Math 134 Course Description

Calculus on Manifolds, Fall 1993

- *Time and place*: MWF 12 noon, Science Center 216.
- *Instructor*: Dror Bar-Natan, Science Center 426G, 5-8797, dror@math.
- *Office hours*: Mondays at 2PM, Wednesdays 1PM, Fridays 11AM.
- *Teaching fellow*: Matteo Paris (paris@math, 3-~~0313~~ 3063).
- *Review sessions*: 8PM Tuesday, Science Center room 309 (tentative). The review sessions will be open ended and students are expected not to leave before all their questions had been answered.
- *Textbooks*: M. Spivak, Calculus on Manifolds (required), and V. Guillemin, A. Pollack, Differential Topology (recommended).
- *Goals*: Develop calculus on differentiable manifolds ("curved spaces") in the framework of differential forms. Hopefully see some good examples for such manifolds.
- *Intended for*: Math and theoretical physics majors. Math 25b graduates have seen most of this material already; graduates of 22b have seen many of the ideas but almost none of the technical details. People that took math 21 *and* any additional 100-level analysis course should do wonderfully here. Those who only took math 21 should be able to survive 134, alas with difficulty.
- *Course plan:* Yet unknown. There are a few possibilities, and only after the first class I will know which of them is the best suited to the audience.

  Things that we *must* know by the end: the general topology of $\mathbf{R}^\mathbf{n}$, continuous functions, derivatives and the differential, the inverse and implicit function theorems, integration of functions, differential form, integration of forms, chains, manifolds, and Stokes' theorem, all as in Spivak's book.
- *Homework* will be assigned weekly and due the following week.
- *Grading*: There is a total of 500 points available to you in this course. During the semester, you can earn up to 350 points: The first midterm can get you up to 150 points, the second midterm is worth 100 points, and the homework assignments are worth an additional 100 points. The final exam then counts for the *difference* between 500 and the number of points you earned during the semester, meaning that at no point along the term do you loose your hope of getting an A for the course, and that at any point along the term you may increase the chance of that happening by *working.* I reserve my right to deviate from this formula in a small number of special cases.
- *Important dates:* First midterm: around October 29th. Second midterm: around November 24th. Final: probably January 21st.

INFORMATION SHEET FOR MATH 134

Name:

Class:

Dorm address:

Electronic mail address:

Dorm phone number:

I want to major in:

I'm taking this class because:

I've taken the following math courses before:

I've taken the following science courses before:

The other math/science courses that I'm taking this term are:

Math 134, Sep 20 1993 "oraganizational meeting"

1. Norm, inner product "the polarization identity"
2. linear trans, matrices, std, basis.
3. Closed & open sets, compact, Heine-Borel Compactness in $\mathbb{R}^n$.
4. Continuity, f^{-1}(open) = open $\cap$ domain, f(compact) = compact; oscillation $o(f,a)$ $\left(= \lim_{\delta \to 0} M(f,a,\delta) - m(f,a,\delta)\right)$

$o = 0 \Rightarrow$ cont, o is "upper semi-continuous".

	Fresh
25+55	13
22	4
21+122,123	2
Freshmen	1
112+115	1

Math 134, Sep 22 1993

No office hours, will rush out.
Is Wed → Wed HW schedule fine?

Math 134 – Course plan:

	week:	contents:
	1	Top review
	2	diff review, inverse
	3	implicit + L, U, meas, content, Integrability thm.
(2 classes)	4	Fubini + Partitions of unity +
	5	
(Midterm Friday)	6	
	7	
	8	
	9	
(Midterm Wed, Free Fri)	10	
	11	
	12	
	13	
(only Monday)	14	

Math 134, Sep 22 1993

Finish compactness—

1. closed subsets of compact sets are compact
2. Products are compact
3. Compactness in $\mathbb{R}^n$ (show only closed bndd $\Rightarrow$ cmpct.)
4. Continuous images of comact sets are compact.

DF(a)
1. Definition (use ~~the small~~ $x=(a+h)$ notation) (very small notation)
2. Uniqueness
3. Constants, linearity
4. the Jacobian matrix, partial derivatives
5. Detailed proof of the chain rule

HW: $\boxed{\text{Read}}$ 1 - last section, pp. 15-22

Do 1.2, 1.7, 1.10, 1.12, 1.17, 1.18, 2.4, 2.5

Math 134, Sep 24 1993

Reminder $f(a+h) = f(a) + (Df)(a)\cdot h + o(h)$ where little o means ---

if $(Df)(a)$ exists, then it is unique.

1. Constants, linear functions, linearity
2. The Jacobian matrix & partial derivatives. $\left(\frac{\partial f}{\partial x_1}, \partial_y f, D_2 f\right.$
3. The chain rule in detail.

<u>Example</u> $\begin{pmatrix} x \\ y \end{pmatrix} \longmapsto \begin{pmatrix} u \\ v \end{pmatrix}$ is "analytic" if $\begin{matrix} u_x = v_y \\ u_y = -v_x \end{matrix}$

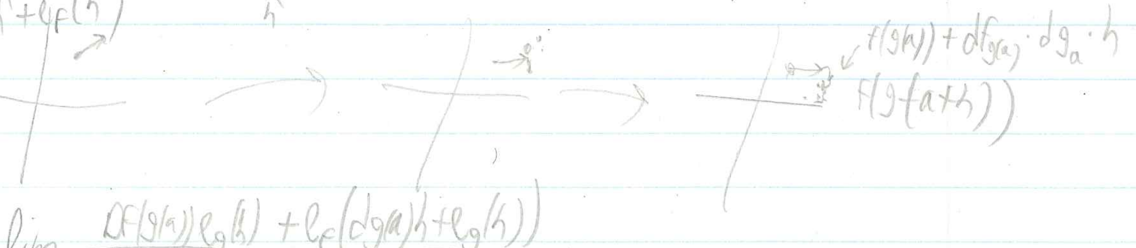
Q prove that the composition of analytic functions is analytic.

1. continuous partials $\Rightarrow$ diffable
2. Cont 2^{nd} partials $\Rightarrow$ commuting.

<u>HW</u>: 1.~~7~~, 10, 18, 2.4, 5, 13, 19, 24

$f(g(a+h)) = f(\underbrace{g(a) + dg(a)h + e_g(h)}_{h'}) = f(g(a)) + Df(g(a))(dg(a)h + e_g(h)) + e_f(dg(a)h + e_g(h))$

$f(a+h') = f(a) + Df(a)h' + e_f(h')$



need: $\lim_{|h|\to 0} \frac{Df(g(a))e_g(h) + e_f(dg(a)h + e_g(h))}{|h|}$

Math 134, Sep 29 1993

Qs about inverse function thm? $(f(x)=y)$

Implicit function thm: "

$$f : \underset{x_0}{V} \times \underset{y_0}{W} \to W \qquad (f(x,y)=0)$$

diffable around, $f(x_0,y_0)=0$; $D_y f\binom{x_0}{y_0}$ invertible

$\Rightarrow \exists$ nbd U of x_0 & $g : U \to W$ s.t. $g(x_0)=y_0$

& $f(x_0, g(x_0))=0$

Pf write $h\binom{x}{y} = \binom{x}{f(x,y)}$. (want $f \circ h = 0$ when

$g(x) = \pi_W \circ h^{-1}\binom{x}{0}$

Similarly, "There's only one kind of 0"

$(n>p)$ $f : \mathbb{R}^n \to \mathbb{R}^p$; diffable around $f(0)=0$; $Df(a)$ has maximal rank

$\Rightarrow \exists\ P : \underset{(\text{nbd of } 0)}{U} \to \mathbb{R}^n$; $P(0)=0$;

s.t. $(f \circ P)(x^1 \dots x^n) = (x^1 \dots x^p)$ $\quad \underbrace{1 \dots p}_{x}\ \underbrace{p+1 \dots n}_{y}$

Pf $h\binom{x}{y} = \binom{f(x,y)}{y}$; $P = h^{-1}$

rectangles, partitions, $m_S(f)$, $M_S(f)$, $L(f,P)$, $U(f,P)$,

Lemma $L(f,P_1) \leq U(f,P_2)$

$L(f)$ $U(f)$ sup & inf over P's, integrability, ϵ-cond., Cantor set & rationals.

Math 134 Oct 1 1993

{mention formula for derivative!}

Partitions (into subrectangles)
"S"

HW: 2-36, 37, 38, 41, 3-3, 7, 8.

Math 134, Oct 4 1993

Proof of thm from prev class.

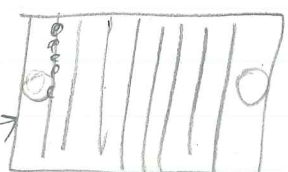
χ_C (& integrability thereof) (even C open may fail)

$\int_C F$

Fubini's: $F: A\times B \to \mathbb{R}$ integrable $\Rightarrow$

$$\int_{A\times B} F = \int_A \left(\genfrac{}{}{0pt}{}{L}{U}\int_B f(x,y)\,dy\right)dx$$

example



Construct on rationals.

example

$$\int = \int\int_{-\sqrt{1-x^2}}^{+\sqrt{1-x^2}}$$

Define partition of unity.

Math 134, Oct 6 1993

Fubini's theorem

$$\int_{A\times B} F = \int_A \left(L\int_B F(x,y)\,dy \right) dx$$

Partitions of unity: $A \subset \mathbb{R}^n$, $\mathcal{U} = \{U_\alpha\}_{\alpha\in I}$ open cover,

$\Rightarrow \exists\ \Phi = \{\varphi_\alpha\}_{\alpha\in I}$; φ_α : open set cont. $A \to [0,1]$ s.t. a C^∞ function,

1. local finiteness
2. $\sum \varphi_\alpha = 1$ on A
3. $\operatorname{Supp} \varphi_\alpha \subset U_\alpha$

"A C^∞ partition of unity for A subordinate to $\mathcal{U}$"

0. A compact, covered by one set U

Pf 1. A compact.
   a. $\mathcal{U} = \{U_1 \dots U_n\}$
   b. $D_i \subset U_i$ compact s.t. $\bigcup D_i \supset A$
   c. $\psi_i \equiv 1$ on D_i, $\operatorname{Supp} \psi_i \in U_i$
   d. $\varphi_i = \dfrac{\psi_i}{\sum \psi_i}$

2. $A = \bigcup A_i$; A_i Compact ; $A_i \subset \operatorname{int} A_{i+1}$

3. A is open

4. A is arbitrary.

Math 134, oct 8 1993

Finish Partitions of unity: $A \subset \mathbb{R}^n$, $\mathcal{V} = \{U_\alpha\}$ open cover,

$\Rightarrow \exists\ \Phi = \{\varphi_\alpha : (\text{nbd of } A) \to [0,1]\}$ s.t.

1. local finiteness
2. $\sum \varphi_\alpha = 1$ on A
3. $\operatorname{supp} \varphi_\alpha \subset U_\alpha$.

middle of the proof

Find $\varphi_{A,V}$ ...

before: Techniques for finding C^∞ on $\mathbb{R}$

After Techniques for assembling these $\varphi_{A,V}$ together.

Step I A compact, many$^{(n)}$ U_i's.

a. shrink the U_i's to get compact D_i's whose interior still covers A

b. Find φ_{D_i,U_i}, compute $\sigma = \sum \varphi_{D_i,U_i}$

c. Set $\varphi_i = F \cdot \frac{\varphi_{D_i,U_i}}{\sigma}$, where $F = \varphi_{A,\{\sigma>0\}}$

Step II $A = \bigcup A_i$; each A_i compact & $A_i \subset \operatorname{int} A_{i+1}$

Step III A is open

Step IV general A.

$\boxed{\text{Integration over open sets.}}$

HW: 3-10, 18, 22, 35, 38 (if integration)

Math 134, Oct 13 1993

Remind Midterm!

1. Integration on open sets.

2. $A \subset \mathbb{R}^n$ open, $g: A \to \mathbb{R}^n$ 1-1 conti. diffable w/ $\det g' \neq 0$ everywhere, $f: g(A) \to \mathbb{R}$ integrable.

$$\int_{g(A)} f = \int_A (f \circ g) |\det g'|$$

We have accumulated enough knowledge to start using it...

A smooth (C^∞) manifold is a (paracompact / compact / countable basis) topological space M together w/ an open cover by sets $\{U_\alpha\}$ and a choice of homeomorphisms $\phi_\alpha : B_n \to U_\alpha$ ...

Math 134, Oct 15 1993

Finish $\mathbb{R}P^n$

A word about the klein bottle  $\subset \mathbb{R}^4$

need to know more about embeddings, thus diffability.

Define diffable; define difeomorphic

Define TM

Define $df: TM \to TN$ for $f: M \to N$.

Define imersion

Define embedding

<u>Thm</u> Every compact Manifold can be embedded in $\mathbb{R}^N$ for some N. $\square$

<u>HW:</u> 3-38 (why doesn't it contradict anything?)

3-41

write RP^n's transition functions explicitly

Midterm

Math 134, Oct 18 1993

Thm every compact manifold can be embedded in $\mathbb{R}^N$ for some N.

Proof, and then complete the definitions of immersion & embedding.

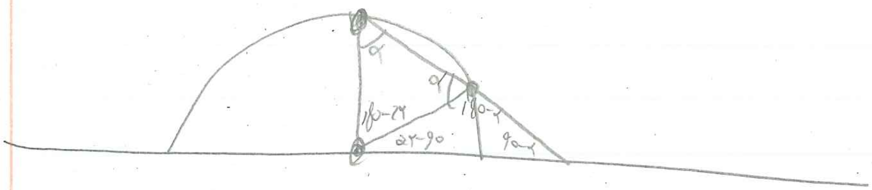
Thm A compact manifold of dimension k can always be embedded in $\mathbb{R}^{2k+1}$

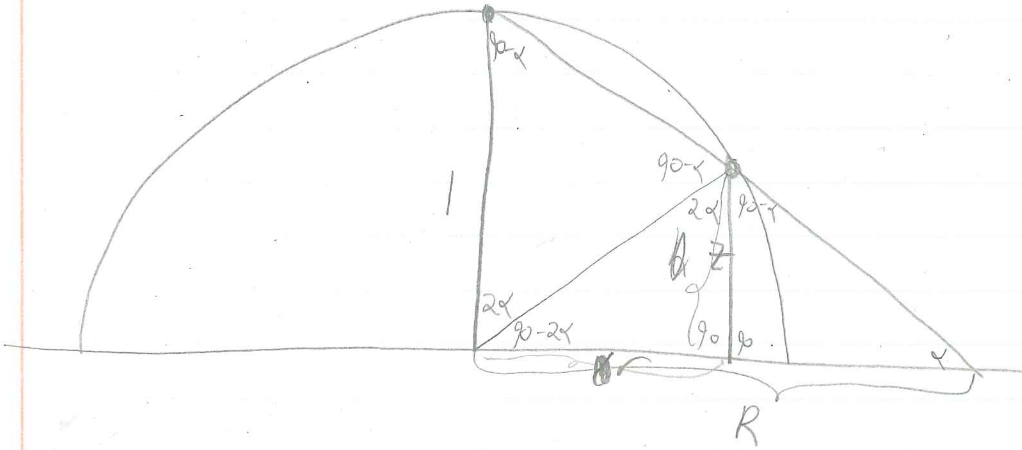
(problems if $\frac{f(x)-f(y)}{t} = a$, or if $df(v) = a$.)

use Sard's theorem: reg value - df is onto at every preimage.

The set of regular values of a smooth map $f: M \to \mathbb{R}^n$ has a complement of measure 0.

Math 134, oct 20 1993





$\frac{R}{1} =$

$$\frac{R-X}{Z} = R \Rightarrow R-X = ZR$$

$$X = (1-Z)R \quad \Rightarrow R =$$

$$R = \frac{X}{1-Z}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} \frac{x}{1-z} \\ \frac{y}{1-z} \end{pmatrix}$$

Math 134 HW, Oct 22 1993

1. Let

$$\mathbb{C}P^n = \{(z_0,\dots,z_n)\in\mathbb{C}^{n+1} : \sum_{i=0}^{n}|z_i|^2>0\} \Big/ \begin{matrix}(z_0,\dots,z_n)\sim\lambda(z_0,\dots,z_n)\\ \text{for all } \lambda\in\mathbb{C}\setminus\{0\}.\end{matrix}$$

Denote the equivalence class of $(z_0,\dots,z_n)\in\mathbb{C}^{n+1}\setminus\{0\}$ in $\mathbb{C}P^n$ by $[z_0,\dots,z_n]$. Set

$(0\le i\le n)$ $\quad U_i = \{[z_0,\dots,\underbrace{1}_{i\text{th slot}},\dots,z_n]\}\subset\mathbb{C}P^n.$

a. Prove that $\mathbb{C}P^n$ with the above open cover is a smooth manifold of dimension $2n$. Compute it transition functions.

b. Let $p=[z_0,z_1,z_2]$ be any point of $\mathbb{C}P^2$. Define $\gamma:\mathbb{R}\to\mathbb{C}P^2$ by:

$$\gamma(t)=[e^{it}z_0,e^{-it}z_1,z_2]$$

Write the vector tangent to γ at $t=0$ in coordinates in all charts of $\mathbb{C}P^2$, and verify that it behaves correctly relative to the differentials of the transition functions.

2. Guillemin-Polack P.P. 55 ex. 7 & 10.

Math 134, Oct 22, 1993

MIDTERM

why are manifolds natural?

Harvard hall 201, Mon Nov 1st, $7^{\underline{30}}$ PM.

Reminder $M = \bigcup U_i = \bigcup_{i=1}^{n} \operatorname{int} D_i$; $\varphi_i : U_i \longrightarrow \mathbb{R}^k$ chart

$\Phi : M \longrightarrow \underbrace{\mathbb{R} \times \dots \times \mathbb{R}}_{n} \times \underbrace{\mathbb{R}^k \times \dots \times \mathbb{R}^k}_{n}$ $\quad \psi_i = \psi_{U_i, D_i}$ $\left(\begin{matrix} =1 \text{ on } D_i \\ 0 \text{ outside } U_i \end{matrix} \right)$

by

$\Phi(x) = \left((\psi_i(x))_{i=1}^{n}, (\psi_i(x)\varphi_i(x))_{i=1}^{n} \right)$

Why is φ_i smooth? Why is ψ_i smooth?

Why is $d\varphi_i$ invertible on D_i?

Silly lemma: $d(f_1, f_2) = (df_1, df_2)$

Φ is an immersion

(counter examples )

Φ is an embedding

Thm A compact manifold of dimension k can always be embedded in $\mathbb{R}^{2k+1}$

(Problems if $\frac{f(x)-f(y)}{t} = a$ or if $df(v) = a$.)

use Sard's thm: (reg value – df is onto at every preimage)

The set of reg. vals of a smooth map $f: M \to \mathbb{R}^n$ has a complement of measure 0.

HW:

Math 112, Oct 25 1993

$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, $\sum \frac{1}{n} = \infty$ what about $\sum \frac{1}{n \log n}$? $\sum \frac{1}{n(\log n)^2}$

$S_n = \sum_{k=1}^{n} a_k$; def of convergence

<u>Thm</u> (Cauchy's crit) $\sum a_n$ converges iff $\forall \epsilon > 0$ $\exists N$ s.t. $\forall\ m \geq n \geq N$ $\quad \left|\sum_{k=n}^{m} a_k\right| \leq \epsilon$

<u>Thm</u> ($\sum a_n$ converges) $\Rightarrow$ ($a_n \to 0$)

<u>Thm</u> A series of non-~~neg~~negative terms converges iff its partial sums are bounded.

"The comparison test": a. If $|a_n| \leq c_n$ for $n \geq N$ and if $\sum c_n$ converges, then $\sum a_n$ converges.

b. $a_n \geq d_n \geq 0$ for $n \geq N$ & $\sum d_n$ diverges $\Rightarrow \sum a_n$ div.

<u>Example</u> $\sum_{n=0}^{\infty} x^n$ $\quad \left(\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x} \text{ for } x \neq 1 \right)$

<u>Thm</u> if $a_1 \geq a_2 \geq a_3 \geq \dots \geq 0$, then $\sum a_n$ converges iff $\sum 2^k a_{2^k}$ converges.

Pf

$2(a_3 + a_4) \geq 2(2a_4) = 4a_4 \geq a_4 + a_5 + a_6 + a_7$ $\qquad \sum 2^k a_{2^k}$

$2(a_4 + \dots a_7) \geq 8a_8 \geq a_8 + \dots + a_{15}$ $\Rightarrow$

<u>Examples</u> $\sum \frac{1}{n^p}$, $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^p}$, $\sum \frac{1}{n \cdot \log n \cdot \log\log n}$

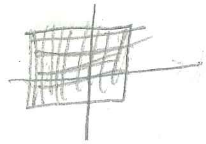
Math 134, Oct 25 1993

Remind midterm.

Def. $f: M \to N$. $y \in N$ is called "regular" if at every $x \in M$ for which $f(x)=y$, $df|_x$ is onto.

Sard's thm $f: \mathbb{R}^m \xrightarrow{\text{smooth}} \mathbb{R}^n$, $\{y \in \mathbb{R}^n : y \text{ isn't regular}\}$ is of measure 0.

Remark this concludes the proof of the "last" theorem.

Proof  if $s \in$ partition intersects $B = \{x : df|_x \text{ isn't onto}\}$ (need to prove $\mu(f(B)) = 0$)

V — v.s. $T: V^k \to \mathbb{R}$ is multilinear, k-tensors,

$T^k(V^*)$ — a v.s. $\otimes: T^k(V^*) \times T^l(V^*) \to T^{k+l}(V^*)$

$\otimes$ is bilinear, associative.

Thm $\dim T^k(V^*) = n^k$ by using dual basis.

$f: V \to W$ induces $f^*: T^k(W^*) \to T^k(V^*)$

V, $T(\cdot,\cdot)$ the $\exists f: V \to \mathbb{R}^n$ s.t. $T(\cdot,\cdot) = f^*\langle\, , \rangle$

$\Lambda^k V^*$; Alt: $T^k(V^*) \to \Lambda^k(V^*)$ $\left(\frac{1}{k!}\sum_{\sigma \in S_k}(-1)^{\sigma} \cdots\right)$

BULLETIN OF THE
ADVANCED MATHEMATICAL SOCIETY
Volume 23, Number 6, November 2001

WHAT EVERY YOUNG MATHEMATICIAN SHOULD KNOW

BY LORD K. ELVIN

ABSTRACT. We evaluate an interesting definite integral.

The purpose of this paper is to call attention to a result of which many mathematicians seem to be ignorant.

THEOREM. *The value of* $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ *is*

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}.$$

PROOF: We have

$$\begin{aligned}
\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 &= \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2}e^{-y^2}\,dx\,dy \qquad \text{by Fubini}\\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy\\
&= \int_{0}^{2\pi}\int_{0}^{\infty} e^{-r^2}\,r\,dr\,d\theta \qquad \text{using polar coordinates}\\
&= \int_{0}^{2\pi}\left[\int_{0}^{\infty} e^{-r^2}\,r\,dr\right]d\theta\\
&= \int_{0}^{2\pi}\left[-\frac{e^{-r^2}}{2}\Bigg|_{r=0}^{r=\infty}\right]d\theta\\
&= \int_{0}^{2\pi}\left[\frac{1}{2}\right]d\theta\\
&= \pi.
\end{aligned}$$

Remark: A mathematician is one to whom *that* is as obvious as that twice two makes four is to you.

INSTITUTE FOR HAUGHTY ATTITUDES

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Homework #4 Handout

Math 134

22 October 1993

3.38 First, some convergence things. We say that a series $\sum_{n=0}^{\infty} a_n$ *converges* if its sequence of partial sums converge. The reason we have to define convergence in this way is the content of the next two definitions. A series $\sum_{n=0}^{\infty} a_n$ converges *absolutely* if it converges and $\sum_{n=0}^{\infty} |a_n|$ also converges. (The second of course implies the first.) For example, any convergent series with all positive terms converges absolutely. On the other hand, a series converges *conditionally* if it converges but $\sum_{n=0}^{\infty} |a_n|$ diverges. The standard example of this is $\sum \frac{-1^n}{n}$. It's a cool fact that a conditionally convergent series can be rearranged to converge to any real number at all.

Therefore, if we choose a partition of unity and want to use it to define the integral of a function f, we have to make sure that $\sum_\alpha \int \phi_\alpha |f|$ converges (see Spivak p. 65), otherwise we're not going to get a well-defined integral: rearranging the order of summation will give a different value.

It can actually happen that for a specific partition of unity $\{\phi_\alpha\}$, the sum $\sum_\alpha \int \phi_\alpha f$ converges absolutely (which means $\sum_\alpha |\int \phi_\alpha f|$ converges), but that the integral still isn't well-defined. In such a case, it doesn't matter how you rearrange the terms in the sum, but the result you get still depends on the partition of unity you chose—an important subtlety! So problem 3.38 shows that to define the integral using partitions of unity, we really do need to be sure that $\sum_\alpha \int \phi_\alpha |f|$ converges, which is the strongest possible statement we could make.

3.41 As I mentioned to many of you, the most important part of this problem was the change o' variables formula. A fun way to do part e is to notice that $B_r \subset C_r \subset B_{\sqrt{2}r}$. From this we get a nice inequality of integrals, which sandwiches when you take the limit.

To show $\mathbf{RP}^n$ is a C^∞ manifold, we just need to specify a set of coordinate charts and show that the transition maps are C^∞. Let $[x_0, ..., x_n]$ be the homogenous coordinates on $\mathbf{RP}^n$. (Think of a point in $\mathbf{RP}^n$ as an equivalence class of points in $\mathbf{R}^{n+1}$, namely those on a line through the origin.). Let U_i be the set in $\mathbf{RP}^n$ where $x_i \neq 0$, and

$$\phi_i([x_0, ..., x_n]) = (\frac{x_0}{x_i}, ..., \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, ..., \frac{x_n}{x_i}).$$

Because of the way we've set up $\mathbf{RP}^n$, ϕ_i is naturally a homeomorphism. The transition function $\tau_{ij} = \phi_j \circ \phi_i^{-1}$ from $\phi_i(U_i)$ to $\phi_j(U_j)$, which is defined only on the intersection, is clearly C^∞:

$$\tau_{ij}(u_0, ..., u_n) = \phi_j([u_0, ..., u_{i-1}, 1, u_{i+1}, ..., u_n]) =$$

$$= (\frac{u_0}{u_j}, ..., \frac{u_{i-1}}{u_j}, \frac{1}{u_j}, \frac{u_{i+1}}{u_j}, ..., \frac{u_{j-1}}{u_j}, \frac{u_{j+1}}{u_j}, ..., \frac{u_n}{u_j}).$$

$((\sigma\tau)T)(v_1,v_2) = (\sigma(\tau T))(v_1,v_2)$

Math 134, Oct 27 1993

k-tensor on V, multilinearity, examples: $\langle\ ,\ \rangle$, the triple product, $\det \in \mathcal{T}^n((\mathbb{R}^n)^*)$

$\mathcal{T}^k(V^*)$ $\quad$ ($\overset{\text{v.s.}}{\mathcal{T}^*(V^*)}$ is a $\overset{\otimes}{\text{ring}}$ (an algebra))

Thm $\{v_i\}_1^n$ basis of V, $\varphi_i(v_j) = \delta_{ij}$

$\{\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k} : 1 \le i_1, \ldots i_k \le n\}$ basis of

$\mathcal{T}^k(V^*)$. Hence $\dim \mathcal{T}^k(V^*) = n^k$

$f: V \to W$, $\quad f^*: \mathcal{T}^k(W^*) \longrightarrow \mathcal{T}^k(V^*)$

The GS thm: If $T \in \mathcal{T}^2(V^*)$ is an inner product, then $\exists\, f: \mathbb{R}^n \to V$ s.t. $f^*T = \langle \cdot, \cdot \rangle$

alternating, $\Lambda^k(V^*)$, Perms, $(\sigma T)(v_i) = T(v_{\sigma i})$ reps, alt in this lan

The signature of a permutation.

$$\mathrm{Alt}\,T = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma (\sigma T)$$

Thm 1. $\operatorname{im} \mathrm{Alt} \subset \Lambda$

2. $\mathrm{Alt}|_\Lambda = I$

Def
$$\omega \wedge \eta = \frac{(k+l)!}{k!\,l!} \mathrm{Alt}(\omega \otimes \eta)$$

Thm bill, pullback, super Associativity, basis

HW: 4.1, 2, 11 $\quad$ 3.39.4

$$((\sigma\tau)T)(v_1 \dots v_k) = T(v_{\sigma\tau 1}, \dots v_{\sigma\tau k})$$

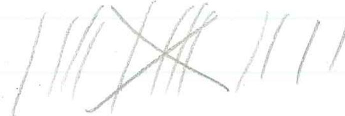
$$(\sigma(\tau T))(v_1 \dots v_k) = (\tau T)(\underset{v_1'}{v_{\sigma 1}} \dots \underset{v_k'}{v_{\sigma k}}) =$$

$$= T(v'_{\tau 1} \dots v'_{\tau k}) = T(v_{\sigma\tau 1} \dots)$$

Math ~~200~~ 134, Nov 1st 1993

Reminder: 1. $\mathrm{Alt}\, T = \frac{1}{k!}\sum_{\sigma\in S_k}(-1)^\sigma(\sigma T)$

2. T is alternating ($T\in\Lambda^k(V^*)$) if $\sigma T = (-1)^\sigma T$

for a σ like 

$$(\sigma\tau)T = \sigma(\tau T) = \sigma((-1)^\tau T) = (-1)^\tau\sigma T = (-1)^\tau(-1)^\sigma T$$
$$= (-1)^{\sigma\tau} T$$

(i.e., can ignore rest)

Thm 1. $\mathrm{im}\,\mathrm{Alt} \subset \Lambda$ (actually, $=$)

2. $\mathrm{Alt}|_\Lambda = Id$

Def $\omega\wedge\eta = \frac{(k+l)!}{k!\,l!}\mathrm{Alt}(\omega\otimes\eta)$

Thm 1. bil

2. Pull back

3. super

4. Associa. (Follows from $T, S \in$ ~~g~~ $\Rightarrow$ $\mathrm{Alt}(T\otimes S) = \mathrm{Alt}((\mathrm{Alt}\,T)\otimes S) = \ldots$

5. basis

Math 134 Midterm

November 1st, 1993

Dror Bar-Natan

You have 120 minutes to solve 5 out of the following 6 equally weighted questions. Plan your time wisely! It is a good idea to read the entire exam before answering any question. You may not use any material other than a pen or a pencil. Don't forget to write your name on anything you submit.

Question 1. (1) $f : U \to \mathbf{R}^m$ is a continuous function, defined in some neighborhood U of a point $a \in \mathbf{R}^n$. Give a precise definition of "f is differentiable at a" and of "the differential of f at a".

(2) State and prove the chain rule for differentiable functions between Euclidean spaces.

Question 2. State clearly the inverse function theorem and the implicit function theorem, and explain in detail how the latter follows from the former.

Question 3. Define "immersion" and give an example of an injective immersion of one manifold into another which is not an embedding.

Question 4. Compute $\int_{-\infty}^{\infty} e^{-x^2/2}dx$, giving a brief statement (i.e., just the main point, don't bother about the precise conditions) of each theorem that you use along the way.

Question 5. A *Riemannian metric* on a a smooth manifold M is a smoothly varying choice of a positive definite inner product on each of its tangent spaces TM_p. (The words "smoothly varying" can be given a precise meaning, but let us not bother about that at this point). Sketch a proof of the fact that every smooth manifold M (or, at least, every compact smooth manifold M) admits a Riemannian metric. *Hint:* any linear combination with positive coefficients of positive definite inner products is a positive definite inner product.

Question 6. The *fake two dimensional complex projective space* $F\mathbf{CP}^2$ is hereby defined as follows:

$$F\mathbf{CP}^2 = (\mathbf{C}^3 \setminus 0) \Big/ \left(\begin{array}{c} (z_0, z_1, z_2) \sim (\lambda z_0, \bar{\lambda} z_1, \lambda z_2) \\ \text{for any } \lambda \in \mathbf{C} \setminus 0 \end{array} \right)$$

(Notice the slight difference from the definition of the usual $\mathbf{CP}^2$, and notice that $\bar{\lambda}$ is the complex conjugate of λ.)

(1) Show that $F\mathbf{CP}^2$ is a 4-dimensional smooth manifold by finding three coordinate charts $U_{0,1,2} \subset F\mathbf{CP}^2$, computing all transition functions between them and showing that these transition functions are all smooth.

(2) Is $F\mathbf{CP}^2$ diffeomorphic to $\mathbf{CP}^2$? In other words, can you find a smooth bijection $\psi : \mathbf{CP}^2 \to F\mathbf{CP}^2$ whose inverse is also smooth?

— GOOD LUCK —

Math 134, Nov 3rd 1993

Reminder Alt T, $\wedge$, bill, asso, super-comm.,

Basis. φ_i basis of $V^* \Rightarrow \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$ $1 \le i_1 < i_2 < \dots < i_k \le n$
basis of $\Lambda^k(V^*)$ $\boxed{\dim}$

Thm $w_i = \sum a_{ij} v_j \Rightarrow$

$$\omega(w_1 \dots w_n) = \det(a_{ij})\, \omega(v_1 \dots v_n)$$

$\boxed{\text{Orientation}}$ $[v_1 \dots v_n]$, $-[v_1 \dots v_n]$; $\Lambda^{\text{top}}(V)$ $\dim V = n$

Volume element given a s.p. & orientation

Cross product. (given a volume ω)

HW: 4.1, 2, 4, 5, 11, 12 3.39

Math 134, ~~Oct~~ Nov 5, 8 1993

$w_i = \sum a_{ij} v_j \Rightarrow w(w_1 \ldots w_n) = \det(a_{ij})\, w(v_1 \ldots v_n)$

Thm $0 \neq w \in \Lambda^{top} V^*$ determines an orientation.

Vol element given a s.p. & an orientation

Cross product given a vol. element.

Pullback.

differential forms on $\mathbb{R}^n$ & on a manifold $(\Omega^k(\mathbb{R}^n))$

$f : \mathbb{R}^n \to \mathbb{R}$, $df \in \Omega^1(\mathbb{R}^n)$

Theorem

$$df = \sum (\partial_i f)\, dx^i$$

Pull backs. (first push forwards)

Thm

$$f^*(dx^i) = \sum \frac{\partial f^i}{\partial x^j} dx^j = d(f^* x^i)$$

linearity, agreement w/ $\wedge$.

Thm $w \in \Lambda^{top}(\mathbb{R}^{n*}) \Rightarrow f^*(hw) = (h \circ f)(\det Df)\, w$

Def d

Thm linearity, Leibnitz, $d^2 = 0$, pullback

HW: 3.39, 4.1, 2, 4, 5, 11, 12

Math 134, Nov 10 1923

do d!

Math 134, Nov 12 1993

Review d

Talk about pullback. Talk about d on M.

example $d(\operatorname{atg} y/x)$

singular n-cube: $c : [0,1]^n \to M$ (or A)

n-chains (over $\mathbb{Z}$)

__bndry:__ $I^n_{(i,\alpha)} : I^{n-1} \to I^n$ by

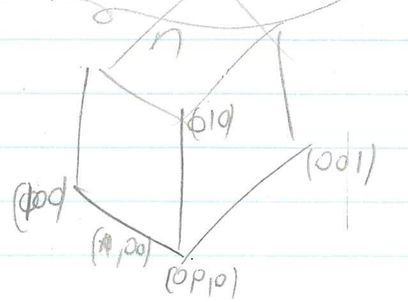
$$(x^1 \ldots x^{n-1}) \longmapsto (x^1 \ldots x^{i-1}, \alpha, x^i, \ldots x^{n-1})$$

$$c_{(i,\alpha)} = c \circ I^n_{(i,\alpha)}$$

$$\partial c = \sum_{i=1}^{n} \sum_{\alpha=0,1} (-1)^{i+\alpha} c_{(i,\alpha)} \quad \text{(extend linearly)}$$

__Theorem__ $\partial(\partial c) = 0$

k-face $\underbrace{(* * * 0 * 0 1 * 1)}_{n}$ k stars.



HW: 4.13, 16, 18, 23, 24

Math 134, Nov 15 1993

Second midterm:
Wed Dec 1st
Final: ~~Dec~~ Jan 21st

Discuss d on M.

n-cubes, n-chains,

Talk about cubes:

k-face: $(* \ldots 0, \ldots *, 1)$ k-stars

$(0,0,0)$

Each k-face comes w/ a natural parametrization by I^k;
i.e. w/ a natural orientation.

$$\overset{\text{naive}}{\partial} f = \sum \ldots$$

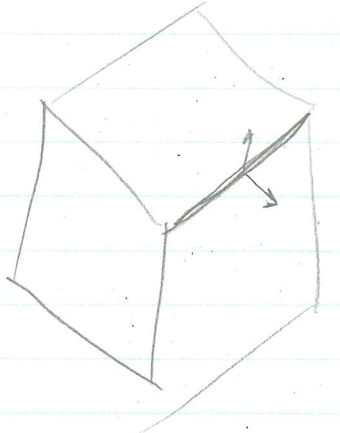
$$\partial f = \sum_{F} \pm \begin{pmatrix} \text{replace one of} \\ \text{the } * \text{ in} \\ \text{by } \alpha \in \{0,1\} \end{pmatrix} = \sum_{i=1}^{k} \sum_{\alpha=0,1}^{1} \overset{(-1)^{i+\alpha}}{\pm} f_{i,\alpha}$$

<u>claim</u> $\partial(\partial f) = 0$.

back to text: $I^n_{(i,\alpha)} : I^{n-1} \to I^n$ by

1. Finish ∂
2. Define $\int$
3. Prove baby Stokes'.

Math 134, Nov 17 1993



1. each face has a "god given" par & orientation. (just read $*$'s in order)
2. That orientation may or may not agree w/ the natural orie as a body.

$\partial(\partial f) = 0$

Concretely, $\partial c = \sum_{i,\alpha} (-1)^{i+\alpha} c \circ I_{i,\alpha}$ $\quad I_{(i,\alpha)}$ parametrizes $(* \dots \underset{i}{\alpha} \dots *)$

$$\int_c w \triangleq \int \cancel{\frac{c^* w}{dx^1 \wedge \dots \wedge dx^k}}$$

means nothing!

$$\int_{\partial c} w = \int_c dw$$

Stokes' thm:

1. May as well restrict to the cube.
2. may as well take $w = F\, dx^1 \wedge \dots \wedge \widehat{dx^i} \wedge \dots \wedge dx^k$

$dw = (-1)^{i-1} \frac{\partial f}{\partial x^i} dx^1 \wedge \dots \wedge dx^k$ $\qquad \int_c dw = (-1)^{i-1} \int_{[0,1]^k} \frac{\partial f}{\partial x^i}$

$$\int_{\partial c} w = (-1)^{i+1} \int_{(**\underset{i}{1}**)} F + (-1)^i \int_{(**\underset{i}{0}**)} F$$

Notice implicit use of Fubini!

Math 134, Nov 19 1993

$$\int_D dw = \int_{\partial D} w$$

Example $\displaystyle\int_{[0,1]^3} d\left[\left(\sin \frac{\pi x}{y+1}\right) dx \wedge dz\right] =$

Problem: As is, integration depends on the parametrization, not just on the domain

Definition: orientation preserving map $\mathbb{R}^n \to \mathbb{R}^n$
orientation for a manifold.
(orientable → has two pos. orientations.
may be non-orientable

orientable $\Leftrightarrow$ ~~smooth~~ cont. choice of of a ~~class~~ class of basis in each TM_p

Examples: T^2, S^2, $\mathbb{RP}^2$, $\mathbb{CP}^2$

Thm $c_{1,2}: [0,1]^k \to M^k_{\text{oriented}}$ orientation preserving,

$w \in \Omega^k(M)$, $w = 0$ outside $\operatorname{im} c_1 \cap \operatorname{im} c_2$

$\Rightarrow \int_{c_1} w = \int_{c_2} w$

~~Thm~~ allows defining $\int_M w$ on manifolds!

Pf

HW: 4.31, 33, 34.

Math 134, Nov 22 1993

Def Oriented manifold, orientable manifold

orientation preserving maps.

Each orientable manifold has exactly two orientations.

Examples T^2, S^2, $\mathbb{RP}^2$, $\mathbb{CP}^2$ (every holomorphic manifold is orientable)

Thm A manifold is orientable iff there exist a nowhere-vanishing top form on it.

No choice of a basis on S^2 !

(While proving add "admitting partitions of unity")

Thm if $\varphi_i : B^n \to U_i \subset M$ $(i=1,2)$ are charts, and $\omega \in \Omega^n(M)$ is 0 outside of $\varphi_1(B^n) \cap \varphi_2(B^n)$, then $\int_{B^n} \varphi_1^* \omega = \int_{B^n} \varphi_2^* \omega$

Pf: enough to show (f 1-1 & loc. inv. & $\det(df) > 0$)

on $\int_{B^n} f^* \omega = \int_{B^n} \omega$

Conclusion $\int_M \omega$ [M orientable] makes sense if M admits...

1. how
2. independence of P.O.U

Def Man. w/ bndry. Model $[X_1 \leq 0]$

Math 134, Nov 24 1993

what should we do next?

Possibilities:
1. Guillemin-Pollack
2. Bott-Tu
3. other...
4. Lawson; very hard.

Commented Oriented $\Leftrightarrow$ Cont. varying orientation on $(TM)_p$

Thm $F: \mathbb{R}^n \to \mathbb{R}^n$ 1-1, $\det dF > 0$ at every pt, ω w/ compact support

$$\int \omega = \int F^* \omega.$$

Def $\int_M \omega$ Prove indep.

Def M w/ bndry; model $[x_1 \geq 0]$.

M orientable (oriented) $\Rightarrow$ ∂M orientable (oriented)

Stokes' thm $\int_M d\omega = \int_{\partial M} \omega$ (M compact oriented !)

(Thm is certainly not true for non-compact mflds!)

Math 134, Nov 29 1993

Midterm at 12noon exactly! be there a few minutes before.

Any question about the midterm?

1. M, ∂M, boundaryness is well defined.
2. The two ways of orienting a boundary agree.
3. Pf of Stokes' thm: 1. First for a single patch
   2. then in general.

$$\int_{M} d\omega \ \ldots \int_{\partial M} \omega =$$

Math 134 Midterm

December 1st, 1993

Dror Bar-Natan

You have 60 minutes to solve the following 4 questions, whose total value is 100 points. Plan your time wisely! It is a good idea to read the entire exam before answering any question. You may not use any material other than a pen or a pencil. Don't forget to write your name on anything you submit.

Question 1. (25 points) Let V be an n-dimensional vector space, and let ϕ be some fixed non-zero linear functional in $V^\star$. For any $0 \leq k < n$ define a map $e_\phi^k = e_\phi : \Lambda^k V^\star \to \Lambda^{k+1} V^\star$ (elsewhere called *exterior multiplication by* ϕ) by $e_\phi \omega = \phi \wedge \omega$. Notice that these maps e_ϕ form a nice chain

$$\Lambda^0 V^\star \xrightarrow{e_\phi} \Lambda^1 V^\star \xrightarrow{e_\phi} \Lambda^2 V^\star \xrightarrow{e_\phi} \ldots \xrightarrow{e_\phi} \Lambda^n V^\star,$$

similar to the chain

$$\Omega^0 V \xrightarrow{d} \Omega^1 V \xrightarrow{d} \Omega^2 V \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n V$$

mensioned in class.

(1) Prove that the e_ϕ chain is "a complex". I.e., prove that $e_\phi \circ e_\phi = 0$. Notice that this implies that $\operatorname{im} e_\phi^{k-1} \subset \ker e_\phi^k$.
(2) Prove that the e_ϕ complex is "exact". I.e., that for any $0 < k < n$, $\operatorname{im} e_\phi^{k-1} = \ker e_\phi^k$.

Question 2. (25 points) Define a map $\pi_1 : \mathbf{R}^3 \to \mathbf{R}$ by

$$\pi_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = x_2 - x_3,$$

and then define the two maps $\pi_{2,3} : \mathbf{R}^3 \to \mathbf{R}$ by cyclicly permuting the indices in the definition of π_1; $\pi_2 = x_3 - x_1$ and $\pi_3 = x_1 - x_2$. Let $\omega \in \Omega^1(\mathbf{R}_x)$ be defined by

$$\omega = d(\log x).$$

(1) Compute $\omega_i = \pi_i^\star \omega$ for $i = 1, 2, 3$.
(2) Compute $\omega_1 \wedge \omega_2 + \omega_2 \wedge \omega_3 + \omega_3 \wedge \omega_1$. If the answer you got is more complicated than the expression you started with, try again!

Question 3. (25 points) Explain in some detail how integration of top forms on an oriented manifold is defined, why it is well-defined and where exactly is orientability used in the process of integration.

Question 4. (25 points) Let $\omega \in \Omega^2 \mathbf{R}^3$ be the form

$$x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy,$$

and let λ be the pullback of ω to the sphere S^2 by the standard embedding of S^2 in $\mathbf{R}^3$ as the unit sphere. Compute $\int_{S^2} \lambda$. You may use Stokes' theorem (in any form) if you so wish, but you don't have to.

— GOOD LUCK —

Math 134 Midterm

December 1st, 1993

Dror Bar-Natan

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$$\Lambda^0 V^\star \xrightarrow{e_\phi} \Lambda^1 V^\star \xrightarrow{e_\phi} \Lambda^2 V^\star \xrightarrow{e_\phi} \ldots \xrightarrow{e_\phi} \Lambda^n V^\star,$$

similar to the chain

$$\Omega^0 V \xrightarrow{d} \Omega^1 V \xrightarrow{d} \Omega^2 V \xrightarrow{d} \ldots \xrightarrow{d} \Omega^n V$$

mensioned in class.

(-3) didn't choose basis explicitly 10 (1) Prove that the e_ϕ chain is "a complex". I.e., prove that $e_\phi \circ e_\phi = 0$. Notice that this implies that $\operatorname{im} e_\phi^{k-1} \subset \ker e_\phi^k$.

15 (2) Prove that the e_ϕ complex is "exact". I.e., that for any $0 < k < n$, $\operatorname{im} e_\phi^{k-1} = \ker e_\phi^k$.

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$$\omega = d(\log x).$$

10 (1) Compute $\omega_i = \pi_i^\star \omega$ for $i = 1, 2, 3$.

15 (2) Compute $\omega_1 \wedge \omega_2 + \omega_2 \wedge \omega_3 + \omega_3 \wedge \omega_1$. If the answer you got is more complicated than the expression you started with, try again! (-10) for not doing so.

Question 3. (25 points) Explain in some detail how integration of top forms on an oriented manifold is defined, why it is well-defined and where exactly is orientability used in the process of integration.

13: right thing, lost in formulas.

Question 4. (25 points) Let $\omega \in \Omega^2 \mathbf{R}^3$ be the form

$$x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy,$$

and let λ be the pullback of ω to the sphere S^2 by the standard embedding of S^2 in $\mathbf{R}^3$ as the unit sphere. Compute $\int_{S^2} \lambda$. You may use Stokes' theorem (in any form) if you so wish, but you don't have to.

— GOOD LUCK —

Math 134, Dec 6 1993.

1. Finish off Stokes'

2. Green's thm $M \subset \mathbb{R}^2$ compact 2D man w/ bndry, $\alpha, \beta: M \to \mathbb{R}$ diffable

$$\int_{\partial M} \alpha\, dx + \beta\, dy = \int_M \left(\frac{\partial \beta}{\partial x} - \frac{\partial \alpha}{\partial y}\right) dx$$

Pf: use $w = \alpha\, dx + \beta\, dy$.

Reminder

$$\begin{array}{ccccccc} \Omega^0(\mathbb{R}^3) & \xrightarrow{d} & \Omega^1(\mathbb{R}^3) & \xrightarrow{d} & \Omega^2(\mathbb{R}^3) & \xrightarrow{d} & \Omega^3(\mathbb{R}^3) \\ \text{Fun} & \xrightarrow{\text{grad}} & \text{V.F.} & \xrightarrow{\text{curl}} & \text{V.F.} & \xrightarrow{\text{div}} & \text{Fun} \\ & & \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} & & & & \end{array}$$

$\underbrace{\text{Thm}}^{\text{(Div thm)}}$ $M \subset \mathbb{R}^3$ cmpct 3D man w/ bndry, n - unit outward normal

$$\int_M (\operatorname{div} F)\, dV = \int_{\partial M} (F \cdot n) \cdot dA$$

$$dA(v_1, v_2) = \|(n \times v_1 \cdot v_2)\|$$

Pf Take $w = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy =$

$$= \det\left(F \,\middle|\, \begin{matrix} dx \\ dy \\ dz \end{matrix} \,\middle|\, \begin{matrix} dx \\ dy \\ dz \end{matrix}\right)$$

Thm (Stokes') $M \subset \mathbb{R}^3$ cmpct 2D oriented man w/ bndry, n - unit normal det by orn, F V.F.

$$\int_M (\operatorname{curl} F \cdot n)\, dA = \int_{\partial M} F \cdot T\, ds$$

Math 134, Dec 8 1993

Finish stokes' ? where was compactness used?

$$\Omega^0 \xrightarrow{d} \Omega^1 \xrightarrow{d} \Omega^2 \xrightarrow{d}$$

$\Omega^2 \supseteq Z^2 = \ker d \supseteq \operatorname{im} d = B^2$

$H^2 = Z^2 / B^2$

How big is H^2?

* integrating things in Z^k on boundaries gives 0.
* integrating things in B^k on anything depends only on its bndry, and if there's no bndry, the integral is 0

S, High groups, H^0.

Examples: 1. $ye^{xy}dx + xe^{xy}dy$ on $\mathbb{R}^2$

2. $\frac{x\,dy}{x^2+y^2} - \frac{y\,dx}{x^2+y^2}$ on $\mathbb{R}^2 - \{0\}$

3. dt on $S^1 = \mathbb{R}/\mathbb{Z}$

4 $dx \wedge dy$ on $T^2 = \mathbb{R}^2/\mathbb{Z}^2$

Problem: Compute $H^k(M)$ in general

we're after proving $H^k(\mathbb{R} \times M) = H^k(M)$

Functoriality, $(g \circ f)^* = f^* \circ g^*$

Math 134, Dec 10 1993.

Reminder: $H^k_{dR}(M) = \ker d/\operatorname{im} d$ = a subquotient of $\Omega^k(M)$

"Categorical mathematics": $f: M\to N \Rightarrow f^*: H^k(N)\to H^k(M)$

f^* is well defined ; $(g\circ f)^* = f^*\circ g^*$

We'll after: $H^k(M\times\mathbb{R}) = H^k(M)$; in particular,

$$H^k(\mathbb{R}^n) = H^k(pt) = \begin{cases}\mathbb{R} & k=0\\ 0 & \text{otherwise}\end{cases}$$

claim dim $H^0(M)$ = # of connected components of M

Def $P:\Omega^k(\mathbb{R}_t\times\mathbb{R}^n_{x_1\dots x_n}) \longrightarrow \Omega^{k-1}(\mathbb{R}_t\times \mathbb{R}^n_{x_1\dots x_n})$ by

$$\sum_{|I|=k-1} f_I(t,x)\,dt\wedge dx^I + \sum_{|J|=k} g_J(t,x)\,dx^J \xmapsto{P} \sum_I \left(\int_0^t f_I(s,x)\,ds\right)dx^I$$

claim If $\Phi = id\times\phi$, then $\Phi^* P\omega = P\Phi^*\omega$

cor P is well defined for manifolds (algebraic proof — use i_v)

geometrical interpretation

claim $\pi:\mathbb{R}\times M\to M$, $i_0: M\to \mathbb{R}\times M$ (hope — show that these are inverses in cohom)

$$dP\omega + Pd\omega = \omega - \pi^* i_0^*\omega$$

Thm i_0^* and π^* are inverses of each other.

HW: 5.31–33; do as much as you can; attempt ~~to do~~ everything the rest will be discussed in class.

Math 134, Dec 13 1923

$W = \sum_{|I|=k-1} f_I \, dt \wedge dx^I + \sum_{|J|=k} g_J \, dx^J \quad PW = \sum_I \left(\int_0^t f_I(s,x)\, ds \right) dx^I$

Suppres summations!!

1. PW is well defined on manifolds

$\Phi^* PW = P \Phi^* W \quad$ if $\Phi = \mathrm{id} \times \phi$

2. $dPW + PdW = W - \pi^* i^* W$ Compute....

Corollary: $H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k=0 \\ \{0\} & \text{otherwise} \end{cases}$

cor 2. Homotopy invariance of deRham Cohomology for maps

cor 3 For spaces.

cor Cohom of contractible spaces.

The Mayer-Victoris sequence: $M = U_1 \cup U_2$ (open contractible sets)

$\cdots \to H^k(U_1) \oplus H^k(U_2) \to H^k(U_1 \cap U_2) \to H^{k+1}(M) \to H^{k+1}(U_1) \oplus H^{k+1}(U_2) \to \cdots$

is exact!

cor: $H^k(S^n)$

Leonid
CUE

Math 134, Dec 15 1993

~~The Mayer-Vietoris sequence; $M = U_1 \cup U_2$ ~~contract~~ open sets~~

~~$\to H^k(U_1) \oplus H^k(U_2) \to H^1$~~

Finish Dec 13 1993
(recheck Mayer-Vietoris)

Math 134, Dec 17 1993

Reading period timetable:

Wd 1/12 : Class as usual, review session — prepare questions
FRI 1/14 : Possible overflow review
TUe 1/18 : Matteo's review session (time & place as usual)
WED 1/19 : " office hours, 4PM @ greenhouse
Thu 1/20 : Joint office hours 3PM my office
FRI 1/21 2:15 PM, Harvard Hall 103, Final
All along I will hold office hours as usual, but they're mainly intended for math 20 students.

$$\to H^k(M) \xrightarrow[\alpha]{(i_1^*,\, i_2^*)} H^k(U_1)\oplus H^k(U_2) \xrightarrow[\beta]{j_1^*-j_2^*} H^k(U_1\cap U_2) \xrightarrow[\delta]{w \mapsto \underbrace{d(\rho_1 w)}_{\text{on } U_2} = -\underbrace{d(\rho_2 w)}_{\text{on } U_1}} H^{k+1}(M) \xrightarrow{(i_1^*,\, i_2^*)} H^{k+1}(U_1)\oplus H^{k+1}(U_2) \to$$

$$\begin{tikzcd}
 & \underset{\rho_1}{U_1} \arrow[dr, "i_1"] & \\
U_1\cap U_2 \arrow[ur, hook, "j_1"] \arrow[dr, hook, "j_2"'] & & M \\
 & \underset{\rho_2}{U_2} \arrow[ur, hook, "i_2"'] &
\end{tikzcd}$$

Dec 20: 1. well definedeness of δ

2. Exactness at $\oplus$ done
3. Exactness at $H^k(U_1\cap U_2)$:
   a. im $\subset$ ker : easy
   im $\supset$ ker

Math 134 Final

January 21st, 1994

Dror Bar-Natan

You have 180 minutes to solve 6 of the following equally weighted 8 questions (but notice that there are restrictions on your choice). Plan your time wisely! It is a good idea to read the entire exam before answering any question. You may not use any material other than a pen or a pencil. Don't forget to write your name on anything you submit.

Do exactly one of questions 1 and 2:

Question 1. It is well known to every beginning topologist that $\dim H^k_{dR}(S^{17}) = 0$ for all k except 0 and 17, and that $\dim H^0_{dR}(S^{17}) = \dim H^{17}_{dR}(S^{17}) = 1$. Use this fact as well as a cleverly set Mayer-Vietoris sequence to compute the de-Rham cohomology of S^{18}. Justify all the steps of your computations by referring to lemmas and theorems proven in class.

Question 2.

$$\begin{tikzcd}
 & {} \arrow[d, "d"'] & {} \arrow[d, "d"'] & {} \arrow[d, "d"'] & \\
0 \arrow[r] & A^{n-1} \arrow[r, "\alpha"] \arrow[d, "d"'] & B^{n-1} \arrow[r, "\beta"] \arrow[d, "d"'] & C^{n-1} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & A^{n} \arrow[r, "\alpha"] \arrow[d, "d"'] & B^{n} \arrow[r, "\beta"] \arrow[d, "d"'] & C^{n} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & A^{n+1} \arrow[r, "\alpha"] \arrow[d, "d"'] & B^{n+1} \arrow[r, "\beta"] \arrow[d, "d"'] & C^{n+1} \arrow[r] \arrow[d, "d"'] & 0 \\
0 \arrow[r] & A^{n+2} \arrow[r, "\alpha"] \arrow[d, "d"'] & B^{n+2} \arrow[r, "\beta"] \arrow[d, "d"'] & C^{n+2} \arrow[r] \arrow[d, "d"'] & 0 \\
 & {} & {} & {} &
\end{tikzcd}$$

In the above diagram,

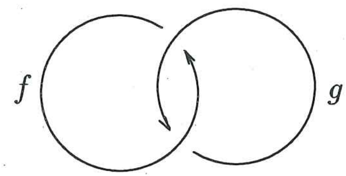
- the squares are commutative,
- the rows are exact,
- and going down two steps in a column gives 0 ($d \circ d = 0$).

Define a map $\delta : H^n(C) \to H^{n+1}(A)$, explaining along the way why:

(1) The image of δ is really in $H^{n+1}(A)$.
(2) δ is independent of a choice made in A^{n+1}.
(3) δ is independent of a choice made in B^n.
(4) δ depends only on a class in $H^n(C)$, and not on a particular representative of it.

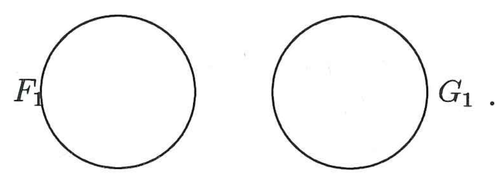
Do exactly one of questions 3 and 4:

Question 3.



Explain why are the two circles f and g in the above figure are *linked.* I.e., explain why there does not exist a pair of smooth homotopies $F_t, G_t : S^1 \to \mathbf{R}^3$ ($t \in [0,1]$) for which

- For each fixed $t \in [0,1]$, F_t and G_t are both embeddings of S^1 in $\mathbf{R}^3$, and their images are disjoint.
- The pair (F_0, G_0) is the pair (f,g) in the above figure.
- The pair (F_1, G_1) is described by the figure below:



You don't need to show computational details, but the plan of your proof should be clear and complete.
Hint: A pair (F_t, G_t) as above defines a map $(F_t - G_t) : [0,1] \times S^1 \times S^1 \to \mathbf{R}^3 - 0$. Use this map to pull back the closed form

$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2+y^2+z^2)^{3/2}} \in \Omega^2\left(\mathbf{R}^3 - 0\right),$$

integrate, and try to get useful information regarding the integrals on the two boundaries of that cylinder. Now re-evaluate these integrals by other means; the integral I_2 for the second figure can be done explicitly without computation, and the integral I_1 for the first figure came out to be 4π on my computer. Why is it relevant? You can earn some extra credit by proving that my computer was actually right this time.

Question 4. Prove that if M is a smooth oriented compact n-dimensional manifold with no boundary, then both $\dim H^0_{dR}(M)$ and $\dim H^n_{dR}(M)$ are at least 1.

Do all of the following questions (5-8):

Question 5. Formulate and prove the "chains" ("cubes") version of Stokes' theorem. You may assume as known all the necessary definitions and lemmas regarding differentiation, integration, forms, orientations, etc.

Question 6. Explain in some detail why every compact smooth manifold of dimension n can be embedded in $\mathbf{R}^{2n+1}$. (You are allowed to skip technical details, but the main points should be clear).

Question 7. Let V be an n-dimensional vector space, and let v be some fixed non-zero vector in V. For any $0 < k \leq n$ define a map $i_v : \Lambda^k V^\star \to \Lambda^{k-1} V^\star$ (elsewhere called *interior multiplication by* v) by $(i_v \omega)(v_1, \dots, v_{k-1}) = \omega(v, v_1, \dots, v_{k-1})$. Notice that these maps i_v form a nice chain

$$\Lambda^n V^\star \xrightarrow{i_v} \Lambda^{n-1} V^\star \xrightarrow{i_v} \Lambda^{n-2} V^\star \xrightarrow{i_v} \dots \xrightarrow{i_v} \Lambda^0 V^\star.$$

Prove that this chain of maps is exact.

Question 8. Let $F : \mathbf{R}^n \to \mathbf{R}^n$ be a smooth map whose differential at 0 is the identity ($df|_0 = I_{n\times n}$). Prove that the image of any neighborhood of 0 via F contains some neighborhood of $F(0)$. You are *not* allowed to use the inverse function theorem as this statement is a lemma used in the proof of that theorem.

— GOOD LUCK —

Math 134 Final

January 21st, 1994

Dror Bar-Natan

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In the above diagram,

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6 Define a map $\delta : H^n(C) \to H^{n+1}(A)$, explaining along the way why: 3-way 3-possibility ability to invert.

5 (1) The image of δ is really in $H^{n+1}(A)$.

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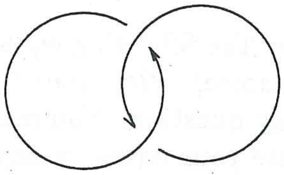
f  g

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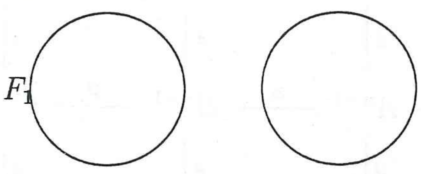
F_1  G_1 .

You don't need to show computational details, but the plan of your proof should be clear and complete.

5 : using stokes!

Hint: A pair (F_t, G_t) as above defines a map $(F_t - G_t) : [0,1] \times S^1 \times S^1 \to \mathbf{R}^3 - 0$. Use this map to pull back the closed form

$$\omega = \frac{x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy}{(x^2+y^2+z^2)^{3/2}} \in \Omega^2\left(\mathbf{R}^3 - 0\right),$$

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8 17

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easy: 8
hard part: 17

Question 7. Let V be an n-dimensional vector space, and let v be some fixed non-zero vector in V. For any $0 < k \leq n$ define a map $i_v : \Lambda^k V^* \to \Lambda^{k-1} V^*$ (elsewhere called *interior multiplication by* v) by $(i_v \omega)(v_1, \dots, v_{k-1}) = \omega(v, v_1, \dots, v_{k-1})$. Notice that these maps i_v form a nice chain

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— GOOD LUCK —

Professor is Dror Bar-natan

Enrollment is 12

Obs	Variable	Label	Mean	Std Dev
11	INTEREST	stimulated interest in the subject matte	3.91	0.83
	READING	quality of the reading	3.00	1.00
	WORKLOAD	course workload overall	3.27	0.65
	COMPLETE	fraction of work completed	4.45	0.69
	DIFFICUL	difficulty overall	3.73	0.90
	ATMOS	competitive atmosphere	2.18	0.75
	PACE	pace of course overall	3.45	0.93
	OVERALL	overall course rating	3.55	0.82
	APPROP	written assignments were well chosen	3.64	1.12
	HELPFUL	comments were helpful	3.91	1.04
	PROMPT	were returned promptly	4.09	0.54
	PACLEAR	gave clear well structured presentations	3.64	1.12
	PAQUEST	answered questions well	3.73	1.10
	PAPARTIC	encouraged participation	3.45	1.21
	PAAVAIL	was available outside class	3.70	1.16
	PABLACK	used the blackboard well	3.36	1.29
	PAOVER	overall rating for Dror Bar-Natan	3.73	1.01
	ATTENDED	fraction attended	2.55	1.04
	ORGANIZE	sections were well integrated	3.78	1.20
	CONTRIB	sections contributed to course meaning	3.37	1.30
	LAUNDER	understands subject matter	4.36	0.92
	LACLEAR	gave clear presentations	4.11	0.93
	LAEFFECT	was an effective discussion leader	3.78	0.97
	LAQUEST	answered questions well	3.80	0.92
	LAPARTIC	encouraged participation	3.89	0.93
	LAAVAIL	was available outside class	4.09	0.94
	LABLACK	used blackboard well	3.78	1.09
	LAOVER	overall rating for Matteo Paris	4.10	0.88

Professor is Dror Bar-Natan

stimulated interest in the subject matte

INTEREST	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
3	1	9.1	2	18.2
4	7	63.6	9	81.8
5	2	18.2	11	100.0

quality of the reading

READING	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	4	36.4	4	36.4
3	4	36.4	8	72.7
4	2	18.2	10	90.9
5	1	9.1	11	100.0

course workload overall

WORKLOAD	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
3	6	54.5	7	63.6
4	4	36.4	11	100.0

fraction of work completed

COMPLETE	Frequency	Percent	Cumulative Frequency	Cumulative Percent
3	1	9.1	1	9.1
4	4	36.4	5	45.5
5	6	54.5	11	100.0

Professor is Dror Bar-Natan

difficulty overall

DIFFICUL	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
3	3	27.3	4	36.4
4	5	45.5	9	81.8
5	2	18.2	11	100.0

competitive atmosphere

ATMOS	Frequency	Percent	Cumulative Frequency	Cumulative Percent
1	2	18.2	2	18.2
2	5	45.5	7	63.6
3	4	36.4	11	100.0

pace of course overall

PACE	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	2	18.2	2	18.2
3	3	27.3	5	45.5
4	5	45.5	10	90.9
5	1	9.1	11	100.0

overall course rating

OVERALL	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
3	4	36.4	5	45.5
4	5	45.5	10	90.9
5	1	9.1	11	100.0

Professor is Dror Bar-Natan

written assignments were well chosen

APPROP	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	2	18.2	2	18.2
3	3	27.3	5	45.5
4	3	27.3	8	72.7
5	3	27.3	11	100.0

comments were helpful

HELPFUL	Frequency	Percent	Cumulative Frequency	Cumulative Percent
1	1	9.1	1	9.1
4	8	72.7	9	81.8
5	2	18.2	11	100.0

were returned promptly

PROMPT	Frequency	Percent	Cumulative Frequency	Cumulative Percent
3	1	9.1	1	9.1
4	8	72.7	9	81.8
5	2	18.2	11	100.0

gave clear well structured presentations

PACLEAR	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	2	18.2	2	18.2
3	3	27.3	5	45.5
4	3	27.3	8	72.7
5	3	27.3	11	100.0

Professor is Dror Bar-Natan

answered questions well

PAQUEST	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	2	18.2	2	18.2
3	2	18.2	4	36.4
4	4	36.4	8	72.7
5	3	27.3	11	100.0

encouraged participation

PAPARTIC	Frequency	Percent	Cumulative Frequency	Cumulative Percent
1	1	9.1	1	9.1
2	1	9.1	2	18.2
3	3	27.3	5	45.5
4	4	36.4	9	81.8
5	2	18.2	11	100.0

was available outside class

PAAVAIL	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	2	18.2	2	18.2
3	2	18.2	4	36.4
4	3	27.3	7	63.6
5	3	27.3	10	90.9
NA	1	9.1	11	100.0

used the blackboard well

PABLACK	Frequency	Percent	Cumulative Frequency	Cumulative Percent
1	1	9.1	1	9.1
2	2	18.2	3	27.3
3	2	18.2	5	45.5
4	4	36.4	9	81.8
5	2	18.2	11	100.0

Professor is Dror Bar-Natan

overall rating for Dror Bar-Natan

PAOVER	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
3	4	36.4	5	45.5
4	3	27.3	8	72.7
5	3	27.3	11	100.0

fraction attended

ATTENDED	Frequency	Percent	Cumulative Frequency	Cumulative Percent
1	2	18.2	2	18.2
2	3	27.3	5	45.5
3	4	36.4	9	81.8
4	2	18.2	11	100.0

sections were well integrated

ORGANIZE	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	2	18.2	2	18.2
3	1	9.1	3	27.3
4	3	27.3	6	54.5
5	3	27.3	9	81.8
NA	2	18.2	11	100.0

sections contributed to course meaning

CONTRIB	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	3	27.3	3	27.3
3	1	9.1	4	36.4
4	2	18.2	6	54.5
5	2	18.2	8	72.7
NA	3	27.3	11	100.0

Professor is Dror Bar-Natan

understands subject matter

LAUNDER	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
4	4	36.4	5	45.5
5	6	54.5	11	100.0

gave clear presentations

LACLEAR	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
4	5	45.5	6	54.5
5	3	27.3	9	81.8
NA	2	18.2	11	100.0

was an effective discussion leader

LAEFFECT	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
3	2	18.2	3	27.3
4	4	36.4	7	63.6
5	2	18.2	9	81.8
NA	2	18.2	11	100.0

answered questions well

LAQUEST	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
3	2	18.2	3	27.3
4	5	45.5	8	72.7
5	2	18.2	10	90.9
NA	1	9.1	11	100.0

Professor is Dror Bar-Natan

encouraged participation

LAPARTIC	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
3	1	9.1	2	18.2
4	5	45.5	7	63.6
5	2	18.2	9	81.8
NA	2	18.2	11	100.0

was available outside class

LAAVAIL	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
3	1	9.1	2	18.2
4	5	45.5	7	63.6
5	4	36.4	11	100.0

used blackboard well

LABLACK	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	2	18.2	2	18.2
4	5	45.5	7	63.6
5	2	18.2	9	81.8
NA	2	18.2	11	100.0

overall rating for Matteo Paris

LAOVER	Frequency	Percent	Cumulative Frequency	Cumulative Percent
2	1	9.1	1	9.1
4	6	54.5	7	63.6
5	3	27.3	10	90.9
NA	1	9.1	11	100.0

Professor is Dror Bar-Natan

AFFILIAT	Frequency	Percent	Cumulative Frequency	Cumulative Percent
H/R	10	90.9	10	90.9
GSAS	1	9.1	11	100.0

Professor is Dror Bar-Natan

YEAR	Frequency	Percent	Cumulative Frequency	Cumulative Percent
freshman	2	18.2	2	18.2
sophomore	2	18.2	4	36.4
junior	5	45.5	9	81.8
senior	2	18.2	11	100.0

Professor is Dror Bar-Natan

SEX	Frequency	Percent	Cumulative Frequency	Cumulative Percent
F	1	10.0	1	10.0
M	9	90.0	10	100.0

Frequency Missing = 1

Professor is Dror Bar-Natan

REASON	Frequency	Percent	Cumulative Frequency	Cumulative Percent
elective	1	11.1	1	11.1
concentration re	4	44.4	5	55.6
elective within	4	44.4	9	100.0

Frequency Missing = 2