd}
\mathbb{R}^{2n}_{q,\dot q} \arrow[r, shift left, "p = F_{\dot q}"] \arrow[dr, dashed, "F"'] & \mathbb{R}^{3n}_{q,\dot q,p} \arrow[l, shift left] \arrow[r, shift right] \arrow[d, "F\ H"] & \mathbb{R}^{2n}_{q,p} \arrow[l, shift right, "p = F_{\dot q}"'] \arrow[dl, "H"] \\
& \mathbb{R} &
\end{tikzcd}$$

<u>Def</u> observable

<u>Def</u> P.B. of observables of p & q:

$$\{f, g\} = \sum \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

<u>Claim</u> All you ever wanted to know about a mechanical system can be recovered from H & $\{\,,\}$

1. $\frac{dF}{dt} = \{F, H\}$ 2. $\{H, H\} = 0$ 3. $\frac{\partial F}{\partial t} = 0 \Rightarrow$ cons. of energy (have we seen that before?-)
4. $\{q_i, p_j\} = \delta_{ij}$

All of QM: replace obs. by matrices (operators) & P.B by commutators

$$[Q, P] = i\hbar I$$

The Von-Neumann: $\Rightarrow \mathcal{H} = L^2(\mathbb{R}_x)$, $Q = x$, $P = -i\hbar\frac{\partial}{\partial x}$

$H = -\hbar^2 \frac{\partial^2}{\partial x} + V(x)$

$i\hbar\frac{dF}{dt} = [F, H] \Rightarrow F_t = e^{-i\hbar t H} F_0 e^{i\hbar t H} \Rightarrow \psi_t \longrightarrow e^{-i\hbar t H}\psi_0$ HW: 4.1-3.

$\Rightarrow \frac{d\psi_t}{dt} = -i\hbar H \psi_t \Rightarrow$ Schrödinger's eqn

Math 117, March 2 1994

$\boxed{\text{Discuss grading Policy}}$

Symplectic Geometry:

ω - closed Non-degenerate 2 form on M

examples: 1. $\sum dp_i \wedge dq_i$

2. Any form on S^2, say $x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy$

$f \mapsto X_f$ by $i_{X_f}\omega = -df$

do 2 examples.

$\{f,g\} = \omega(X_f, X_g)\ \left(= -df(X_g) = -X_g f = X_f g\right)$

do example 1.

Thm 1. $\{f,g\}$ has all the properties of prev class

2. $X_{\{f,g\}} = [X_f, X_g]$

3. The flow generated by X_f preserves ω & $\omega^n/n!$

4. if $L_X\omega = 0$, almost $\exists f$ s.t $X = X_f$.

Math 117, March 4 1994. (Announce review session)

Continue symplectic geometry: $i_{X_f} = -df$

$\{f, g\} = X_f g$

example: $f = z$ on S^2.

Thm 1. a $\{,\}$ is bil

b. $\{,\}$ is as.

c. $\{,\}$ sat. Leib.

d. $\{,\}$ sat. Jacobi

2. a. flow generated by X_f preserves ω & $\omega^n/n!$

b. $L_X\omega = 0 \Rightarrow$ almost $\exists f$ s.t. $X = X_f$

3 $X_{\{f,g\}} = [X_f, X_g]$

---

computations:

$$-dz = i_{\alpha\partial_x+\beta\partial_y+\gamma\partial_z}(x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy) = \begin{matrix}(\gamma y-\beta z)dx\\(\alpha z-\gamma x)dy\\(\beta x-\alpha y)dz\end{matrix} = \begin{matrix}-xz\,dx\\-yz\,dy\\(1-z^2)dz\end{matrix}$$

V & $V\cdot p = 0$

claim: $\begin{pmatrix}\alpha\\\beta\\\gamma\end{pmatrix} = \begin{pmatrix}+y\\-x\\0\end{pmatrix}$

$g\} = [X_f, X_g]$ $\quad i_{X_f}\omega = -df;\ i_{X_g}\omega = -dg$

$$i_{[X_f,X_g]}\omega \stackrel{?}{=} -d\{f,g\} = -d(\omega(X_f,X_g))$$

$$\omega([X_f,X_g],Z) \stackrel{?}{=} L_Z(\omega(X_f,X_g))$$

$$=(L_Z\omega)(X_f,X_g) + \omega([Z,X_f],X_g) + \omega(X_f,[Z,X_g])$$

$$=(di_Z\omega)(X_f,X_g) + [Z,X_f]g - [Z,X_g]f$$

$$= X_f\omega(Z,X_g) - X_g\omega(Z,X_f) - \omega(Z,[X_f,X_g]) + [Z,X_f]g - [Z,X_g]f$$

$$= X_f Z g - X_g Z f + [Z,X_f]g - [Z,X_g]f + \omega([X_f,X_g],Z) = ZX_f g - ZX_g f + \omega([X_f,X_g],Z)$$

Math 117, March 7 1994

1. Liouville's thm

2. Nöther's thm. sym $\Longleftrightarrow$ conservation laws.

a. $X = \frac{\partial}{\partial q_i}$

b. $X = q_1 \frac{\partial}{\partial q_2} - q_2 \frac{\partial}{\partial q_1} + p_1 \frac{\partial}{\partial p_2} - p_2 \frac{\partial}{\partial p_1}$

$$\Rightarrow i_X W = -q_1 dp_2 + q_2 dp_1 + p_1 dq_2 - p_2 dq_1$$
$$= d(q_2 p_1 - q_1 p_2)$$

3. Bracket of two v.f:
   1. def
   2. geom. interp.
   3. Lie der. interp.

4. Proof of $X_{\{f,g\}} = [X_f, X_g]$.

start w/ that

HW: 1. Verify that P.B. sat. Jac.

2. Compute P.B. of any pair of x, y, z

on S^2

~~Ho~~mework #2

MATH 117

① We have the basis dx, dy, dz, dt with $(dx,dx)=(dy,dy)=(dz,dz)=1$ and $(dt,dt)=-1$. We have

$$F = E_x\, dx\wedge dt + E_y\, dy\wedge dt + E_z\, dz\wedge dt +$$
$$B_x\, dy\wedge dz + B_y\, dz\wedge dx + B_z\, dx\wedge dy.$$

By linearity + orthogonality (verify if you wish), and skew symmetry of basis + $\wedge$,

$$(dx_1 dx_2, dx_3 dx_4) = (dx_1, dx_3)(dx_2, dx_4) - (dx_2, dx_3)(dx_1, dx_4).$$

Recall that "$(dA \wedge dB) = (*dA, dB)\, dx_1\wedge dx_2\wedge \dots \wedge dx_n$ for all dB" is the definition of $*dA$. (DROR MIGHT HAVE WRITTEN SOMETHING ELSE IN CLASS!)

This gives us $*(dx\wedge dt) = dy\wedge dz$ and cyclic permutations of x, y, z.
$*(dx\wedge dy) = -dz\wedge dt$ " " " " "

$dF = 0$ and $d(*F) = J = \rho\, dxdydz - j_x\, dydzdt - j_y\, dzdxdt - j_z\, dxdydt$, now give us the correct maxwell equations,

$\text{curl } E = -\frac{\partial B}{\partial t}$ (Faraday's Law of Induction)

$\text{curl } B = J + \frac{\partial E}{\partial t}$ (Ampère's Law)

$\text{div } E = \rho$ (Gauss's Law)

$\text{div } B = 0$ (nonexistence of magnetic monopole)

② First, notice that if $t' = (\cosh\mu)t + (\sinh\mu)x$ and $x' = (\sinh\mu)t + (\cosh\mu)x$ then $t'^2 - x'^2 - y'^2 - z'^2 = (\cosh^2\mu - \sinh^2\mu)t^2 - (\cosh^2\mu - \sinh^2\mu)x^2 - y^2 - z^2 = t^2 - x^2 - y^2 - z^2$.

Good, the metric is invariant, so <u>Maxwell's equations will still hold.</u>

Now, $dt' = \cosh\mu\, dt + \sinh\mu\, dx$ and $dx' = \sinh\mu\, dt + \cosh\mu\, dx$

Plug these into $F' = E_x\, dx'\wedge dt' + E_y\, dy\wedge dt' + E_z\, dz\wedge dt' + B_x\, dy\wedge dz + B_y\, dz\wedge dx' + B_z\, dx'\wedge dy$.

This gives us

$$\begin{cases} E_x' = E_x \\ E_y' = E_y\cosh\mu - B_z\sinh\mu \\ E_z' = E_z\cosh\mu + B_y\sinh\mu \end{cases} \qquad \begin{cases} B_x' = B_x \\ B_y' = B_y\cosh\mu + E_z\sinh\mu \\ B_z' = B_z\cosh\mu - E_y\sinh\mu \end{cases}$$

CAN YOU GIVE A PHYSICAL PICTURE?

$$S(A) = \int \tfrac{1}{2}\|dA\|^2 + J \wedge A$$

$$= \underbrace{\int \tfrac{1}{2}(dA \wedge * dA)}_{\text{by definition}} + \underbrace{\int d(*dA) \wedge A}_{\text{found from } d*F = J}$$

Now, $d(*dA) \wedge A = d(*dA \wedge A) - (-1)^{\deg *dA}(*dA \wedge dA)$

(this is like a product rule)

$$= -*dA \wedge dA = -dA \wedge *dA$$

The last step uses $\alpha \wedge *\beta = \beta \wedge *\alpha$, for any forms α, β.

Thus $S(A) = \int \frac{-1}{2} dA \wedge * dA = \frac{-1}{2}\int F \wedge *F = \ldots$

$$\ldots = \frac{1}{2}\int (E^2 - B^2)\, dt \wedge dx \wedge dy \wedge dz$$

4) $i\frac{\partial \psi}{\partial t} = H\psi \Rightarrow \frac{\partial \psi}{\partial t} = -iH\psi \Rightarrow \psi_{t,x} = e^{-itH}\psi_{0,x}$.

Now, $e^{-itH} = \sum_{j=0}^{\infty} \frac{\left[-it\left(\frac{-1}{2}\frac{\partial^2}{\partial x^2}\right)\right]^j}{j!} = \sum_{j=0}^{\infty}\frac{\left(\frac{it}{2}\frac{\partial^2}{\partial x^2}\right)^j}{j!}$.

$\frac{\partial^2}{\partial x^2}\psi_0 = \frac{\partial}{\partial x^2} e^{ipx} = ip \cdot ip\, e^{ipx} = -p^2\psi_0$.

So $e^{-itH}\psi_0 = \psi_0 \cdot \sum_{j=0}^{\infty}\frac{(-itp^2/2)^j}{j!} = \boxed{e^{ipx - \frac{ip^2}{2}t}}$

The exponent here is $ip(x - \frac{p}{2}t)$, so this is a "traveling wave" wavefunction

Math 115, Dec 16 1991

Review statement of Nöether:

Example: if F is t indep, then L is invariant under

$$T = t+\epsilon, \quad Q = q$$

$$q = q^* = \tilde{q}^*(t^*) = \tilde{q}^*(t+\epsilon)$$

$$L^* = \int_{t_0^*}^{t_1^*} F(t, \tilde{q}^*, \dot{\tilde{q}}^*)\,dt = \int_{t_0+\epsilon}^{t_1+\epsilon} F(\ , \tilde{q}(t+\epsilon), \dot{\tilde{q}}(t+\epsilon)),$$

Example: if $F = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - V(q_1^2 + q_2^2)$

L is inv under $\begin{pmatrix} q_1^* \\ q_2^* \end{pmatrix} = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$

intuitive proof of Nöether's

Proof of Noether's

If extra time - start integrating.

Noether's theorem, Dec 13 1991

Proof of Noether's theorem:

$$0 = \frac{\partial}{\partial \epsilon} J\Big|_{\epsilon=0} = \frac{\partial}{\partial \epsilon} \int_{t_0^*}^{t_1^*} F(t, \tilde{q}_i^*, \dot{\tilde{q}}_i^*)\,dt \Big|_{\epsilon=0} =$$

$$= F(t_1, \tilde{q}_i(t_1), \dot{\tilde{q}}_i(t_1)) \frac{\partial T(t_1, \tilde{q}_i(t_1), \dot{\tilde{q}}_i(t_1))}{\partial \epsilon}\Big|_{\epsilon=0} - F(t_0) \frac{\partial T}{\partial \epsilon}(t_0)\Big|_{\epsilon=0}$$

$$+ \int_{t_0}^{t_1} \sum_i \left(F_{q_i} \cdot \frac{\partial \tilde{q}^*}{\partial \epsilon} + F_{\dot{q}_i} \left(\frac{\partial \tilde{q}^*}{\partial \epsilon}\right)^{\cdot} \right) =$$

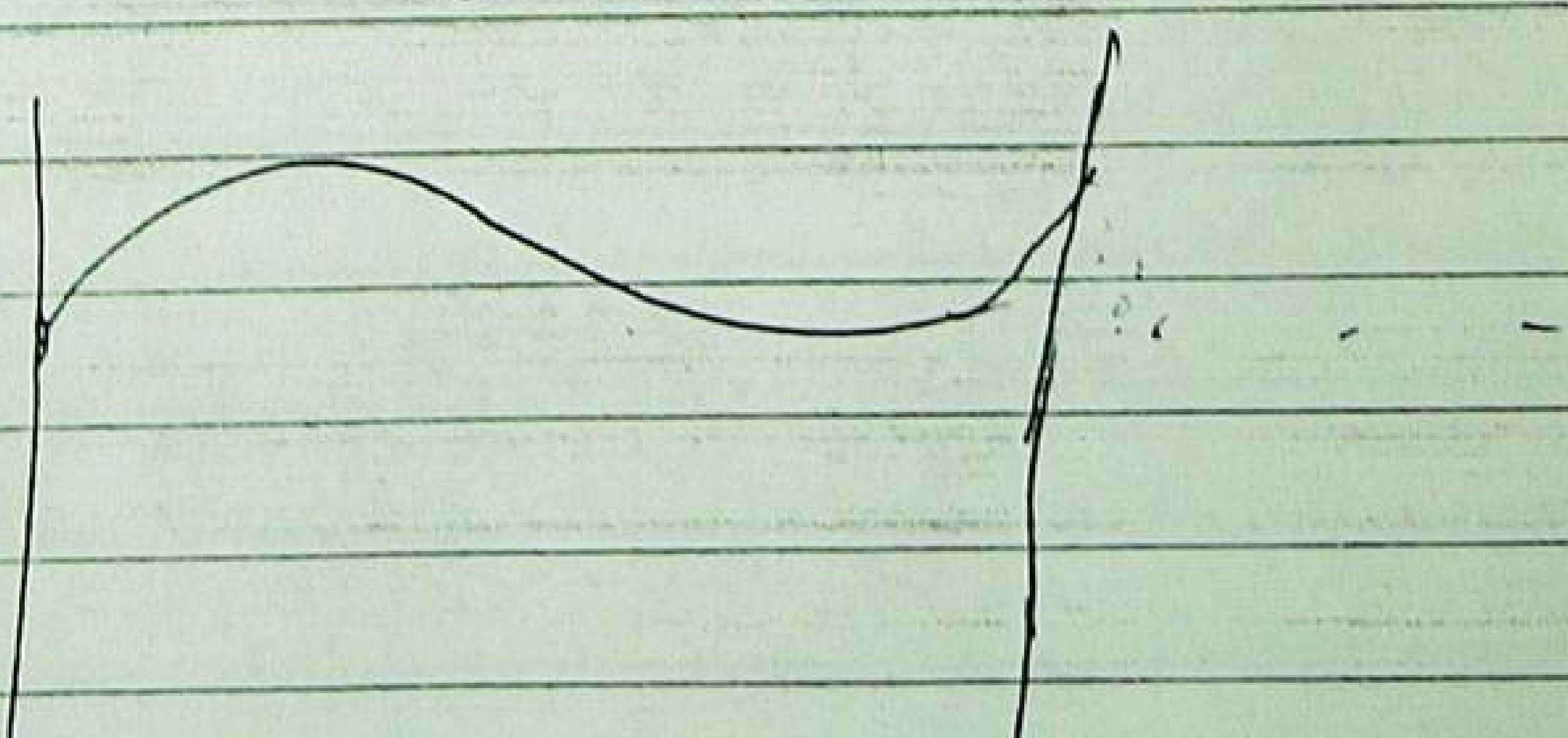
$$= F \frac{\partial T}{\partial \epsilon}\Big|_{t_1} - F \frac{\partial T}{\partial \epsilon}\Big|_{t_0} + \sum p_i \frac{\partial \tilde{q}^*}{\partial \epsilon}\Big|_{t_1} - \sum p_i \frac{\partial \tilde{q}^*}{\partial \epsilon}\Big|_{t_0} + \underbrace{E\text{-}L}_{= 0}$$

$$= \left(F - \sum p_i \dot{\tilde{q}}_i\right) \frac{\partial T}{\partial \epsilon}\Big|_{t_0}^{t_1} + \sum p_i \frac{\partial Q}{\partial \epsilon}\Big|_{t_0}^{t_1}$$

Suppose we hold t fixed. what is $\frac{\partial \tilde{q}^*}{\partial \epsilon}$?

$$q^* = \tilde{q}^*(t^*) \quad \Rightarrow \quad \frac{\partial Q}{\partial \epsilon} = \frac{\partial \tilde{q}^*}{\partial \epsilon} + \dot{\tilde{q}} \cdot \frac{\partial T}{\partial \epsilon}$$

Geometrically:



Math 117, March 9 1994

Short office hours today!

1. Bracket of two v.f.
   a. recall def
   b. recall geom interp
   c. Lie def
   d. Leibnitz' law derivation.

2 $X_{\{f,g\}} = [X_f, X_g]$
   a. Cliff's proof
   b. my proof.

Nöther's theorem in Lagrangian setting:

If $S(\tilde{q}_0) = \int_{t_0}^{t_1} F(t, q_i, \dot{q}_i)\,dt$ is invariant under

$$t^* = T(t, q_i, \dot{q}_i, \dots, \epsilon) \quad q_i^* = Q_i(t, q_i, \dot{q}_i, \epsilon \dots)$$

with

$$T|_{\epsilon=0} = t, \quad Q_i|_{\epsilon=0} = q_i \quad \left(\begin{array}{l}\text{Namely, define } \tilde{q}^* \text{ by} \\ q^* = \tilde{q}^*(t^*)\end{array}\right)$$

then

$$\sum p_i \frac{\partial Q_i}{\partial \epsilon} - H \frac{\partial T}{\partial \epsilon}$$

is conserved (constant along trajectories)

where $p_i = F_{\dot{q}_i}$; $H = \sum \dot{q}_i p_i - F$

Examples: $T = t + \epsilon, Q = q \Rightarrow H$ is conserved if F is t indep

$T = t$, $Q_i = q_i + \epsilon$, $Q_j = q_j$ $j \neq i$ $\Rightarrow p_i$ is conserved if F is q_i indep

Math 117, March 11 1994

1. Announce change in plan

2. Finish off Noether's theorem

    1. General variation: $\delta S =$ (under $q \mapsto q + \delta q$, $t \mapsto t + \delta t$)

    $$\int_{t_0}^{t_1} (E-L)\cdot \delta q + p\,\delta q\Big|_{t_0}^{t_1} - H\,\delta t\Big|_{t_0}^{t_1}$$

3. Hamilton-Jacobi:

Hamilton–Jacobi for Euclidean Path length: Mar 8 1994

$$F = \sqrt{1+\dot q^2} \qquad p = F_{\dot q} = \frac{\dot q}{\sqrt{1+\dot q^2}}$$

$$p\sqrt{1+\dot q^2} = \dot q \qquad p^2(1+\dot q^2) = \dot q^2$$

$$p^2 = \dot q^2(1-p^2)$$

$$\dot q = \sqrt{\frac{p^2}{1-p^2}}$$

$$H = p\dot q - F = p\sqrt{\frac{p^2}{1-p^2}} - \sqrt{1+\frac{p^2}{1-p^2}}$$

$$= p^2\sqrt{\frac{1}{1-p^2}} - \sqrt{\frac{1}{1-p^2}} = (p^2-1)\sqrt{\frac{1}{1-p^2}} = -\sqrt{1-p^2}$$

Hamilton Jacobi:

$$\frac{\partial d}{\partial t} = \sqrt{1-\left(\frac{\partial d}{\partial q}\right)^2}$$

$$\Rightarrow \left(\frac{\partial d}{\partial t}\right)^2 + \left(\frac{\partial d}{\partial q}\right)^2 = 1$$

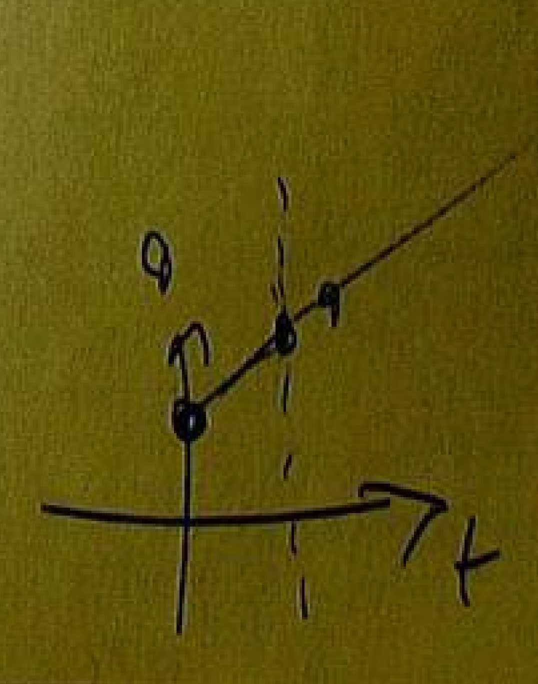
ie.

$$d = \sqrt{(q-\alpha)^2+t^2} \text{ solves.}$$

$$\frac{\partial d}{\partial \alpha} = \frac{\alpha-q}{\sqrt{(q-\alpha)^2+t^2}} = \beta$$

$$p = \frac{\partial d}{\partial q} = \frac{q-\alpha}{\sqrt{(q-\alpha)^2+t^2}}$$

$$\dot q = \sqrt{\frac{(q-\alpha)^2 / (q-\alpha)^2+t^2}{t^2 / (q-\alpha)^2+t^2}} = \sqrt{\frac{(q-\alpha)^2}{t^2}} = \frac{q-\alpha}{t-0}$$



Mar 14 1994

Hamilton-Jacobi for the harmonic oscillator

$$F = \frac{1}{2} m \dot{q}^2 - \frac{1}{2} k q^2 \qquad p = m\dot{q} \qquad \dot{q} = \frac{p}{m}$$

$$H = \frac{p^2}{2m} + \frac{1}{2} k q^2$$

Hamilton-Jacobi:

$$\frac{\partial d}{\partial t} + \frac{1}{2m}\left(\frac{\partial d}{\partial q}\right)^2 + \frac{1}{2} k q^2 = 0$$

Solution: (For $k = m = 1$:)

$$\frac{q\sqrt{2\alpha - q^2}}{2} - \alpha t + \alpha \,\mathrm{atg}\left(\frac{q}{\sqrt{2\alpha - q^2}}\right)$$

Homework for Math 117, March 14 1994.

1. Find the mistake in the formulation and in the proof of theorem 1, page 91 of Gelfand-Fomin.

2. Solve the harmonic oscillator, defined by

$$F = \frac{-q^2 + q'^2}{2},$$

using the Hamilton-Jacobi equation.

3. Let A be a nonsingular symmetric n by n matrix, and let F be the corresponding quadratic form:

$$F(q) = q^T A q.$$

Compute the Legendre transform of F, and the Fourier transform of exp(iF).

4. Do problems 6 and 11 on page 95 of Gelfand-Fomin.

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Compute the Legendre transform of F, and the Fourier transform of exp(iF).

4. Do problems 6 and 11 on page 95 of Gelfand-Fomin.

Math 117, March 14 1994

1. Hamilton Jacobi: $\frac{\partial d}{\partial t} + H(t, q_i, \frac{\partial d}{\partial q_i}) = 0$

<u>Thm</u> Let $d(t, q_1 \ldots q_n, \alpha_1 \ldots \alpha_n)$ be a solution of H.J. with

$$\det\left(\frac{\partial^2 d}{\partial \alpha_i \partial q_k}\right) \neq 0.$$

Fix $\beta_1 \ldots \beta_n$, and Let $\{q_i, p_i\}$ solve:

1. $\frac{\partial}{\partial \alpha_i} d(t, q_1 \ldots q_n, \alpha_1 \ldots \alpha_n) = \beta_i$

2. $p_i = \frac{\partial}{\partial q_i} d(t, q_1 \ldots q_n, \alpha_1 \ldots \alpha_n)\Big|$ evaluated on the q_i's of 1.

then $\{q_i, p_i\}$ solve the Cannonical eqn's.

2. Example: $F = \sqrt{1+\dot{q}^2}$ $\overbrace{\quad}^{p = \frac{\dot{q}}{\sqrt{1+\dot{q}^2}}}$ $H = -\sqrt{1-p^2}$ HJ: $\left(\frac{\partial d}{\partial t}\right)^2 + \left(\frac{\partial d}{\partial q}\right)^2 = 1$

Sol'n: easy: $\alpha t + \sqrt{1-\alpha^2}\, q$

hard: $\sqrt{(q-\alpha)^2 + t^2}$

easy:

1. $t - \frac{\alpha}{\sqrt{1-\alpha^2}} q = 0$
$\Rightarrow q = \frac{\sqrt{1-\alpha^2}}{\alpha} t$

2. $p = \sqrt{1-\alpha^2}$

indeed $\frac{\dot{q}}{\sqrt{1+\dot{q}^2}} = \frac{\sqrt{1-\alpha^2}/\alpha}{\sqrt{1+\frac{1-\alpha^2}{\alpha^2}}}$

$= \sqrt{1-\alpha^2}$

hard:

1. $\frac{q-\alpha}{\sqrt{(q-\alpha)^2+t^2}} = \beta$
$\Rightarrow q = \alpha + \sqrt{\beta^2/1-\beta^2}\; t$

2. $p = \frac{q-\alpha}{\sqrt{(q-\alpha)^2+t^2}} = \frac{\sqrt{\beta^2/1-\beta^2}}{\sqrt{\frac{\beta^2}{1-\beta^2}+1}} = \sqrt{\quad} = \beta$

indeed . . .

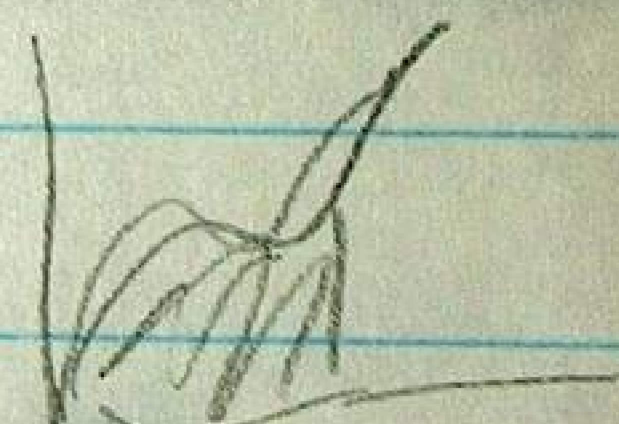
Math 117, March 14 1994 Cont'd.

3. Arnold's example.

4. Proof.

5. The Legendre transform: (appears in: Fourier, Prob. (large dev), stat mech, mech, QFT, probably a few other fields)

   1. The Fourier approach.

   2.  , involutivity, Young's ineq.

   3. relation with S, H

   4. Several variables

Math 117, March 16 1994

$f: V \to \mathbb{R}$ convex e.g.: $f = \frac{a q^2}{2}$

$Lf: V^* \to \mathbb{R}$

$(Lf)(p) = \max_q \, p \cdot q - f(q)$ e.g. $(Lf)(p) = \frac{p^2}{2a}$

connections w/ $H \leftrightarrow F$ if f is diffable, solve $f'(q) = p$, & set $(Lf)(p) = p \cdot q - f(q)\big|_{f'(q)=p}$

Properties: 1. $L(f+c) = L(f) - c$ $L(f-c) = L(f) + c$

Define $f_{q_0}(q) = f(q - q_0)$ 2. $L(f_{q_0}) = L(f) + q_0$ $L(f + p_0) = (Lf)_{p - p_0}$

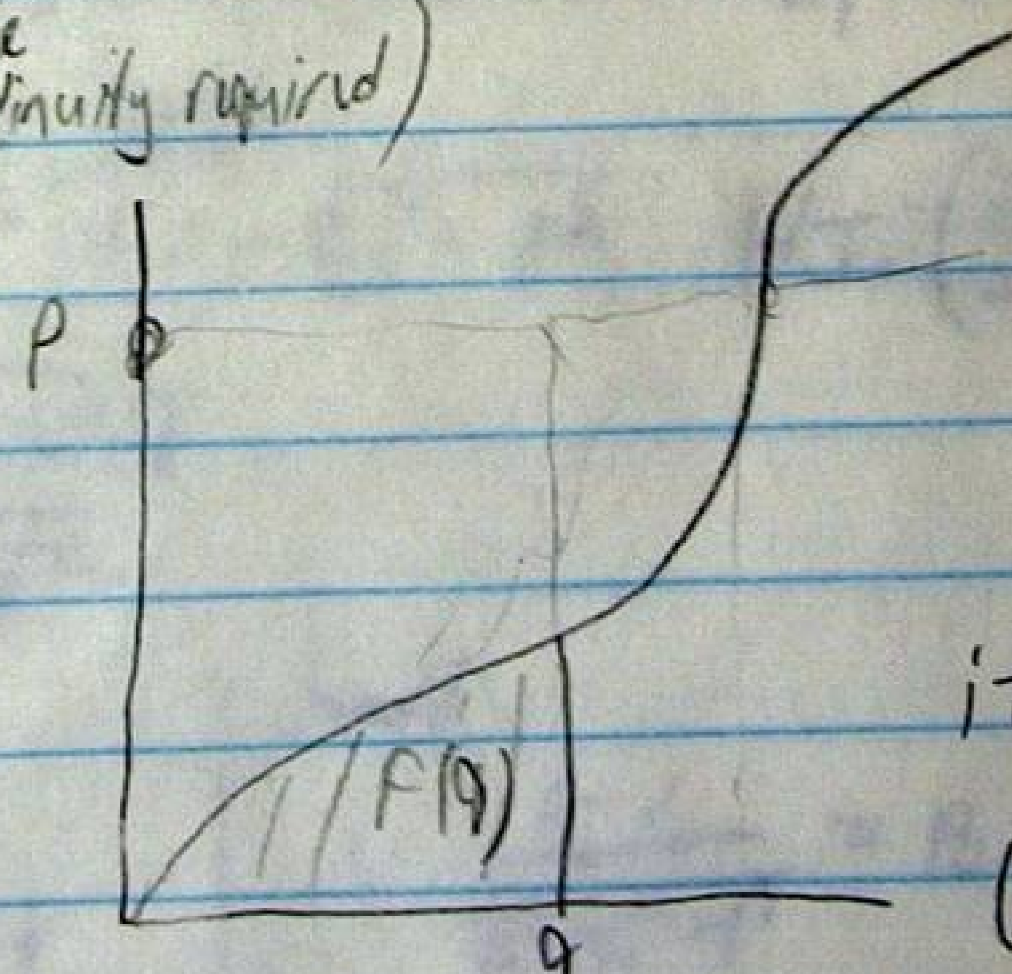
3. $L(f \circ T) = L(f) \circ (T^{-1})^*$

4 $L(f)$ is convex

5. $L(L(f)) = f$ (some continuity required)

Pf of 4&5, 1-D:

A convex function is the integral of an increasin functn



it follows that

$(Lf)''(p) \cdot f''(q) = 1$
if

$\Rightarrow (Lf)(p)$ is the integral of the inverse function?

Pf on $\mathbb{R}^n$, assuming diffability

In the 1-dim case, $(Lf)''(p) \cdot f''(q) = 1$ @ $p = f'(q)$, $q = (Lf)'(p)$

& $(Lf)(p) + f(q) \geq p \cdot q$ for all p, q

"youngs ineq"

Same in n-dim.

example: $\frac{q^a}{a} \Rightarrow \frac{p^b}{b}$ w/ $\frac{1}{a} + \frac{1}{b} = 1$

$\Rightarrow \frac{1}{a} q^a + \frac{1}{b} p^b \geq pq$

Math 117, March 18 1994

April 4: Class in 411
April 25: I'll be away

1. The sphere example.

2. $F = y^2(1-y'^2)$ $\quad y^2(1-y'^2) + y' y^2 2y' = y^2(1+y'^2)$

   $y \equiv 0$ is certainly a solution
   is it really a minimum?

3. Def of weak & strong minima.

---

4. Second Variation

5. $\int (P y'^2 + Q y^2) dx$ $\quad P \geq 0$ is $P > 0$ sufficient?
   (can't be!)

6. Idea: add $(wy^2)'$, get $\int (ay'^2 + 2byy' + cy^2) dx$.
   if $= \int (\alpha y' + \beta y)^2$, we're happy.

   $\Rightarrow b^2 = ac$; i.e. $w^2 = P \cdot (Q + w')$; i.e. $w' = \frac{w^2}{P} - Q$

   $\Rightarrow$ Almost sufficient!

Example: $P = Q = 1 \Rightarrow w' = w^2 + 1 \Rightarrow w = tg(x+c)$

Final Project/Lecture — Suggested Topics

Math 117, March 21, 1994

Dror Bar-Natan

Following is a list of suggested topics for a final project/lecture. I may amend this list later. Notice that all (except for the first and the last) suggestions begin with the word "understand". The word "understand" means:

> Understand very well (to Dror's satisfaction), write something proving that you've really understood, and be ready to give a lecture or two in class about what you've understood.

- Dream up (or look up) your own idea, and have it approved by me.
- Understand the Stone-von-Neumann uniqueness theorem, saying that (in some sense) the only realization of the "canonical commutation relations" $[P, Q] = -iI$ is $P = \frac{1}{i}\frac{\partial}{\partial x}$, $Q = x$. A good place to read about this is ???.
- Understand "strong minimas" as in chapter 6 of Gelfand-Fomin.
- Understand the gravitational two body problem (i.e., the motion of a single planet around a single star) using Lagrangian and/or Hamiltonian mechanics. What is "the Lenz vector"? You may use any mechanics text as a source, or borrow from me a paper by Guillemin and Sternberg titled "Variations on a Theme by Kepler".
- Understand the Hydrogen atom quantum-mechanically from the Schödinger equation.
- Understand Darboux' theorem, which says that locally every symplectic structure looks like $\sum dp_i \wedge dq_i$. Arnold's "Mathematical Methods of Classical Mechanics" is a good source, but there are many others.
- Understand "canonical transformations". Any mechanics book would do.
- Understand "the many-world interpretation of quantum mechanics".
- Understand Brownian motion as an example of a mathematically rigorous path integral. What is "the Feynman-Kac formula"?
- Find and understand as many as possible examples for systems which are the most easily solved using the Hamilton-Jacobi equation.

Please let me know by April 8th which topic you've chosen.

Homework Due April 4th

Math 117, March 21, 1994

Dror Bar-Natan

1. Prove the following particular case of "Young's ineaqulity":

$$pq \leq \frac{p^a}{a} + \frac{q^b}{b} \quad \text{whenever} \quad p > 0,\ q > 0, \text{ and } \frac{1}{a} + \frac{1}{b} = 1.$$

   (This inequality is used in the proof of the famous Hölder inequality, which is itself used in the proof of the famous Minkowski inequality.)

2. Find the mistake in the proof in page 110 of Gelfand-Fomin and briefly indicate how (with some effort) it can be fixed without changing the global structure of the proof (as we did in class).

3. Do exercises 7,8,11,13 on pages 129–130 of Gelfand-Fomin.

Math 117, March 21 1994

Review: $(\delta^2 \int F)(h) = \frac{1}{2}\int (\underbrace{P}_{F_{y'y'}|_{y_{crit}}} h'^2 + \underbrace{Q}_{F_{yy} - \frac{d}{dx}F_{yy'}|_{y_{crit}}} h^2)\,dx$

---

study $\int (Ph'^2 + Qh^2)dx$: nec that $P \geq 0$.

Jacobi eqn: $-\frac{d}{dx}(Ph') + Qh = 0$ (2nd order linear ODE)

Def of conjugate pt.

Lemma if a, b conjugate & $h(a) = h(b) = 0$ & h sat Jac,
then $\int (Ph'^2 + Qh^2)dx = 0$ (by $\int$ Jac $\cdot h\,dx$)

Thm $\int Ph'^2 + Qh^2$ w/ $P > 0$ is P.D. iff $[a, b]$
contains no pts. conjugate to a.

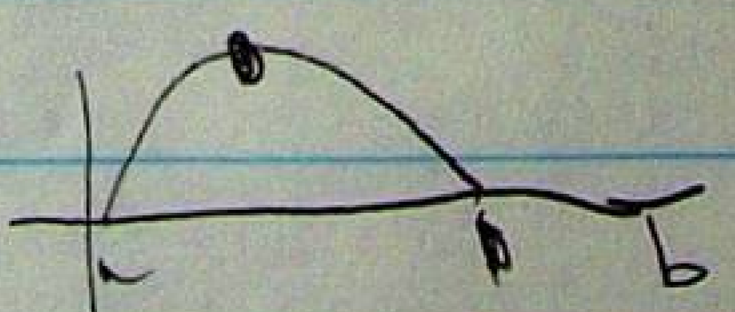
Pf. assume no conjugate pts. Recall our technique of proving positivity

- find w s.t. $w^2 = P(Q + w')$, ~~add $(wh^2)'$ & get~~

$\int Ph'^2 + Qh^2 = \int P(h' + \frac{w}{P}h)^2$ which is P.D.

now just subs. $w = -\frac{u'}{u}P$ into Riccati.

other side easier:



Math 117, March 23 1994

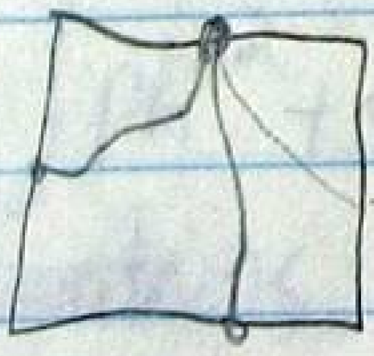
Justify ↑ { Picard's theorem: $y' = F(x,y)$, F cont. diff'able wrt y & cont. wrt. x on $|x - x_0| \le a$ $|y - y_0| \le b$. Let $a^* = \min\left(a, \frac{b}{M}\right)$ where $M = \max \|F\|$ $\Rightarrow \exists!$ solution through x_0, y_0

2^{nd} order eqn's $\rightarrow y'' = F(x, y, y'') \Leftrightarrow$ $\begin{array}{l} v' = F(x,y,v) \\ y' = v \end{array}$

Linear equations are soluble on any compact ~~domain~~ interval (and thus on any interval)

pf: $\{\boxed{\ }\}$ stupid me, no existance problems!

Assume conj + P.D. (by contra.)



$t(Ph'^2 + Qh^2) + (1-t)h'^2$

Back to minimization:

<u>Jacobi's suf. condition.</u> $\int_a^b F\,dx$; Suppose $\tilde{y}$
1. E-L.
2. $F_{y'y'}|_{\tilde{y}} > 0$
3. $[a,b]$ contains no conj. of a.

0, F is defined in $[a-\epsilon, b\epsilon]$

$\Rightarrow \tilde{y}$ is a min.

<u>pf</u> $S(\tilde{y}+h) - S(y) = \int_a^b (Ph'^2 + Qh^2)\,dx + \int_a^b (\xi h'^2 + \eta h^2)\,dx$

w/ $\xi, \eta \xrightarrow[\|h\|_1 \to 0]{} 0$ uniformly. $h^2(x) = \left(\int_a^x h'\,dx\right)^2 \le (x-a)\int_a^x h'^2\,dx \le (b-a)\int_a^b h'^2\,dx$

$\Rightarrow$ for very small $\|h\|_1$, error term is $< \epsilon_0 \int h'^2\,dx$.

Subtract ϵ_0 from P ↑ Next class unusual.

Morse theory.

Is the moon there when nobody looks? Reality and the quantum theory

Einstein maintained that quantum metaphysics entails spooky actions at a distance; experiments have now shown that what bothered Einstein is not a debatable point but the observed behavior of the real world.

N. David Mermin

Quantum mechanics is magic[1]

In May 1935, Albert Einstein, Boris Podolsky and Nathan Rosen published[2] an argument that quantum mechanics fails to provide a complete description of physical reality. Today, 50 years later, the EPR paper and the theoretical and experimental work it inspired remain remarkable for the vivid illustration they provide of one of the most bizarre aspects of the world revealed to us by the quantum theory.

Einstein's talent for saying memorable things did him a disservice when he declared "God does not play dice," for it has been held ever since that the basis for his opposition to quantum mechanics was the claim that a fundamental understanding of the world can only be statistical. But the EPR paper, his most powerful attack on the quantum theory, focuses on quite a different aspect: the doctrine that physical properties have in general no objective reality independent of the act of observation. As Pascual Jordan put it[3]

> Observations not only disturb what has to be measured, they produce it.... We compel [the electron] to assume a definite position.... We ourselves produce the results of measurement.

Jordan's statement is something of a truism for contemporary physicists. Underlying it, we have all been taught, is the disruption of what is being measured by the act of measurement, made unavoidable by the existence of the quantum of action, which generally makes it impossible even in principle to construct probes that can yield the information classical intuition expects to be there.

Einstein didn't like this. He wanted things out there to have properties, whether or not they were measured[4]:

> We often discussed his notions on objective reality. I recall that during one walk Einstein suddenly stopped, turned to me and asked whether I really believed that the moon exists only when I look at it.

The EPR paper describes a situation ingeniously contrived to force the quantum theory into asserting that properties in a space–time region **B** are the result of an act of measurement in another space–time region **A**, so far from **B** that there is no possibility of the measurement in **A** exerting an influence on region **B** by any known dynamical mechanism. Under these conditions, Einstein maintained that the properties in **A** must have existed all along.

Spooky actions at a distance

Many of his simplest and most explicit statements of this position can be found in Einstein's correspondence with Max Born.[5] Throughout the book (which sometimes reads like a Nabokov novel), Born, pained by Einstein's distaste for the statistical character of the quantum theory, repeatedly fails, both in his letters and in his later commentary on the correspondence, to understand what is really bothering Einstein. Einstein tries over and over again, without success, to make himself clear. In March 1948, for example, he writes:

> That which really exists in **B** should ... not depend on what kind of measurement is carried out in part of space **A**; it should also be independent of whether or not any measurement at all is carried out in space **A**. If one adheres to this program, one can hardly consider the quantum-theoretical description as a complete representation of the physically real. If one tries to do so in spite of this, one has to assume that the physically real in **B** suffers a sudden change as a result of a measurement in **A**. My instinct for physics bristles at this.

Or, in March 1947,

> I cannot seriously believe in [the quantum theory] because it cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at a distance.

The "spooky actions at a distance" (*spukhafte Fernwirkungen*) are the acquisition of a definite value of a property by the system in region **B** by virtue of the measurement carried out in region **A**. The EPR paper presents a wavefunction that describes two correlated particles, localized in regions **A** and **B**, far apart. In this particular two-particle state one can learn (in the sense of being able to predict with certainty the

---

David Mermin is director of the Laboratory of Atomic and Solid State Physics at Cornell University. A solid-state theorist, he has recently come up with some quasithoughts about quasicrystals. He is known to PHYSICS TODAY readers as the person who made "boojum" an internationally accepted scientific term. With N. W. Ashcroft, he is about to start updating the world's funniest solid-state physics text. He says he *is* bothered by Bell's theorem, but may have rocks in his head anyway.

0031-9228 / 85 / 0400 38–10 / $01.00

Non-Commutative (Quantum) Probability

Math 117, April 4 1994
Dror Bar-Natan

Claim: In the quantum probability space $(\mathbf{R}^4, v)$ where v is the unit vector $v = \frac{\sqrt{2}}{2}(0\ 1\ -1\ 0)^T$, one has $p(A = B) = p(B = C) = p(C = D) = \frac{3}{4}$ and $p(D = A) = 0$, where A, B, C, and D are the random variables corresponding to the matrices:

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad ; \quad B = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & \frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

$$C = \begin{pmatrix} -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 \\ 0 & -\frac{1}{2} & 0 & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} & 0 \\ 0 & -\frac{\sqrt{3}}{2} & 0 & \frac{1}{2} \end{pmatrix} \quad ; \quad D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

```
Mathematica 2.0 for SPARC
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 -- Terminal graphics initialized --

In[1]:= v=1/2 Sqrt[2] {0, 1, -1, 0}; q=1/2 Sqrt[3];

In[2]:= A1=DiagonalMatrix[{1, 1, -1, -1}]; A4=DiagonalMatrix[{1, -1, 1, -1}];

In[3]:= A2={{-1/2, q, 0, 0}, {q, 1/2, 0, 0}, {0, 0, -1/2, q}, {0, 0, q, 1/2}};

In[4]:= A3={{-1/2, 0, -q, 0},{0, -1/2, 0, -q},{-q, 0, 1/2, 0},{0, -q, 0, 1/2}};

In[5]:= {Eigenvalues[A1], Eigenvalues[A2], Eigenvalues[A3], Eigenvalues[A4]}

Out[5]= {{1, -1, 1, -1}, {1, -1, 1, -1}, {1, -1, 1, -1}, {1, -1, 1, -1}}

In[6]:= {A1.A2==A2.A1, A2.A3==A3.A2, A3.A4==A4.A3, A4.A1==A1.A4}

Out[6]= {True, True, True, True}

In[7]:= pequal[M1_, M2_]:=1-v.(M1-M2).(M1-M2).v / 4

In[8]:= {pequal[A1,A2], pequal[A2,A3], pequal[A3,A4], pequal[A4,A1]}

         3  3  3
Out[8]= {-, -, -, 0}
         4  4  4
```
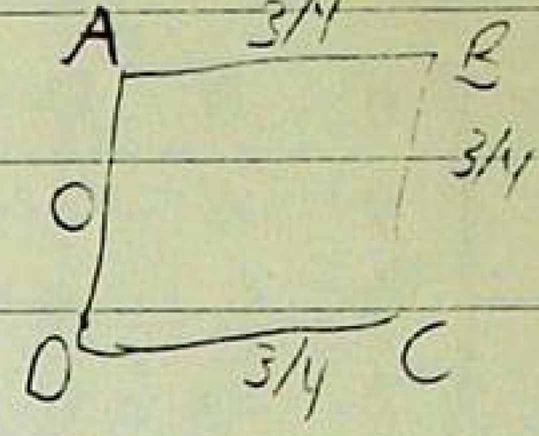

More information can be found at N.D. Mermin, Physics Today 39(4) 38 (1985) and D. Bar-Natan, Foundations of Physics 19(1) 97 (1989).

Math 117, April 4 1994

Non Commutative (Quantum) probability. 3/31/92

5 1. The physical paradox

$$\begin{matrix} A & & B \\ C & & D \end{matrix}$$



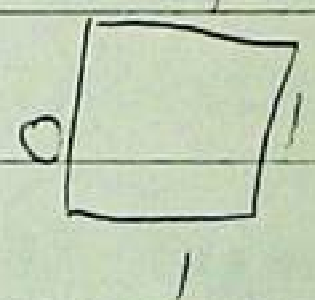
3 2. Fact & Theory of evolution

7 3. Fact of probability stated finitely $\{X_i\}$

5 4 Classical probability theory $(\Sigma, \sigma_\alpha, P_\alpha)$

3 5 leads to a paradox

3 6. minimal probability theory



10 7. Quantum probability theory.

$$P(X_i = \lambda) = \begin{cases} 0 & \lambda \text{ unit} \\ \|P_\lambda(X) V\|^2 & \text{otherwise} \end{cases}$$

$P_\lambda(X) \rightarrow$ orthogonal projection on the λ eigenspace.

3 8. It is known that

$$\{\text{classical}\} \subset \{\text{Quantum}\} \subset \left\{\begin{matrix}\text{minimal} \\ \text{(which is somehow maximal)}\end{matrix}\right\}$$

1 9. Exact relation isn't known

5 10. Do example.

Foundations of Physics, Vol. 19, No. 1, 1989

Two Examples in Noncommutative Probability

Dror Bar-Natan[1,2]

Received June 23, 1987; revised September 14, 1987

A simple noncommutative probability theory is presented, and two examples for the difference between that theory and the classical theory are shown. The first example is the well-known formulation of the Heisenberg uncertainty principle in terms of a variance inequality and the second example is an interpretation of the Bell paradox in terms of noncommutative probability.

1. INTRODUCTION

We shall present here a simple yet representative version of the theory of noncommutative probability, including two examples of the difference between that theory and the classical probability theory. The first is a precise formulation of the Heisenberg uncertainty principle, while the second constitutes a strong indication for the existence of random phenomena in nature explainable only by assuming that the probability space in which we live is noncommutative.

2. THE CLASSICAL PROBABILITY THEORY

Definition. A classical probability space is a triple (X, B, p) consisting of

X, a collection of points,
B, a subcollection of the collection of all functions $f: X \to \{0, 1\}$ satisfying some simple closure properties, and
p, a probability measure on (X, B).

---

[1] Department of Mathematics, Tel Aviv University, Tel Aviv, Israel.
[2] Current address: Department of Mathematics, Princeton University, Princeton, New Jersey 08544.

$G' = G - E(G)I$. We obtain, after some manipulation,

$$V(F') = \langle w,(F-E(F)I)^2w\rangle - \langle w,(F-E(F)I)w\rangle^2 = \langle w, (F-E(F)I)^2w\rangle$$

$$= \langle w,F^2w\rangle - \langle w,Fw\rangle^2 = V(F).$$

In the same manner, one obtains $V(G') = V(G)$. Now:

$$V(F)V(G) = V(F')V(G') = \langle w,F'^2w\rangle\langle w,G'^2w\rangle = \langle F'w,F'w\rangle\langle G'w,G'w\rangle$$

$$\geq |\langle F'w,G'w\rangle|^2 = |\langle w,F'G'w\rangle|^2$$

$$= \left| \langle w, \tfrac{1}{2}[F',G']w\rangle + \langle w, \tfrac{1}{2}\{F',G'\}w\rangle \right|^2$$

$$\geq \left|\langle w, \tfrac{1}{2}[F',G']w\rangle\right|^2 = \frac{1}{4}|E([F',G'])|^2,$$

where we have used the fact that the anticommutator $\{F',G'\}$ is always self-adjoint, while the commutator is always anti-self-adjoint. The corresponding expectation values are therefore repectively real and imaginary. We conclude by noting that $[F',G'] = [F,G]$.

In quantum mechanics, a vector w in $L^2(R)$ represents the state of a quantum particle, the operator $(Fu)(x) = xu(x)$ represents the random variable whose distribution is the position of the particle, and the operator

$$Gu = -i\hbar \frac{\partial}{\partial x} u \quad (\hbar \text{ is the Planck constant})$$

represents the random variable whose distribution is the momentum distribution of the same particle. The identity $[F,G] = i\hbar I$ now makes our theorem the Heisenberg uncertainty principle:

"In any state of a quantum particle:

(variance of position)x(variance of momentum) $\geq \frac{1}{4}\hbar^2$"

5. SECOND EXAMPLE: THE BELL PARADOX[3,4,5,9)]

Phrases like,"choose at random one of three given random variables", have no direct meaning in classical probability, but they can easily be given a meaning by defining a new

Math 117; April 6 1994

Fact of Prob:

$(S = \{a, b, c \ldots\}; J, P)$

Classical Prob:

(Ω, ρ, S, F)

Quantum Prob.

$(\mathcal{H}, v, S, F)$ $\mathcal{H}$, v.s + s.p.

$\Pi_A(F) : \mathcal{H} \to \mathcal{H}$ spectral projection

$$S_t = \begin{pmatrix} \cos 2t & \sin 2t \\ \sin 2t & -\cos 2t \end{pmatrix}$$

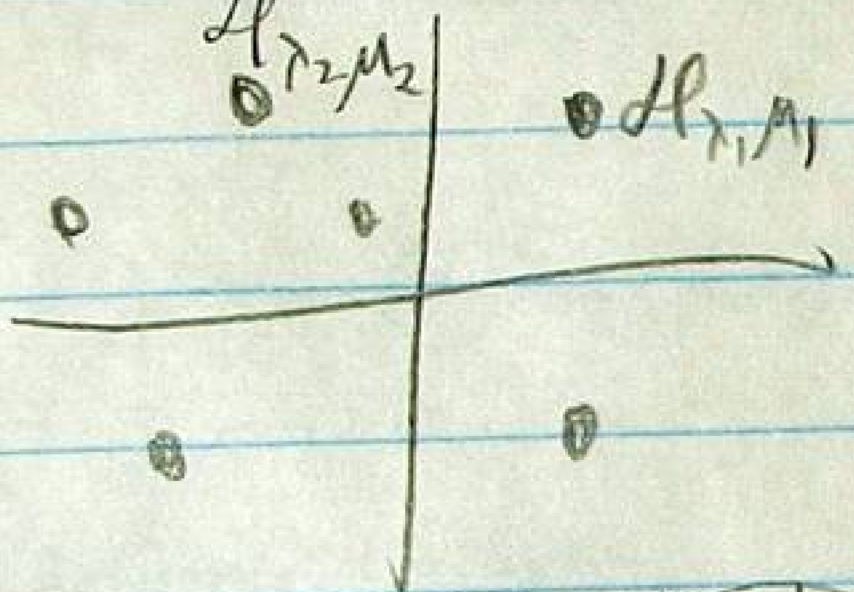
$A_t = S_t \otimes I \qquad B_t = -I \otimes S_t$

$t = 0, 120, 240$

Math 117, April 8 1994

Spectral thm for a pair of commuting S.A. operators:
$[F_1, F_2] = 0$:

$$\mathcal{H} = \bigoplus_{(\lambda,\mu)} \mathcal{H}_{\lambda,\mu} \text{ (orth.)} \qquad F_1|_{\mathcal{H}_{\lambda,\mu}} = \lambda \, Id$$
$$F_2|_{\mathcal{H}_{\lambda,\mu}} = \mu \, Id$$



NCP:
QP: $P(a=\lambda, b=\mu) = \| \Pi_{\mathcal{H}_{\lambda,\mu}} V_0 \|^2$

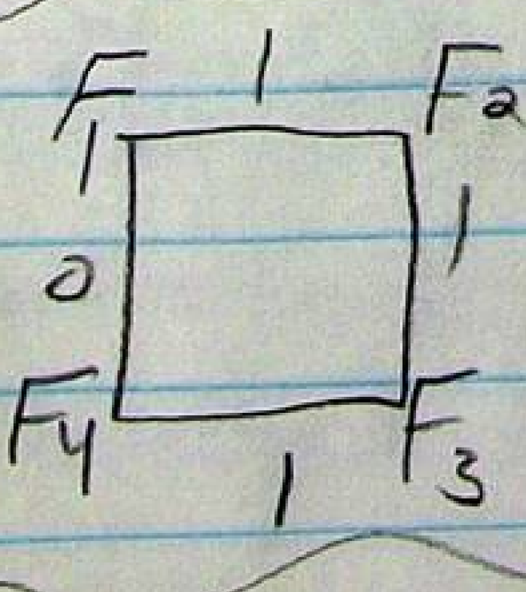
h: pol. in two vars.

Thm $E(h(a,b)) = \langle V_0, h(F_a, F_b) V_0 \rangle$

$$\sum_{(\lambda,\mu)} h(\lambda,\mu) \cdot P(a=\lambda, b=\mu)$$

Pf write $V_0 = \sum_{(\lambda,\mu)} V_{\lambda,\mu}$

Example $P_{eq}(a,b) = \langle V_0, (I - \frac{(a-b)^2}{4}) V_0 \rangle = 1 - \frac{\langle V_0, (a-b)^2 V_0 \rangle}{4}$
(given that a, b are supported on ± 1)

Thm 

Cannot be achieved in Quantum Prob.
Pf: $F_1 V_0 = F_2 V_0 = F_3 V_0 = F_4 V_0$

Q: Give a criterion for a fact to be classical
Give a criterion for a fact to be quantum
(both are open!)

Thm QP ⊃ CP.

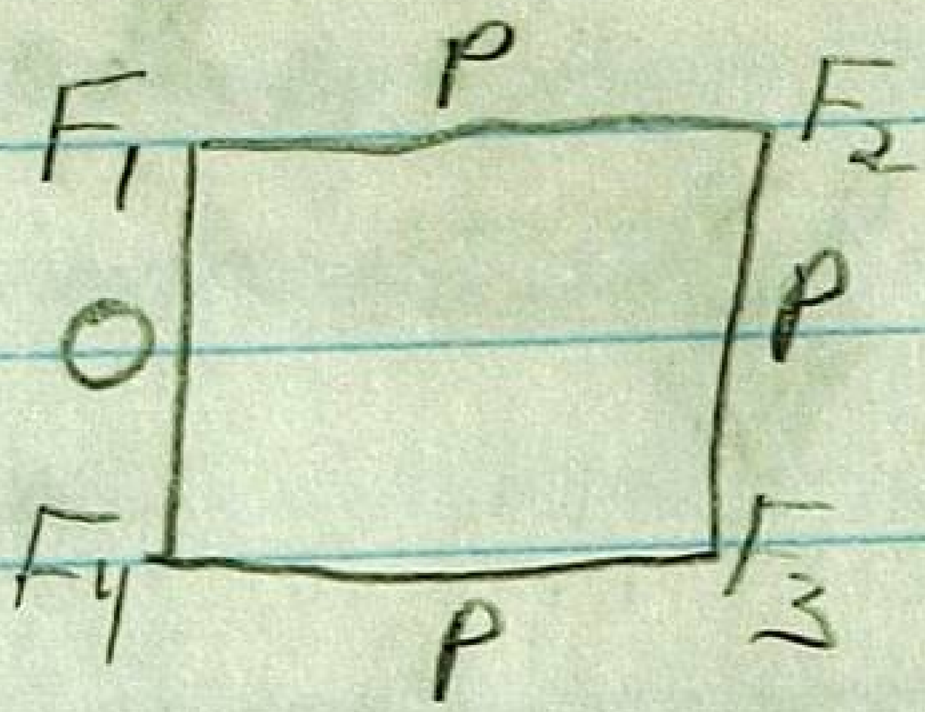
If time — 1. back to Mermin
2. analyze P, Q.

Math 117, April 11 1994 HW - due 4/18/1994.

1. Complete the proof of the fact that every classical probability fact is also a quantum probability fact.

*2. In class we've shown that the square

S:



(definitions as in class)

cannot be represented by a quantum prob. space for $p=1$, and can be represented as such for $p=3/4$. What is "the critical p" in which S becomes non-quantum? And when does it become classical?

Math 117, April 11 1994

$(\mathcal{H} = L^2(\mathbb{R}),\ V = \psi,\ Q = \text{m. by } q,\ P = -i\hbar \frac{\partial}{\partial q})$

1. is Q S.A ?
2. What is the dist of Q ? $\left(\widetilde{P\psi} = \hbar p \tilde{\psi} \right)$
3. is Q S.A. ?
4. What is the dist of Q ?

$$\begin{array}{ccc} L^2(\mathbb{R}_q) & \xrightarrow{P} & L^2(\mathbb{R}_p) \\ \downarrow & & \downarrow \\ \mathbb{R}_p & \xrightarrow{\hbar p} & \mathbb{R}_p \end{array}$$

Expectation value, Variance

CCR : $[P, Q] = i\hbar I$

Thm $V(P)V(Q) \geq \frac{1}{4}\hbar^2$

The EPR paradox.

Gaussians.

Math 117 - Quantum mechanical plan: (11 classes left)

1. Finish heizenberg uncertainty.
2. Square well, Potential barier.
3. Harmonic Ocscillator.
4. Rotations, SU(2), SO(3), ... Zeeman effect?
5. Von-Neumann uniqueness thm (Math. annals 104 1931 570-578 (Von-Neumann's paper)
6. Thy Hydrogen Atom
7. Justification of rel. with Path integrals
8. $A \wedge dA$, linking, self linking
9. Rud. pert theory, My RSS thm!.

Sep 26, 1987,
Dror Bar-Natan
Source unknown

Stone-von-Neuman uniqueness theorem:

We are given two continuous unitary representations U & V of $\mathbb{R}^n$ in a seperable Hilbert space $\mathcal{H}$, satisfying the weyl form of the CCR:

$$U(\alpha)V(\beta)=e^{i\alpha\cdot\beta}V(\beta)U(\alpha),$$

and we wish to prove that $\mathcal{H}$ can be decomposed into countably (possibly finitely) many invariant subspaces, each of them unitarity equivalent to $\mathcal{H}_0=L^2(\mathbb{R}^n)$,

$$(U_0(\alpha)\phi)(q)=\phi(q+\alpha) \qquad (V_0(\beta)\phi)(q)=e^{i\beta\cdot q}\phi(q)$$

(E.g. there exists unitary $A:\mathcal{H}\to\mathcal{H}_0$ s.t. for all $\alpha,\beta\in\mathbb{R}^n$:
$U(\alpha)=A^{-1}V_0(\alpha)A$ and $V(\beta)=A^{-1}V_0(\beta)A$)

Equivalently, one defines $S(\alpha,\beta)=e^{-i\frac{\alpha\cdot\beta}{2}}U(\alpha)V(\beta)$, and proves a uniqueness theorem for irreducible continuous unitary projective representations $S(\alpha,\beta)$ of $\mathbb{R}^{2n}$ satisfying:

$$(*)\qquad S(\alpha,\beta)S(\gamma,\delta)=e^{\frac{i}{2}(\alpha\cdot\delta-\beta\cdot\gamma)}S(\alpha+\gamma,\beta+\delta)$$

The idea of the proof: To "reconstruct" $\mathcal{H}$ using only the $S(\alpha,\beta)$-s and their property $(*)$. E.g: To show that the $S(\alpha,\beta)$-s determine canonicaly a cyclic vector Φ for $\mathcal{H}$, in such a way that the norms of all the $S(\alpha,\beta)\Phi$-s is given explicitly. E.g: We shall show that the operator E defined by:

$$(\#)\quad E=\frac{1}{(2\pi)^n}\iint d^n\alpha\, d^n\beta\, e^{-\frac14(\alpha^2+\beta^2)}S(\alpha,\beta)$$

(one can verify that on $\mathcal{H}_0$ E is the projection on the one dimensional subspace of $\mathcal{H}_0$ generated by: $\Phi_0(q)=\pi^{-n/4}e^{-\frac12 q^2}$)

is a projection operator satisfying $(\phi,\psi\in E\mathcal{H})$

$$\langle S(\alpha,\beta)\phi, S(\gamma,\delta)\psi\rangle=e^{-\frac14(\alpha-\gamma)^2-\frac14(\beta-\delta)^2+\frac{i}{2}(\beta\cdot\gamma-\alpha\cdot\delta)}\langle\phi,\psi\rangle.$$

After the general idea of the proof is clear, we can proceed to the technical details:

First, a lemma that will enable us to "integrate" operators:

Lemma 1: If $\lambda(\psi, \phi)$ is a sesquilinear form defined on $\mathcal{H}$ and satisfying

$$|\lambda(\psi,\phi)| \leq C \|\psi\| \|\phi\| \quad \forall \phi, \psi \in \mathcal{H}$$

for a constant C, then there exists a uniquely determined operator A on $\mathcal{H}$ with:

$$\lambda(\psi,\phi) = \langle \psi, A\phi \rangle \; ; \; \|A\| \leq C.$$

Proof: a trivial application of Riesz's theorem.

Definition 1: For $a \in L^1(\mathbb{R}^{2n})$ define

$$S(a) = \text{"}\int d^n\alpha \, d^n\beta \, a(\alpha,\beta) \, S(\alpha,\beta)\text{"}$$

to be the operator A corresponding to the sesquilinear form:

$$\lambda_a(\phi,\psi) = \int d^n\alpha \, d^n\beta \, a(\alpha,\beta) \langle \phi, S(\alpha,\beta)\psi \rangle$$

one can verify that:

$$|\lambda_a(\phi,\psi)| \leq \|a\|_{L^1} \cdot \|\phi\| \|\psi\| \; ; \; \|S(a)\| \leq \|a\|_{L^1}$$

Lemma 2: a) $S(a+b) = S(a) + S(b)$

b) $S(ra) = rS(a) \qquad (r \in \mathbb{C})$

c) $S(a)^* = S(\hat{a})$ where $\hat{a}(a,b) = \bar{a}(-\alpha,-\beta)$

Proof: a), b) are trivial. For prooving c), notice that by (*), $S(\alpha,\beta)^* = S(-\alpha,-\beta)$, and

$$\langle \phi, S(a)^*\psi \rangle = \overline{\langle \psi, S(a)\phi \rangle} = \int d^n\alpha \, d^n\beta \, \overline{a(\alpha,\beta) \langle \psi, S(\alpha,\beta)\phi \rangle} =$$

$$= \int d^n\alpha \, d^n\beta \, \bar{a}(\alpha,\beta) \langle \phi, S(-\alpha,-\beta)\psi \rangle = \lambda_{\hat{a}}(\phi,\psi)$$

Lemma 3: a) $S(a)S(\gamma,\delta)=S(a')$ where $a'(\alpha,\beta)=e^{\frac{i}{2}(\alpha\cdot\delta-\beta\cdot\gamma)}a(\alpha-\gamma,\beta-\delta)$

b) $S(\gamma,\delta)S(a)=S(a'')$ where $a''(\alpha,\beta)=e^{-\frac{i}{2}(\alpha\cdot\delta-\beta\cdot\gamma)}a(\alpha-\gamma,\beta-\delta)$

Proof: we shall prove only a:

$$\langle\phi,S(a)S(\gamma,\delta)\psi\rangle=\int d^n\alpha\, d^n\beta\, a(\alpha,\beta)\langle\phi,S(\alpha,\beta)S(\gamma,\delta)\psi\rangle=$$

$$=\int d^n\alpha\, d^n\beta\, e^{\frac{i}{2}(\alpha\cdot\delta-\beta\cdot\gamma)}a(\alpha,\beta)\langle\phi,S(\alpha+\gamma,\beta+\delta)\psi\rangle=$$

$$=\int d^n\alpha\, d^n\beta\, e^{\frac{i}{2}((\alpha-\gamma)\cdot\delta-(\beta-\delta)\cdot\gamma)}a(\alpha-\gamma,\beta-\delta)\langle\phi,S(\alpha,\beta)\psi\rangle$$

$$=\langle\phi,S(a')\psi\rangle.$$

Lemma 4: $S(a)S(b)=S(c)$, where $c(\gamma,\delta)=\int d^n\alpha\, d^n\beta\, e^{\frac{i}{2}(\gamma\cdot\beta-\delta\cdot\alpha)}a(\gamma-\alpha,\delta-\beta)b(\alpha,\beta)$

Proof: As usual:

$$\langle\phi,S(a)S(b)\psi\rangle=\langle S(a)^*\phi,S(b)\psi\rangle=$$

$$=\int d^n\alpha\, d^n\beta\, b(\alpha,\beta)\langle S(a)^*\phi,S(\alpha,\beta)\psi\rangle=$$

$$=\int d^n\alpha\, d^n\beta\, b(\alpha,\beta)\langle\phi,S(a)S(\alpha,\beta)\psi\rangle=$$

$$=\int d^n\alpha\, d^n\beta\, b(\alpha,\beta)\int d^n\gamma\, d^n\delta\, e^{\frac{i}{2}(\beta\cdot\gamma-\alpha\cdot\delta)}a(\gamma-\alpha,\delta-\beta)\langle\phi,S(\gamma,\delta)\psi\rangle=$$

$$=\langle\phi,S(c)\psi\rangle$$

Lemma 5: I $S(a)=0$, then $a=0$ a.e.

Proof: $S(a)=0\Longrightarrow S(-\gamma,-\delta)S(a)S(\gamma,\delta)=0\Longrightarrow\forall\gamma,\delta,\phi,\psi$

$$\int d^n\alpha\, d^n\beta\, e^{i(\alpha\cdot\delta-\beta\cdot\gamma)}a(\alpha,\beta)\langle\phi,S(\alpha,\beta)\psi\rangle=0\Longrightarrow$$

$$\xRightarrow[\text{transform)}]{\text{(Fourier}} a(\alpha,\beta)\langle\phi,S(\alpha,\beta)\psi\rangle=0 \text{ a.e.} \xRightarrow[\text{separability}]{\phi\in\text{cons,}}$$

$$\Longrightarrow a(\alpha,\beta)S(\alpha,\beta)\psi=0 \text{ a.e.} \xRightarrow[\text{is unitary}]{S(\alpha,\beta)} a(\alpha,\beta)=0 \text{ a.e.}$$

Definition 2:

$$E = \frac{1}{(2\pi)^n}\int d^n\alpha\, d^n\beta\, e^{-\frac{1}{4}(\alpha^2+\beta^2)} S(\alpha,\beta)$$

This is not a part of the proof, but however, it is interesting to find what is E in the standard representation:

Begin with

$$(S(\alpha,\beta)\phi)(q) = e^{-\frac{i}{2}\alpha\cdot\beta}(V(\alpha)V(\beta)\phi)(q) = e^{-\frac{i}{2}\alpha\cdot\beta}(V(\beta)\phi)(q+\alpha) =$$

$$= e^{\frac{i}{2}\alpha\cdot\beta} e^{i\beta\cdot q}\phi(q+\alpha)$$

to find that:

$$(E\phi)(q) = \int d^n\alpha\, d^n\beta \frac{1}{(2\pi)^n} e^{-\frac{1}{4}(\alpha^2+\beta^2)} e^{\frac{i}{2}\alpha\cdot\beta} e^{iq\cdot\beta}\phi(q+\alpha) =$$

$$= \pi^{-\frac{n}{2}}\int d^n\alpha\, e^{-\frac{1}{2}(\alpha+q)^2-\frac{1}{2}q^2}\phi(\alpha+q) =$$

$$= \pi^{-\frac{n}{4}}e^{-\frac{1}{2}q^2}\int d^n\alpha\, \pi^{-n/4}e^{-\frac{1}{2}\alpha^2}\phi(\alpha) = \Phi_0\cdot\langle\Phi_0,\phi\rangle$$

where $\Phi_0(q) = \pi^{-\frac{n}{4}}e^{-\frac{1}{2}q^2}$.

Back to our main line:

Lemma 6: $E = E^* \neq 0$

Proof By lemma 2c and lemma 5

Lemma 7: $ES(\gamma,\delta)E = e^{-\frac{1}{4}(\gamma^2+\delta^2)}E$

Proof a) $E = S(a_0)$ where $a_0(\alpha,\beta) = (2\pi)^{-n}e^{-\frac{1}{4}(\alpha^2+\beta^2)}$

b) $S(\gamma,\delta)E = S(a_1)$ where $a_1(\alpha,\beta) = e^{-\frac{i}{2}(\alpha\cdot\delta-\beta\cdot\gamma)}\frac{1}{(2\pi)^n}e^{-\frac{1}{4}((\alpha-\gamma)^2+(\beta-\delta)^2)}$

c) $ES(\gamma,\delta)E = S(a_0)S(a_1) = S(a_2)$ where

$$a_2 = \frac{1}{(2\pi)^{2n}}\int d^n\rho\, d^n\sigma\, e^{\frac{i}{2}(\alpha\cdot\sigma-\beta\cdot\rho)}e^{-\frac{1}{4}((\alpha-\rho)^2+(\beta-\sigma)^2)}e^{-\frac{i}{2}(\rho\cdot\delta-\sigma\cdot\gamma)}e^{-\frac{1}{4}((\rho-\gamma)^2+(\sigma-\delta)^2)} =$$

by calculating gaussian integrals (or avoiding this, as in the text):

$$= \frac{1}{(2\pi)^n}e^{-\frac{1}{4}(\alpha^2+\beta^2)}\cdot e^{-\frac{1}{4}(\gamma^2+\delta^2)} = e^{-\frac{1}{4}(\gamma^2+\delta^2)}a_0(\alpha,\beta)$$

Lemma 8: E is an orthogonal projection.

Proof: Immidiate from lemma 6 & lemma 7.

Lemma 9: If $\phi, \psi \in E\mathcal{H}$ then

$$\langle S(\alpha,\beta)\phi, S(\gamma,\delta)\psi \rangle = e^{-\frac{1}{4}(\alpha-\gamma)^2 - \frac{1}{4}(\beta-\delta)^2 + \frac{i}{2}(\beta\cdot\gamma - \alpha\cdot\delta)}$$

Proof: $\text{l.h.s.} = \langle S(\alpha,\beta)E\phi, S(\gamma,\delta)E\psi \rangle = \langle \phi, ES(-\alpha,-\beta)S(\gamma,\delta)E\psi \rangle =$

$$= e^{\frac{i}{2}(\beta\cdot\gamma-\alpha\cdot\delta)} \langle \phi, ES(\gamma-\alpha, \delta-\beta)E\psi \rangle = \text{r.h.s.}$$

And strange as it might seem to be, this is

Q.E.D.

Math 117, April 13 1994

Reminder - gradables should talk to me.

$Q = q \cdot$, $P = -i\hbar \frac{\partial}{\partial q}$ $\quad [P, Q] = i\hbar I$

Prove

$V(P)V(Q) \geq |\langle v_0, [P_1, Q_1] v_0 \rangle + \langle v_0, \{P_1, Q_1\} v_0 \rangle|^2$

Claims
1. P_1, Q_1 are S.A.
2. $\{P_1, Q_1\}$ is SA, 2^{nd} summand is real.
3. $[P_1, Q_1] = i\hbar I$
4. QED.

EPR paradox, EPR form

Gaussians $v_0 \psi_0 = \frac{1}{\sqrt{\pi\sigma}} e^{-\frac{q^2}{2\sigma}} \Rightarrow V(Q) = \frac{\sigma}{2}$

$\hbar = 1 \Rightarrow V(P) = \frac{1}{2\sigma}$

Von-Neumann uniqueness thm: $(\hbar = 1)$

H S.A $\rightarrow V_t = e^{itH}$ is unitary, sat. $V_{t_1+t_2} = V_{t_1} \cdot V_{t_2}$

examples $U_\alpha = e^{i\alpha P}$ $\quad (U_\alpha \phi)(q) = \phi(q+\alpha)$

$V_\beta = e^{i\beta Q}$ $\quad (V_\beta \phi)(q) = e^{i\beta q}\phi(q)$

claim $[P, Q] = iI \Rightarrow U_\alpha V_\beta = e^{i\alpha\beta} V_\beta U_\alpha$

Pf First show $U_\alpha Q U_\alpha^{-1} = Q - i\alpha I$

Math 117, April 18 1994

The Weyl form of the CCR:

$$U_\alpha V_\beta = e^{i\alpha\beta} V(\beta) U(\alpha)$$

Let $S(\alpha,\beta) = e^{-\frac{i\alpha\cdot\beta}{2}} U_\alpha V_\beta$, (1. S is unitary in S; 2. $S(\alpha,\beta)\cdot S(\gamma,\delta) \propto \text{scalar}\cdot S(\alpha+\gamma,\beta+\delta)$)

$$E = \frac{1}{2\pi} \text{``}\iint d\alpha\, d\beta\, e^{-\frac{1}{4}(\alpha^2+\beta^2)} S(\alpha,\beta)\text{''}$$

We will prove:

1. E is an orthogonal projection

2. $\mathrm{im}\,E \neq \{0\}$

3. $\phi,\psi \in \mathrm{im}\,E \Rightarrow$

$$\langle S(\alpha,\beta)\phi, S(\gamma,\delta)\psi\rangle = e^{-\frac{1}{4}(\alpha-\gamma)^2 - \frac{1}{4}(\beta-\delta)^2 + \frac{i}{2}(\beta\gamma-\alpha\delta)} \langle \phi,\psi\rangle$$

4. $\mathrm{im}\,E_0$ is 1-D.

Claim: This proves the von-Neuman uniqueness thm.

Indeed: 1. $\mathrm{span}\{S(\alpha,\beta)\phi : \alpha,\beta\in\mathbb{R},\ \phi\in \mathrm{im}\,E\} = \mathcal{H}$

2. $\mathcal{H} = \mathcal{H}_1 \oplus \dots \mathcal{H}$; $\mathcal{H}_i = \mathrm{span}\{S(\alpha,\beta)v_i\}$

3. Each $\mathcal{H}_i$ is iso to $\mathcal{H}_0$;

map $v_i \xmapsto{L_i} v_0$

$S(\alpha,\beta)v_i \xmapsto{L_i} S(\alpha,\beta)v_0$

Math 117, Apr 20 1994

Comments: 1. Ours is von-Neumann's, but ...

2. Following Mackey, ... Jared

$$E = \frac{1}{2\pi}\int d\alpha\, d\beta\, e^{-\frac{1}{4}(\alpha^2+\beta^2)} S(\alpha,\beta)$$

by a messy computation

$$E S(\gamma,\delta) E = e^{-\frac{1}{4}(\gamma^2+\delta^2)} E \quad \Rightarrow E^2 = E$$

$\Rightarrow \langle S(\alpha,\beta)\phi, S(\gamma,\delta)\psi\rangle = e^{\cdots} \langle \phi,\psi\rangle$

for $\phi,\psi \in E\mathcal{H}$

E is non-zero!

End of statics, begin dynamics. $H(p,q) = \frac{1}{2}(p^2+q^2)$

Heiz. pic.	Schro. pic
$\mathcal{H}, \psi_0$ remain cons.	ψ evolving
~~opt~~ observables evolving	$\mathcal{H}$, observables fixed

$$H = -\frac{1}{2}\frac{\partial^2}{\partial q^2} + \frac{1}{2}q^2 \; ; \; \theta \; ; \; i\frac{d\theta}{dt} = [\theta, H] \Rightarrow$$

$\theta_t = e^{-itH}\theta_0 e^{itH}$ in Heiz. pic.

In sch. pic: $\psi_t = e^{-itH}\psi_0 \quad \frac{d\psi_t}{dt} = -iH\psi$ (sch. eqn)

Eigvalues of H - energy levels. Cons. of energy.

Math 117, Apr 20 1994, cont. $[P,Q] = -iI$

Most important example: $H = \frac{1}{2}(P^2 + Q^2)$

Comment: $[\ ,\ ]$ sat. Leib: $[A,BC] = [A,B]\cdot C + B[A,C]$

$\Rightarrow [P,H] = -iQ,\ [Q,H] = +iP$ (P, Q ~ sin, cos; better look at exp)

set

$a = (Q+iP)/\sqrt{2}$ (annihilation op.) $\quad a^+ = a^* = (Q-iP)/\sqrt{2}$ (creation op)

& get

$[H,a] = -a;\ [H,a^+] = a^+ \qquad [a,a^+] = I$

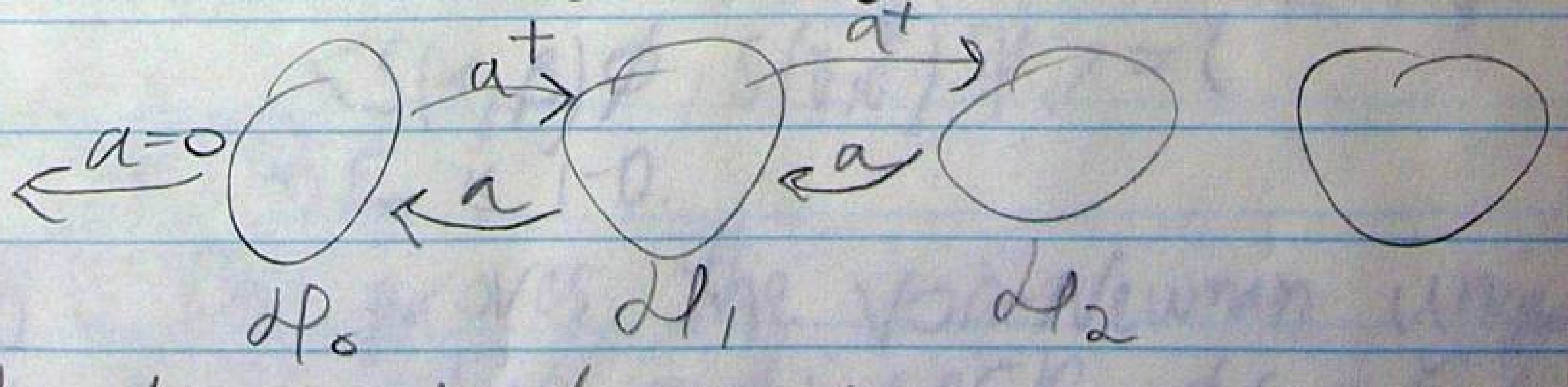
Finally, set $N = a^+a = H - \frac{1}{2}I$ "the number operator".

Let $\mathcal{H}_n = \{\psi \in \mathcal{H} : N\psi = n\psi\}$ $(n \geq 0!)$ (energy $\geq \frac{1}{2}$)!

claim $\quad a\mathcal{H}_n \subset \mathcal{H}_{n-1} \quad a^+\mathcal{H}_n \subset \mathcal{H}_{n+1}$

$\|a\psi_n\| = \sqrt{n}\,\|\psi_n\| \quad \|a^+\psi_n\| = \sqrt{n+1}\,\|\psi_n\|$

Concl: n is a non-negative integer.



"Quantum mechanics"!

enough to understand $\mathcal{H}_0$:

On $L^2(\mathbb{R})$:

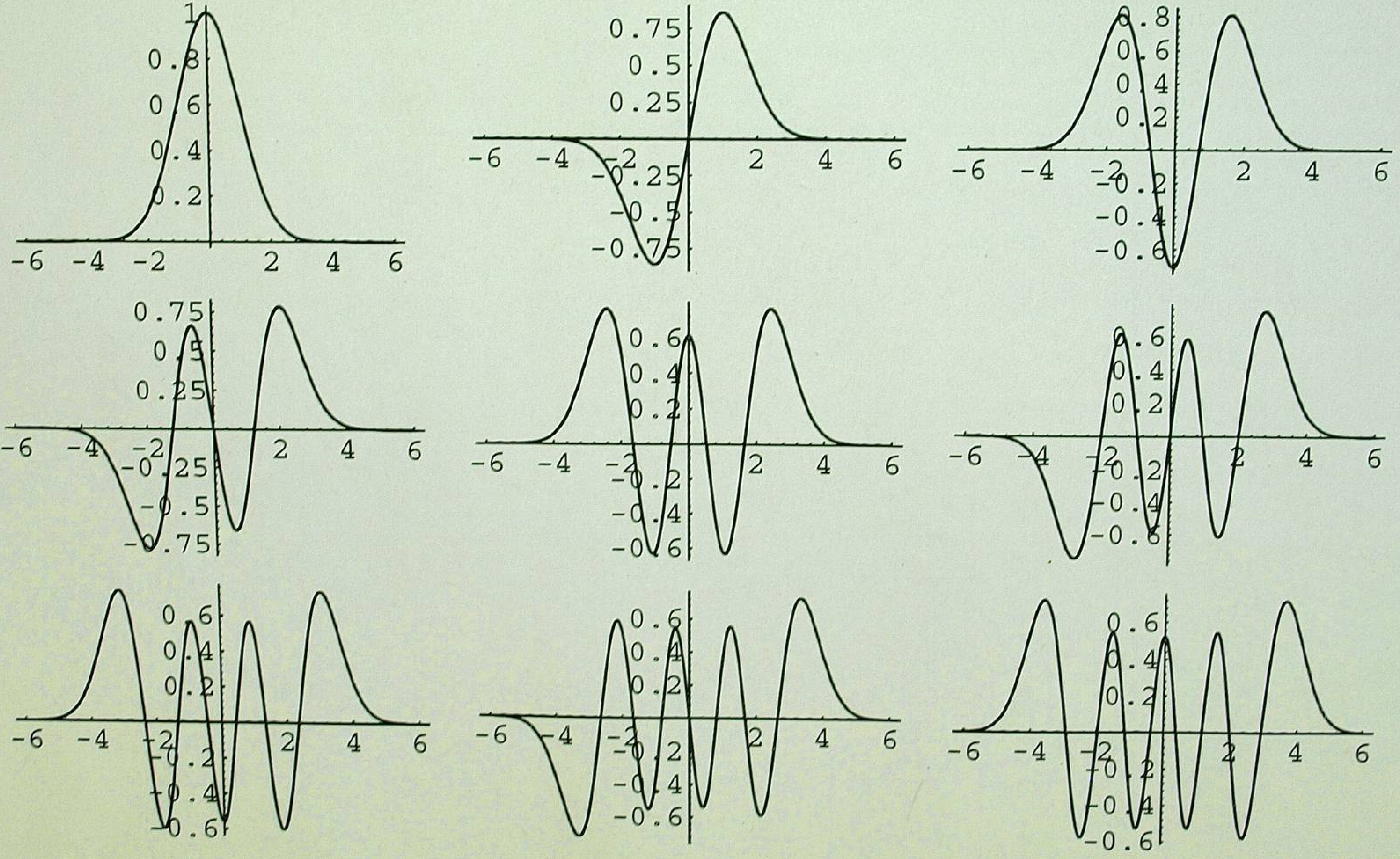
$$\left(q + \frac{\partial}{\partial q}\right)\psi_0 = 0$$

$$\Rightarrow \psi_0 = \frac{1}{\pi^{1/4}} e^{-q^2/2}$$

Hermite Pol.

$$\Rightarrow \psi_n = \frac{1}{\sqrt{2n!}}\left(q - \frac{\partial}{\partial q}\right)^n e^{-\frac{q^2}{2}} = e^{-\frac{q^2}{2}} H_n(q)$$

```
Mathematica 2.2 for SPARC
Copyright 1988-93 Wolfram Research, Inc.

In[1]:= psi[n_] := Simplify[1/Sqrt[2^n n!] *
        Nest[Expand[q*#1 - D[#1, q]] & , Exp[-(q^2/2)], n]
        ]

In[2]:= Do[
        Print["psi[",n,"] = ",psi[n]];
        Print["\n"],
        {n,0,12}]

             2
          -q /2
psi[0] = E


          Sqrt[2] q
psi[1] = ---------
             2
            q /2
           E


                    2
          -1 + 2 q
psi[2] = -------------
                  2
                 q /2
         Sqrt[2] E


                     2
         q (-3 + 2 q )
psi[3] = -------------
                  2
                 q /2
         Sqrt[3] E


                 2      4
         3 - 12 q  + 4 q
psi[4] = ----------------
                    2
                   q /2
         2 Sqrt[6] E


                  2      4
         q (15 - 20 q  + 4 q )
psi[5] = ---------------------
                      2
                     q /2
           2 Sqrt[15] E


                  2       4      6
         -15 + 90 q  - 60 q  + 8 q
psi[6] = -------------------------
                       2
                      q /2
            12 Sqrt[5] E


                           2       4      6
          q (-105 + 210 q  - 84 q  + 8 q )
psi[7] = --------------------------------
                              2
                             q /2
                 6 Sqrt[70] E


                       2        4        6       8
          105 - 840 q  + 840 q  - 224 q  + 16 q
psi[8] = -------------------------------------
                                2
                               q /2
                   24 Sqrt[70] E


                           2        4       6       8
          q (945 - 2520 q  + 1512 q  - 288 q  + 16 q )
psi[9] = ---------------------------------------------
                                   2
                                  q /2
                      72 Sqrt[35] E


                           2         4        6       8       10
           -945 + 9450 q  - 12600 q  + 5040 q  - 720 q  + 32 q
psi[10] = ------------------------------------------------------
                                         2
                                        q /2
                             720 Sqrt[7] E


                               2         4        6       8       10
           q (-10395 + 34650 q  - 27720 q  + 7920 q  - 880 q  + 32 q  )
psi[11] = -----------------------------------------------------------
                                            2
                                           q /2
                              360 Sqrt[154] E


psi[12] =

                         2          4          6         8         10       12
     10395 - 124740 q  + 207900 q  - 110880 q  + 23760 q  - 2112 q   + 64 q
>
   -------------------------------------------------------------------------
                                                 2
                                                q /2
                                 1440 Sqrt[231] E

In[3]:= Show[
        GraphicsArray[Table[Plot[Evaluate[psi[3i+j]],{q,-6,6}],{i,0,2},{j,0,2}]]
        ]
```


1
0.8
0.6
0.4
0.2
-6 -4 -2 2 4 6
0.75
0.5
0.25
-0.25
-0.5
-0.75
-6 -4 -2 2 4 6
0.8
0.6
0.4
0.2
-0.2
-0.4
-0.6
-6 -4 -2 2 4 6
0.75
0.5
0.25
-0.25
-0.5
-0.75
-6 -4 -2 2 4 6
0.6
0.4
0.2
-0.2
-0.4
-0.6
-6 -4 -2 2 4 6
0.6
0.4
0.2
-0.2
-0.4
-0.6
-6 -4 -2 2 4 6
0.6
0.4
0.2
-0.2
-0.4
-0.6
-6 -4 -2 2 4 6
0.6
0.4
0.2
-0.2
-0.4
-0.6
-6 -4 -2 2 4 6
0.6
0.4
0.2
-0.2
-0.4
-0.6
-6 -4 -2 2 4 6

Math 117, Apr 22 1994

Finish Harmonic oscillator:

$$\vdots$$

$$\psi_n = \frac{1}{\sqrt{2^n n!}}\left(q - \frac{\partial}{\partial q}\right)^n e^{-q^2/2} = \underbrace{H_n(q)}_{\text{Hermite}} e^{-q^2/2} \qquad \left(\text{Let } \tilde{\psi}_n = \sqrt{2^n n!}\,\psi_n\right.$$

$$\mathcal{F}\hat{\psi}_n = \mathcal{F}\left(q - \frac{d}{dq}\right)\hat{\psi}_{n-1} = i\left(\frac{d}{dq} - q\right)\mathcal{F}\hat{\psi}_{n-1}$$

$$= -i\left(q - \frac{d}{dq}\right)\mathcal{F}\hat{\psi}_{n-1}$$

$$\Rightarrow \mathcal{F}\psi_n = (-i)^n \psi_n$$

on the other hand

$$e^{-itH}\psi_n = e^{-it(n+\frac{1}{2})}\psi_n = e^{-\frac{it}{2}} e^{-itn}\psi_n$$

$t = \pi/2$

$$= e^{-\frac{i\pi}{4}} e^{-\frac{i\pi n}{2}}\psi_n = (-i)^n \psi_n \cdot e^{-\frac{i\pi}{4}}$$

$\Rightarrow$ our path integral calculation was off

only by a phase!

$H = 2i\theta_3'$ $\quad X^+ = (\theta_1' + i\theta_2')$ $\quad X^- = (\theta_1' - i\theta_2')$

$[H, X^+] = 2X^+ \quad [H, X^-] = -2X^-$

$[X^+, X^-] = -H$

$\Rightarrow$ v_0 max vector eigval λ ; $v_j = \frac{(-1)^j}{j!}(X^-)^j v_0$

$Hv_j = (\lambda - 2j)x_j$

$X^- v_j = -(j+1)v_{j+1}$

$X^+ v_j = (\lambda - j + 1)v_{j-1}$

$j = 0 \ldots \lambda$

Math 117, April 27 1994

Just a little on Rotations & spin

$\{p_i, q_j\} = \delta_{ij} \longrightarrow L^2(\mathbb{R}^3) \oplus \ldots \ ; L^2(\mathbb{R}^3 \longrightarrow \mathbb{C}^n)$

How does one quantize angular momenta?

$$\theta_x = q_y p_z - q_z p_y$$
$$\theta_y = q_z p_x - q_x p_z \qquad (\theta = q \times p)$$
$$\theta_z = q_x \cdot p_y - q_y p_x$$

$\{\theta_x, \theta_y\} = \{q_y p_z - q_z p_y, q_z p_x - q_x p_z\} = q_y p_x - q_x p_y = \theta_z$

$\{\theta_y, \theta_z\} = \theta_x$

$\{\theta_z, \theta_x\} = \theta_y$ $\qquad \{\theta_i, p_i\} = \ldots$

$\qquad \{\theta_j, \theta_i\} = \ldots$

easy quantization:

$$\theta_i \longmapsto Q_j P_k - Q_k P_j \quad (\theta_i')$$

non-unique!

Get $[\theta_i', \theta_j'] = \theta_k'$

Prob: classify reps of $[\theta_i', \theta_j'] = \theta_k'$ in $\mathbb{C}^n$ by anti-self-adjoint matrices.

Soln: One in each dim; (+ direct sums)

1	spin-0	
2	spin-1/2	electrons, !
3	spin 1	

cont. on other side.

$$\frac{d}{dt}E(F_t) = \frac{d}{dt}E\langle F_t\psi_0, \psi_0\rangle = \langle -i[F_0, H]\psi_0, \psi_0\rangle$$

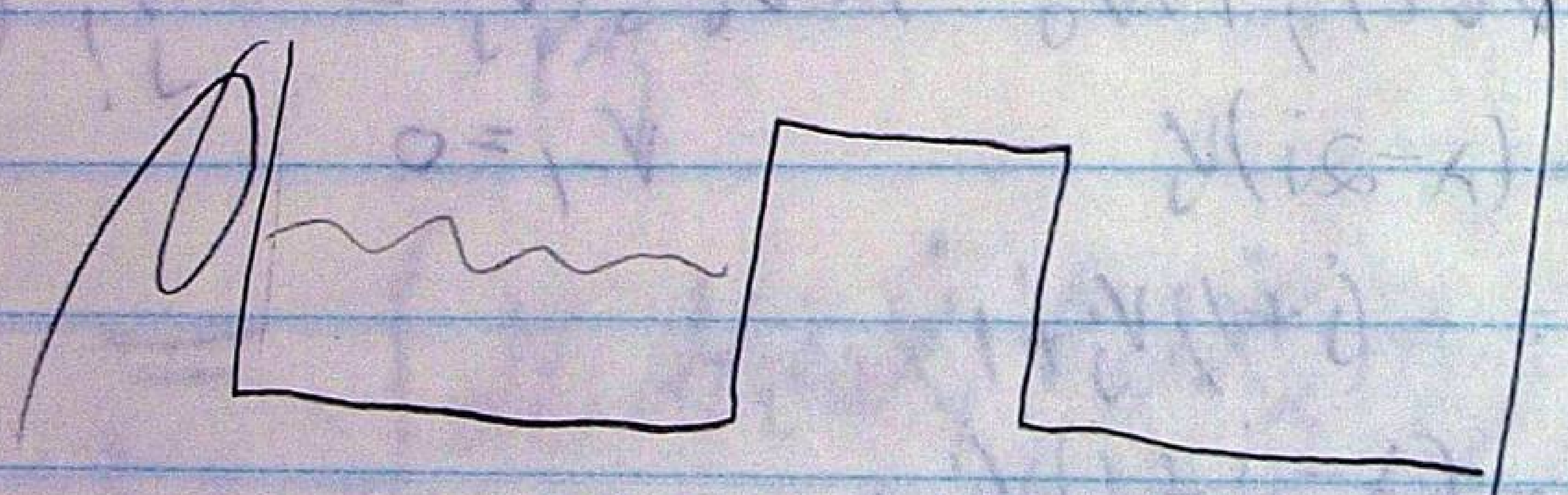
in particular $H = \frac{1}{2m}P^2 + V(Q)$

$$\frac{d}{dt}E(Q) = \langle -i[Q, \tfrac{1}{2m}P^2]\psi_0, \psi_0\rangle = m\langle P\psi_0, \psi_0\rangle = mE(P)$$

$$\frac{d}{dt}E(P) = \qquad = E(V'(Q))$$

Similarly, using path integrals

Tunneling:



Math 117, April 29 1994

Reminder:

$$\theta_i \longmapsto Q_j L_k - Q_k P_j + i\theta_i'$$

$[\theta_i', \theta_j'] = \theta_k'$ reps of that: (No invariant subspace other than $\{0\}$ & V)

1. all $= 0$

2. $\theta_1' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$ $\theta_2 = \begin{pmatrix} 0 & 0 & 1 \\ & & \\ -1 & & \end{pmatrix} \cdots$

$SO(3)!$

$H = 2i\theta_3'$; $X^+ = \theta_1' + i\theta_2'$ $X^- = \theta_1' - i\theta_2'$

$[H, X^+] = 2X^+$ $[H, X^-] = -2X^-$

$[X^+, X^-] = -H$

$\Rightarrow v_0$ max vect, $Hv_0 = \lambda v_0$; $v_j = \frac{(-1)^j}{j!}(X^-)^j v_0$

<u>claim</u> $Hv_j = (\lambda - 2j)v_j$ $v_{-1} = 0$

$X^- v_j = -(j+1)v_{j+1}$

$X^+ v_j = (\lambda - j + 1)v_{j-1}$

<u>Cor</u> if λ is not a natural number, $d\ell$ is infinite d.
otherwise, it is of dim $\lambda + 1$. (or infinite)

~~One~~ Two last examples $\lambda = 1$ (spin $1/2$)

try to compute

$$e^{2\pi i\theta_3'} = e^{-2\pi\theta_3'} = e^{\pi i H} = e^{\pi i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} !$$

---

spin along z: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$; spin along x

$$e^{-\alpha\theta_1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}e^{\alpha\theta_1} = \begin{pmatrix} \cos\frac{1}{2}\alpha & -\sin\frac{1}{2}\alpha \\ \sin\frac{1}{2}\alpha & \cos\frac{1}{2}\alpha \end{pmatrix}\begin{pmatrix} 1 & 0 \\ \cdot & -1 \end{pmatrix}\begin{pmatrix} \cos\frac{1}{2}\alpha & \sin\frac{1}{2}\alpha \\ -\sin\frac{1}{2}\alpha & \cos\frac{1}{2}\alpha \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ \sin\alpha & -\cos\alpha \end{pmatrix}$$

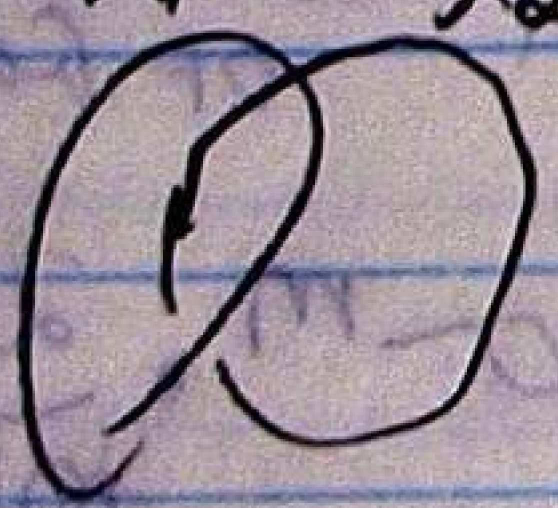
$$\theta_1 = \frac{1}{2}(X^+ + X^-) = \frac{1}{2}\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

similarly,

$$\int_{\{A\}} A_i(x) A_j(y)\, e^{2\pi i \int A \wedge dA} \propto \frac{1}{4\pi}\epsilon_{ijk}\frac{x^k - y^k}{|x-y|^3}$$

$\Rightarrow$

$x_1 : S^1 \to \mathbb{R}^3$, x_2



$$\int_{\{A\}} \left(\int_{x_1} A\right)\left(\int_{x_2} A\right) e^{CS}$$

$$= \int_{\{A\}} \left(\int_{S^1} A_i(x_1(t_1))\cdot \dot{x}_1^i(t_1)\,dt_1\right)\left(\int_{S^1} A_j(x_2(t_2))\cdot \dot{x}_1^j(t_2)\,dt_2\right) e^{CS}$$

$$= \int dt_1\, dt_2\, \epsilon_{ijk}\, \dot{x}_1^i(t_1)\, \dot{x}_2^j(t_2)\, \frac{x_1^k(t_1) - x_2^k(t_2)}{|x_1(t_1) - x_2(t_2)|^3}$$

$$= ?$$

Math 117, ~~April 30~~ May 6 1994

Mon ~~5/9~~ : JP
B.M., Feynman-Kac

Wed 5/11 : Pat
many worlds

FRi 5/13: Joel
strong minima

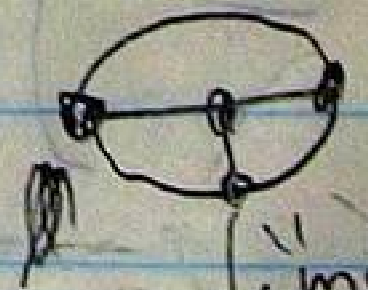
Mon! 5/16: Mark W.
2-body problems.

$\int \mathcal{D}\phi\, e^{-\text{Quad}} = \sum$

$\int x^2 e^{-\frac{x^2}{2\sigma}} dx = \; = \sigma$ (std deviation of Gaussians)

$\int x^{2m} e^{-\frac{x^2}{2\sigma}} dx = \; = \sigma^m \cdot \frac{2m!}{m!\, 2^m}$

$\int (\lambda_{ijk} x^i x^j x^k)^m e^{-\frac{1}{2}\lambda_{ij} x^i x^j} = \underbrace{Y\,Y\,Y}_{m}$ = Fey. Diags.

$\int \mathcal{D}A \left(\int_{\mathbb{R}^3} tr A \wedge A \wedge A \right)^m e^{\int A \wedge dA} =$ 

"messy expr"

vertices $\longrightarrow \int_{\mathbb{R}^3}$, algebra

edges $\longrightarrow (\wedge d)^{-1}$ Huh?

Example: $\Lambda = \frac{1}{2}\frac{d^2}{dx^2}$ on $L^2(\mathbb{R})$

$(\Lambda^{-1} f)(s) = \int k(s,t) f(t)\, dt \quad k(s,t) = |s-t|$

Example in QM

$$\begin{matrix} E(Q(t)P(t+\epsilon)) \\ - E(P(t)Q(t+\epsilon)) \end{matrix} \longrightarrow 1 \text{ instead of } 0 \text{ !}$$

cont. →