Math 131 Course Description

Topology, Spring 1992

Time and place: 9AM (sorry) MWF, Emerson 108.
Instructor: Dror Bar-Natan, Science Center 426G, 5-8797, dror@math.
Office hours: Monday at noontime, Wednesday 10AM, Friday 1PM.
Teaching fellow: Jason Fulman (fulman@math, 3-0313).
Review sessions: 8PM Tuesday, Science Center room 309 (tentative). The review sessions will be open ended and students are expected not to leave before all their questions had been answered.
Textbooks: J. R. Munkres, Topology — a first course, and W. S. Massey, A Basic Course in Algebraic Topology.
Goals: Understand the foundations of the notion of "continuity" — understand "topological spaces". Play a little with ultrafilters — not-too-useful but very pretty gadgets that live on and also make topological spaces, and then study some mathematically more useful topological creatures — surfaces, fundamental groups, and covering spaces.
Intended for: Math major, and anyone else who likes math for its own sake.
Prerequisites: Knowing well what $\{\}, \cap$ and $\cup$ are, having seen $\forall, \epsilon, \exists$, and δ (preferably in this order), and knowing what the quotient of a group by a normal subgroup is.
Course plan: The plan is quite ambitious. We'll try to do it all, but we might have to settle with a somewhat smaller subset:

	M	W	F	Topics
Feb		3	5	Topological spaces and continuous functions, bases, closed sets,
	8	10	12	subspaces, product spaces, quotient spaces, metric spaces.
	XX	17	19	
	22	24	26	Connectedness, connected components, the intermediate value theorem.
Mar	1	3	5	Compactness, compactness in metric spaces, the maximal value theorem.
	8	10	12	The axiom of choice, ultrafilters, compactness and the Stone-Cech
	15	17	19	compactification, Tychonoff's and Hindemann's theorems.
	ME	24	26	ME stands for Midterm Examination; Baire category.
Apr	XX	XX	XX	Spring recess - no classes.
	5	7	9	Two-dimensional manifolds.
	12	14	16	The fundamental group, the circle, the Brouwer fixed point theorem.
	19	21	23	Van-Kampen's theorem, the complement of the trefoil knot.
	26	28	30	A little about covering spaces.
May	3	5	7	Reading period - I plan to finish everything before that,
	10	12	14	but plans are there only so that they can be changed later.

Homework will be assigned weekly and due the following week. Homework assignments will be hard and challenging.

Grading: There is a total of 600 points available to you in this course. During the semester, you can earn up to 450 points: The midterm (ME) can get you up to 150 points, the homework assignments are worth an additional 150 points, and during the semester you *may* be able to earn up to 150 additional points by collecting various prizes for solving various *hard* extra problems that I will assign. The final exam then counts for the *difference* between 600 and the number of points you earned during the semester, meaning that at no point along the term do you loose your hope of getting an A for the course, and that at any point along the term you may increase the chance of that happening by *working*. I reserve my right to deviate from this formula in a small number of special cases.

INFORMATION SHEET FOR MATH 131

Name:

Class:

Dorm address:

Electronic mail address:

Dorm phone number:

I want to major in:

I'm taking this class because:

I've taken the following math courses before:

I've taken the following science courses before:

The other math/science courses that I'm taking this term are:

~~Spring~~ Math 131 – Spring 1993

Fringe

~~Questions~~: Side benefits

Main questions:
1. What is continuity?
2. Classify objects up to continuous maps.

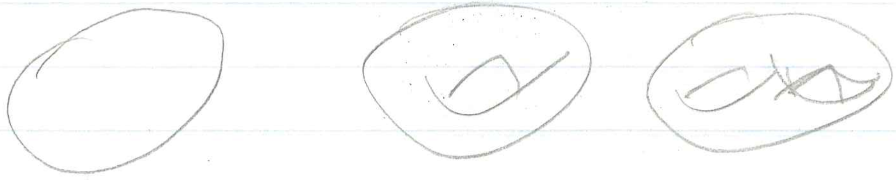
1. Is there an effective algorithm that given a set of real numbers chooses one of its numbers?

2. Color the natural numbers w/ 17 diffent colors. Is it alway possible to find ~~a triple~~ two natural number a & b st. . . .

3. Hindemann's

4. Is there a cont. onto map $\phi : \mathbb{R} \longrightarrow \mathbb{R}^2$?

5. Is there a nowhere differentiable ~~cont~~ every where cont. ~~real~~ function $F : \mathbb{R} \rightarrow \mathbb{R}$?

6.  . . .

are there any other 2-D surfaces?

7. Can you devise a theory of infinite integers?

Math 131, Feb 3 1993

Read "weekly". Add Fringe benefits

Define continuity for $f: \mathbb{R} \to \mathbb{R}$

$$\forall x_0 \in \mathbb{R}\ \forall \epsilon > 0\ \exists \delta > 0\ \forall x \in \mathbb{R},\ |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$$

<u>idea</u> express in terms of "open sets", Find properties of open sets, abstractize.

<u>Definition</u> a set $V \subset \mathbb{R}$ is open if
$x_0 \in V \Rightarrow \exists \epsilon$ s.t. $\forall x \in \mathbb{R},\ |x - x_0| < \epsilon \Rightarrow x \in V$

<u>example</u> (a,b) is open, $[0,1)$ is not open.

<u>Thm</u> f is cont. $\Leftrightarrow \forall U \subset \mathbb{R},\ (U \text{ is open}) \Rightarrow (f^{-1}(U) \text{ is open})$

<u>pf</u> $\Rightarrow \ldots \Leftarrow$

<u>Thm</u> 1. $\emptyset$ is open, $\mathbb{R}$ is open
2. Arbitrary unions of open sets are open $\left(\forall \alpha \in I\ U_\alpha \text{ open}\right) \Rightarrow \bigcup_{\alpha \in I} U_\alpha$ is open
3. finite intersections of open sets are open
$(U_1, \ldots, U_n \text{ open}) \Rightarrow \bigcap_{i=1}^{n} U_i$ is open

<u>Definition</u> a topological space is a set X together with a set $\mathcal{O}$ (called "a topology") of subsets of X, s.t.

<u>Definition</u> $f: X \to Y$ cont iff

H.W. understand everything I said
refresh 1.1-1.7, read 2.1

Math 131, Feb 5 1923

incomplete logic of prev. time was:

We want to study continuity

f cont $\Leftrightarrow$ (U open $\Rightarrow$ $f^{-1}(U)$ is open)

opennes is important,

open sets have a list of properties.

Re-define "topology on a set X"

$(X, \mathcal{T})$ - top. space, $U \in \mathcal{T} \Leftrightarrow$ "U is open"

Examples: 1. $\mathcal{T} = \{\emptyset, X\}$ - the trivial topology

2. $\mathcal{T} = 2^X$ - the discrete topology

3. $\mathcal{T} = \{U \subset \mathbb{R}^n : U$ is open is the usual sense$\}$ -

most fund. top $\Rightarrow$ has no name $\Rightarrow$ top of $\mathbb{R}^n$

<u>Definition</u> A basis $\mathcal{B}$ for a topology on a set X is a collection $\mathcal{B} \subset 2^X$ s.t.

1. $\forall x \in X \ \exists B \in \mathcal{B}$ s.t. $x \in B$
2. $(B_{1,2} \in \mathcal{B}, x \in B_1 \cap B_2) \Rightarrow \exists B_3 \in \mathcal{B}$ s.t. $x \in \mathcal{B}_3 \subset B_1 \cap B_2$

<u>Examples</u> 1. $\{(a, b)\}$ are bases for topologies ~~$\mathcal{T}_1$ & $\mathcal{T}_2$~~ on $\mathbb{R}$

2. $\{[a, b)\}$ 3. open balls is basis in $\mathbb{R}^n$

<u>Thm</u> 1. X set, $\mathcal{B}$ basis a. There exists a minimal topology $\mathcal{T} \supset \mathcal{B}$

b. $U \in \mathcal{T} \Leftrightarrow \forall x \in U \ \exists B \in \mathcal{B}$ s.t. $x \in B \subset U$

2. X set, $\mathcal{T}$ topology then a sub collection $\mathcal{B}^{c\mathcal{T}}$ is a basis for $\mathcal{T}$ iff

$\forall U \in \mathcal{T} \ \forall x \in U \ \exists B \in \mathcal{B} \ x \in B \subset U$

claim $\mathcal{T}_1 \supsetneq \mathcal{T}_2$ ~~$\mathcal{T}$~~ or $\mathcal{T}_1$ is finer / stronger / larger than $\mathcal{T}_2$, which is coarser / weaker / smaller than $\mathcal{T}_1$

HW 1 Read 1.1–1.7, 2.1, 2 Do 2.2 Q's 1, 3, 4, 6, 7.

Math 131, Feb 8 1993

1. Tom's review session
2. Reading will be my policy
3. books, ~20

Topics: examples, making new spaces from old ones, closed sets.

2.3 Let X be a simply-ordered set. The order topology on X is the top. whose basis are the sets (a,b), $[a_0, b)$, $(a, b_0]$

simply ordered
1. $x \neq y \Rightarrow x<y$ or $y<x$
2. never $x<x$
3. $x<y,\ y<z \Rightarrow x<z$

Examples: 1. $\mathbb{R}$ 2. $\mathbb{Z}$ 3. $\dots$ 4. $\mathbb{R}\times\mathbb{R}$ in dictionary order.

X, Y, want a top. on $X\times Y$. Recall $f: Z \to W$ is cont if $f^{-1}(\text{open})$ is open.

We want

$$X\times Y \xrightarrow{\pi_X} X, \quad X\times Y \xrightarrow[\text{cont.}]{\pi_Y} Y \qquad f,g: Z \to X, Y \text{ cont} \Rightarrow f\times g: Z \to X\times Y \text{ cont.}$$

2.4 **Thm** There is a unique topology $\mathcal{T}_{X\times Y}$ on $X\times Y$ for which π_X, π_Y & $f\times g$ are continuous for every cont. f, g. $\boxed{\text{write down}}$ A basis for it is the collection $\{U\times V : U\subset X \text{ is open},\ V \text{ is open in } Y\}$

Pf. 1. subbasis & basis 2. $(f\times g)^{-1}(\text{base set})$ 3. $\mathcal{T}_{X\times Y}$ does the job.
4. uniqueness: $Z = (X\times Y)_{\mathcal{T}_2} \xrightarrow[\pi_X\times\pi_Y]{id} (X\times Y)_{\mathcal{T}_1}$ is cont.

X top space, $Y\subset X$ want a topology $\mathcal{T}_Y$ on Y s.t.
$i: Y \hookrightarrow X$ is cont, & whenever

$$\begin{tikzcd} & Z \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, hook, "i"] & & X \end{tikzcd}$$

commutes, g cont $\Rightarrow f$ cont. $(g = i\circ f)$ explain!

2.5 <u>claim</u> 1. such a $\mathcal{T}_Y$ exist & is unique
2. $\mathcal{T}_Y = \{Y\cap U : U \in \mathcal{T}_X\}$
3. $\mathcal{T}_Y$ is "well behaved"
i.e. $A\subset X$, $B\subset Y$, the two ways of defining $\mathcal{T}_{A\times B}$ agree.
$Z\subset Y\subset X \dots$

pf: exercise.

2.6 <u>closed sets</u>

HW: Read 2.3

Math 131, Feb 10 1993

Reminder - Abstractness will remain about same.

2.5, Fast

X top space, $Y \subset X$ (set), want a topology $\mathcal{T}_Y$ on Y. (basis of $X \times Y$)

$$\mathcal{T}_Y = \{ Y \cap U : U \in \mathcal{T}_X \}$$

Thm $\mathcal{T}_Y$ is the unique topology on Y for which

1. $i : Y \hookrightarrow X$ is cont,
2. if $g : Z \to X$ satisfies $g(Z) \subset Y$, then $g : Z \to Y$ is continuous

Thm 1. $Z \subset Y \subset X$
2. $A \times B \subset X \times Y$
3. $(a, b) \subset X$ (ordered)

2.6

Closed sets; the 3 props of closedness; skip thing related to the subspace topology.

$\mathrm{Int}_{\mathcal{T}} A = \mathring{A}$ $\quad \mathrm{Cl}_{\mathcal{T}} A = \bar{A}$

Thm $x \in \bar{A}$ iff every neighborhood of x intersects A iff every basic neigh...

Limit points if $x \in \overline{A - \{x\}}$, x is called a limit point

Thm $\bar{A} = A \cup A'$, but union is not nec. disjoint.

ex 1. $A = \{\frac{1}{n}\}$ $\quad$ 2. $A = \mathbb{Q}$

Hausdorff spaces: Definition. ex 1. $\mathbb{R}$ is Haus.
2. $\mathcal{B} = \{(a, \infty)\}$ not Haus.

Thm Singletons are closed in a Haus. space

Thm X Haus, $x \in A' \iff$ every neighborhood of x intersects A in infinitely many pts.

Math 131, Feb 12 1993

Reminder: $x \in \bar{A}$ iff every nbd of x intersects A

2.6
Limit pts: if $x \in \overline{A - \{x\}}$, x is called a limit pt.
Thm $\bar{A} = A \cup A'$, union not nec. disjoint.

Hausdorff spaces Definition. ex. 1. $\mathbb{R}$ is Haus.
Thm Basic ops, preserve Haus'ness 2. $\mathcal{D} = \{(a, \infty)\}$ isn't.
Thm singletons are closed in a Haus. space
Thm X Haus, $x \in A' \iff$ every nbd on x intersect A in infinitely many pts.

2.7
Cont. functns:
TFAE: 1. f is cont. 4. Chk on a basis.
2. $f(\bar{A}) \subset \overline{f(A)}$ 5. Cont @ a point.
3. f^{-1}(closed) is closed.

Thm f, g cont $\Rightarrow f \circ g$ cont, but f cont $\not\Rightarrow f^{-1}$ cont.

Definition X & Y are call homeomorphic if $\exists f: X \to Y$ cont s.t. f^{-1} exists and is also cont.

2.9
Metric $d: X \times X \to \mathbb{R}$ 1. Positivity
2. symmetry
3. $\Delta \leq$

$B_d(x, \epsilon)$, This is a basis.

HW: Read, do 2.7. 2, 4, 7, 8 (supp 82), 10, 14

MATH 131 — PRIZES! PRIZES! PRIZES!

DROR BAR-NATAN

February 17, 1993

Remember, during term each student may accumulate up to 150 points by solving various (normally rather hard) problems and collecting prizes. Here are the first two such problems.

(1) 50 points to be awarded to the first few who solve this problem, provided all the solutions are original. When (and if) the solution will become general knowledge, I will remove this problem from the list.
Let R be a finite closed rectangle in the plane, and let $f : R \to \mathbf{R}^2$ be a distance non-increasing map — a function satisfying $d(f(x), f(y)) \leq d(x, y)$ for every $x, y \in R$, where d is the standard Euclidean distance function. Is it always the case that the length of the boundary of the image $f(R)$ of R under f is smaller than the length of the boundary of R? Prove or give a counterexample. If you don't feel like playing with distance non-increasing maps, feel free to consider shadows cast by folded envelopes.

(2) 100 points to anyone who convinces me that she/he *fully* understands the proof of the following theorem, which I think is one of the most amazing theorems in mathematics: Any continuous function $f : \mathbf{R}^2 \to \mathbf{R}$ is a finite composition of functions $\phi_i : \mathbf{R} \to \mathbf{R}$ and the function $+ : \mathbf{R}^2 \to \mathbf{R}$. (For example, $xy = e^{\log x + \log y}$ and $x/y = e^{\log x + (-\log y)}$). This theorem is the solution of one of the famous 23 problems Hilbert posed in the 1900 conference. It is due to Kolmogorov, and finding a reference is a part of your challenge. The proof is beautiful and uses no more than what we've studied, but it's not quite obvious.

Math 131, Feb 17 1993

Askabout Massey.

<u>Thm</u> f, g cont. ⇒ f∘g cont, but f^{-1} might not be so. (example)

2.7 <u>Definition</u> X & Y are called homeomorphic if

$\exists\, f: X \to Y$ - - - -

<u>Example</u>

<u>Thm</u> $+: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $\cdot: \mathbb{R}\times\mathbb{R} \to \mathbb{R}$, $\div: \mathbb{R}\times(\mathbb{R}-\{0\}) \to \mathbb{R}$

are continuous.

(⇒ $f+g$, fg, f/g are cont.)

on $\prod_{\alpha\in I} X_\alpha = \{(p_\alpha)_{\alpha\in I} : p_\alpha \in X_\alpha\} = \{p: I \to \bigcup X_\alpha : p(\alpha)\in X_\alpha \ \forall \alpha\in I\}$

there are two natural topologies:

<u>generalize the topology</u>

2.8 $\mathcal{B} = \{\prod U_\alpha : U_\alpha \subset X_\alpha\}$

The Box topology

<u>exercise</u> show that the box top does not satisfy reqs.

<u>generalize the requirements</u>

$\mathcal{B} = \pi_{\alpha_1}(U_{\alpha_1}) \cap \dots \cap \pi_{\alpha_n}(U_{\alpha_n})$

(π_α cont, $f: Z \to \prod X_\alpha$ cont iff $\pi_\alpha \circ f$ cont for each α)

The product topology

↑

<u>Wins</u>

Define Metric 1. pos 2. Sym 3 Δ   define $B_r(x)$ show basis

2.9 <u>Thm</u> The metric top is alway Haus

<u>Def</u> Metrizable

<u>Thm</u>

HW Read $\prod_{i\in\mathbb{N}} X_i$ is metrizable iff each X_i is.

Math 131, Feb 19 1993

Define metric
1. pos
2. sym
3. Δ

Examples
1. $\mathbb{R}^n$
2. discrete
3. $\mathbb{R}^\infty = (\mathbb{R}^\omega)_{uniform}$

Define $B_r(x)$ show basisness

Thm the metric topology is always Haus.

Def Metrizable

do not prove but exercise { Thm $\prod_{i \in \mathbb{N}} X_i$ is metrizable $\Leftrightarrow$ $\forall i$ X_i is metrizable.

Def $x_i \longrightarrow x$ (in a top. space)

Thm $x \in \bar{A}$ iff $\exists\, x_i \in A$ s.t. $x_i \longrightarrow x$

Examble $(\prod_{i \in \mathbb{N}} \mathbb{R})_{box} = (\mathbb{R}^{\mathbb{N}})_{box} = (\mathbb{R}^\omega)_{box}$ is not metrizable

Def $C_0(X,Y)$, $d(f,g) = \text{lub}_{x \in X}(d_Y(f(x), g(x)))$

Thm $C_0(X,Y)$ is closed in $C_0(X_{discrete}, Y)$

HW
1. Read 2.9, 10, w/ [illegible]
2. Do 2.8.5, 6, 7 ; 2.9.2, 5 ; 2.10.3, 8, 9.

Math 131, Feb 22 1993   No office hours!

Example $\mathbb{R}^{\mathbb{R}}$ is not metrizable.

Def $C_0(X,Y)$, $d(f,g) = \operatorname{lub}_{x \in X} d_Y(f(x), g(x))$

top space, metric space

Thm $C_0(X,Y)$ is closed in $C_0(X_{discrete})$

in other words — a uniform limit of cont. functions is cont.

use "ε-δ" def of cont. $x_0 \in X$, $\epsilon > 0$ find n s.t. $d(f_n, f) < \frac{\epsilon}{3}$, U s.t. $f_n(U) \subset B_{\epsilon/3}(f_n(x_0))$, $x \in U \Rightarrow d(f(x), f(x_0)) < \epsilon$ ...

Def clopen, connected space; connected set

connected: (no clopens other than $\emptyset, X$) $\Leftrightarrow \nexists\, U, V$ open $U \cup V = X$, $U \cap V \neq \emptyset$

Thm a cont. image of a connected set is connected.

Thm $[0,1]$ is connected

PF ~~$x \in U, y \in V, x < y$~~

$\exists\, U$ clopen $\neq \emptyset, [0,1]$, $x = \inf U = \operatorname{glb} U$

claim $x = 0$.

PF suppose $x > 0 \Rightarrow$ by closedness $x \in U$; by openness $x - \epsilon \in U \Rightarrow \times$

claim proves thm

Cor all intervals & rays are connected.

Cor the intermediate value theorem.

Math 131, Feb 24 1993

Reminder: clopen; connected

<u>Thm</u> X connected, $f: X \to Y$ cont & onto $\Rightarrow$ Y connected.

<u>Cor</u> A cont. image of a connected set is connected.

<u>Thm</u> $[0,1]$ is connected <u>Pf</u> <u>claim</u> U clopen $\neq \emptyset$, $x = \inf U$; $x = 0$

<u>Pf</u> $x > 0 \Rightarrow$ by closedness $x \in U$; by openness ~~$x \in U$~~ $\cancel{\epsilon U}$

<u>Cor</u> all closed intervals are connected.

<u>Cor</u> The intermediate value thm ($X = [a,b]$

$f: \underset{\text{connected}}{X} \to \mathbb{R}$; $f(a) < 0$ $f(b) > 0$ $\Rightarrow \exists x \in X$ $f(x) = 0$.

<u>Thm</u> U_α connected, $p \in U_\alpha \ \forall \alpha \Rightarrow \bigcup U_\alpha$ is connected.

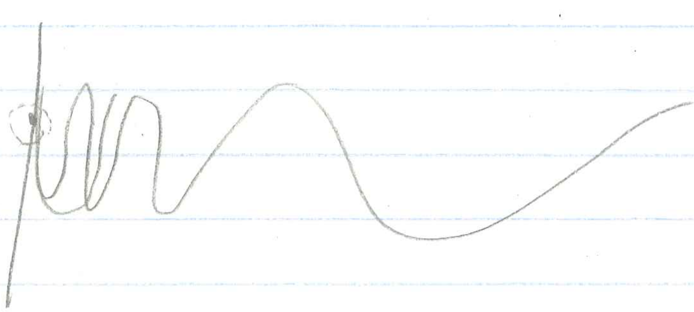
<u>Cor</u> 1. open intervals are connected.
2. Finite products are connected

<u>Thm</u> A connected, $A \subset B \subset \bar{A} \Rightarrow B$ connected.

<u>Thm</u> X_α connected $\Rightarrow \prod_{\alpha \in I} X_\alpha$ connected.

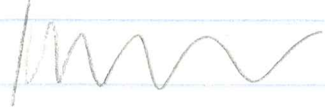
Example

Connected but path connected?



offer book
ballot ?

Math 131, Feb 26 1993.

Reminder: $A \subset B \subset \bar{A}$  connected (but not path connected)

<u>Thm</u> X_α path connected $\Rightarrow \prod X_\alpha$ path connected

X_α connected $\Rightarrow \prod X_\alpha$ connected.

skipping: components, local connectivity

- interesting, usefull, will be done if/when necessary.

<u>compactness</u> Want to prove prop P, here is a strategy: eg: bnddness

1. Show that it ~~happens~~ is true in a nbd of every pt.
2. Show that ~~this~~ if it is true on U, V, $\Rightarrow U \cup V$
3. Deduce finite unions
4. be stuck.

<u>Define</u> open cover $\mathcal{U}$, compactness.

<u>Thm</u> X compact, $f: X \to \mathbb{R}$ cont $\Rightarrow$ f bndd.

<u>Thm</u> $[0,1]$ is compact.

Fix an open cover $\mathcal{U}$ of $[0,1]$; $\mathcal{U} = \{U_\alpha\}$

<u>Pf</u> $x_0 = \sup\{x : [0,x]$ can be covered by finitely many of the $U_\alpha\}$

<u>claim</u> $x_0 > 0$ <u>pf</u> trivial, notice that $(0,1)$ fails

<u>claim</u> $x_0 = 1$ <u>Pf</u> assume $x_0 < 1$, ...

<u>claim</u> we are done.

<u>HW:</u> 2.10.8, 9 3.5.2, 3
3.1.3, 9, 11
3.2.3, 9, 10

collect ballot ?

Math 131 - Prize ideas

1. a Folding folding, infinite gain folding
2. write a chapter on Uniform structures.
3. Raise Kolmogorov prize.
4. Hindermann's thm
5 V.D.W thm.

Math 131, Mar 1 1993

Wed → Wed, Sci Ctr, Lorentz, Approximation theory

Remind compactness

Remainder: $[0,1]$ is compact Pf Fix $\mathcal{U}$, $x_0 = \sup\{x : [0,x]$ has subcov$\}$

$x_0 > 0 \Rightarrow$ easy

Assume by $\Rightarrow\Leftarrow$ $x_0 < 1$

$x_0 = 1 \Rightarrow$ compactness

Prop. ~~Thm~~ A closed subset of a compact space is compact

Thm A compact subset of a T2 space is closed.

Follows from

Lemma X T2, $A \subset X$ compact, $x_0 \in X - A \Rightarrow$
$\exists$ separation, i.e. $\exists\, U, V$ open s.t. $U \cap V = \emptyset$, $A \subset U$, $x_0 \in V$

Thm $A \subset \mathbb{R}$ is compact $\Leftrightarrow$ A is closed & bndd.

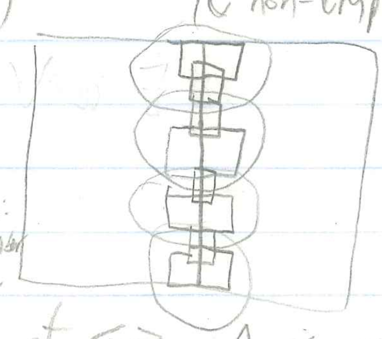
Thm A cont. image of a compact space is compact.

Thm The maximal value thm

Thm $X \times Y$ compact (& non-empty) $\Leftrightarrow$ X & Y compact (& non-empty)

Proof By drawing:

Step I: Find a tube: Covered by finite number of the U_α's.



Step 2 Finitely many tubes are sufficient.

Corol. $A \subset \mathbb{R}^n$ is compact $\Leftrightarrow$ A is closed & bndd

Example The Cantor set.
(Use F.I.P)

Math 131, Mar 3 1993
Moving to sci ctr 309?

Finish Product spaces from prev. time.

The F.I.P & the Cantor set. { Problem (30 pts) in what sense is dim $C = \log_3 2$? 30 pts for a full discussion, written.

Define uniform continuity for $f: X \to Y$

<u>Thm</u> X compact $\Rightarrow f$ uniformly cont.

<u>Lemma</u> (The Lebesgue number lemma) $\mathcal{U}$ open covering $\Rightarrow \exists\, \delta$ s.t. diam $A < \delta \Rightarrow A \subset U_\alpha$ for some α)

<u>Pf</u> of thm from lemma

<u>Pf</u> of lemma Pick a finite subcover $\mathcal{U}'$

$$\Delta(x) = \max_{V \in \mathcal{U}'} \underbrace{\sup_{r>0} \{ r>0 : B_r(x) \subset V \}}_{\text{cont.}} > 0.$$

$\delta = \min \Delta(x)$

<u>Definition</u> limit pt compact $|A| = \infty \Rightarrow A' \neq \emptyset$
sequentially compact
totally bounded

<u>Thm</u> compact $\to$ l.p.c $\to$ seq com $\to$ tot bndd + Lebesgue $\to$ compact.

<u>HW</u>: 3.5.5, 9, 11   Read 3.5–3.7
3.6.3, 4
3.7.1, 2, 6, 7

Math 131, Mar 5 1993

Define limit pt. compactness, sequential compactness

Thm ~~in a metric space,~~

In a metric space:

$$\begin{tikzcd}
 & & & \text{tot bndd}^{5} \arrow[dr, "7"] & \\
\text{compact} \arrow[r, "1"] & \text{l.p.c} \arrow[r, "2"] & \text{seq. compact} \arrow[ur, "6"] \arrow[d, "3"] & + & \text{compact} \\
 & & \text{max val thm.} \arrow[r, "4"] & \text{Lesbegue number lemma} \arrow[ur] &
\end{tikzcd}$$

do parts in order.

Definition A space is called locally compact iff every pt in it has a nbd w/ a compact closure.

Definition $\alpha X = X \cup \{\infty\}$, w/ $\mathcal{T} = \mathcal{T}_X \cup \{\alpha X - C : C \text{ compact}\}$

Claim this space is Hausdorff iff X is locally compact.

Example

$$\alpha \mathbb{R} = S^1$$

ULTRAFILTERS, COMPACTNESS, AND THE STONE-ČECH COMPACTIFICATION

DROR BAR-NATAN

March 7, 1993

1. The axiom of choice and Zorn's lemma

Axiom 1. *Whenever $\{X_\alpha\}_{\alpha\in I}$ is an arbitrary indexed collection of non-empty sets, their cartesian product*

$$\prod_{\alpha\in I} X_\alpha$$

is non-empty. In other words, whenever $\{X_\alpha\}_{\alpha\in I}$ is an arbitrary indexed collection of non-empty sets, there is a so-called ***choice function*** *$f : I \to \bigcup_{\alpha\in I} X_\alpha$ satisfying $f(\alpha)\in X_\alpha$ for every α in I.*

Warning: This axiom is far less innocent than it first seems!!!

Definition 1.1. A *partially ordered set* is a set $\mathcal{S}$ together with a binary relation $\leq$ on it, which is:

(1) *Reflexive*: $s \leq s$ for every $s\in\mathcal{S}$.
(2) *Anti-symmetric*: if $s\leq t$ and $t\leq s$ for $s,t\in\mathcal{S}$, then $t=s$.
(3) *Transitive*: If $s\leq t$ and $t\leq u$ for $s,t,u\in\mathcal{S}$, then $s\leq u$.

A *chain* in a partially ordered set $\mathcal{S}$ is a subset $\mathcal{C}$ of $\mathcal{S}$ which is *simply ordered*, namely, a subset $\mathcal{C}$ for which whenever $s,t\in\mathcal{C}$, either $s\leq t$ or $t\leq s$. A chain $\mathcal{C}$ in a partially ordered set $\mathcal{S}$ is called *bounded from above* if there exists some $m\in\mathcal{S}$ for which $s\leq m$ whenever $s\in\mathcal{C}$.

Lemma 1.2. *(Zorn's lemma) If $\mathcal{S}$ is a partially ordered set in which every chain is bounded from above, then $\mathcal{S}$ contains (at least one)* ***maximal element*** *M — an element $M\in\mathcal{S}$ for which $s\in\mathcal{S}$ and $M\leq s$ implies $s=M$.*

Remark 1.3. Zorn's lemma is an equivalent and sometimes more convenient version of the axiom of choice. A proof of this equivalence can be found, for example, in [5].

2. Filters, ultrafilters, and compactness

Definition 2.1. A *filter* on a set X is a collection $\mathcal{F}$ of subsets of X satisfying:

(1) $X\in\mathcal{F}$, but $\emptyset\notin\mathcal{F}$.
(2) If $A\in\mathcal{F}$ and $A\subset B\subset X$, then $B\in\mathcal{F}$.
(3) A finite intersection of sets in $\mathcal{F}$ is in $\mathcal{F}$: if $A_{1,2}\in\mathcal{F}$, then $A_1\cap A_2\in\mathcal{F}$.

Example 2.2. Let X be a set, x be a member of X, and $\mathcal{F}_x$ be the collection $\mathcal{F}_x=\{A\subset X : x\in A\}$. Then $\mathcal{F}_x$ is a filter on X, called "the *principal* filter on X at x".

Example 2.3. The collection of all sets containing some neighborhood of a fixed point in a topological space is a filter on that space.

Example 2.4. Let $\mathbf{N}$ be the natural numbers, and let $\mathcal{F} = \{A \subset \mathbf{N} : \mathbf{N} - A \text{ is finite}\}$. Then $\mathcal{F}$ is a filter on $\mathbf{N}$.

Definition 2.5. Let X be a topological space, $\mathcal{F}$ a filter on X, and x a point in X. We say that $\mathcal{F}$ *converges* to x and write $\mathcal{F} \to x$ if every neighborhood of x is in $\mathcal{F}$. If $\mathcal{F}$ converges to exactly one point x of X, we will call that point "the *limit* of $\mathcal{F}$" and write $x = \lim \mathcal{F}$.

Example 2.6. If X is a topological space, x is a point in X and $\mathcal{F}_x$ is the principal filter at x, then $\mathcal{F}_x \to x$.

Proposition 2.7. *A filter on a Hausdorff space X may converge to at most one point in X.*

Definition 2.8. Let X and Y be sets, $f : X \to Y$ be any function, and let $\mathcal{F}$ be a filter on X. The collection

$$f_*\mathcal{F} = \{A \subset Y : f^{-1}(A) \in \mathcal{F}\}$$

is a filter on Y, called "the *pushforward* of the filter $\mathcal{F}$ via the map f".

Example 2.9. Let $f : \mathbf{N} \to X$ be an arbitrary sequence in a topological space X, let x be a point in X, and let $\mathcal{F}$ be the filter of example 2.4. Then $f_*\mathcal{F} \to x$ iff $f_n \to x$ as a sequence.

leave as HW.

Theorem 1. *Let X and Y be topological spaces. A function $f : X \to Y$ is continuous iff whenever a filter $\mathcal{F}$ on X converges to a point $x \in X$, the filter $f_*\mathcal{F}$ on Y converges to $f(x)$.*

Definition 2.10. An *ultrafilter* on a set X is a filter $\mathcal{F}$ on X which is maximal with respect to inclusion. I.e., it is a filter $\mathcal{F}$ for which any other filter $\mathcal{F}'$ on X satisfying $\mathcal{F}' \supset \mathcal{F}$ actually satisfies $\mathcal{F}' = \mathcal{F}$.

Example 2.11. Every principal filter is an ultrafilter. The filter of example 2.4 is not an ultrafilter.

Theorem 2. *Every filter is contained in some ultrafilter.*

Theorem 3. *The following are equivalent for a filter $\mathcal{F}$ on a set X:*

(1) *$\mathcal{F}$ is an ultrafilter.*
(2) *For every set $A \subset X$ either $A \in \mathcal{F}$ or $A^c = X - A \in \mathcal{F}$.*
(3) *For every finite cover $\{A_i\}_{i=1}^n$ of X, $A_i \in \mathcal{F}$ for some i.* of a set $A \subseteq X$,

Problem 2.12. Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbf{N}$. Determine if the set

$$A_{\mathcal{F}} = \left\{ \sum_{n \in F} \frac{1}{2^n} \ : \ F \in \mathcal{F} \right\}$$

is Lebesgue measurable and if it is measurable, determine its Lebesgue measure. (Said differently, $A_{\mathcal{F}}$ is the collection of all numbers $x \in [0,1]$ for which the set of 1s in the binary expansion of x is in $\mathcal{F}$).

Theorem 4. *A topological space X is compact iff every ultrafilter on X is convergent.*

Proposition 2.13. *If $\mathcal{F}$ is an ultrafilter on a set X and $f : X \to Y$ is a function, then $f_*\mathcal{F}$ is also an ultrafilter.*

Tired of non-convergent sequences? You might like the following theorem: (Recall that l^∞ is the set of all bounded sequences of real numbers)

Theorem 5. *There exists a functional $l : l^\infty \to \mathbf{R}$ (called a **generalized limit**) satisfying:*

(1) *l is defined on **all** bounded sequences.*

BCS

(2) *If* (x_n) *is a sequence whose limit exists in the usual sense, then* $l((x_n)) = \lim_{n\to\infty} x_n$.
(3) l *is linear and multiplicative; whenever* (x_n) *and* (y_n) *are bounded sequences and* a *and* b *are real numbers,* $l((ax_n + by_n)) = al((x_n)) + bl((y_n))$ *and* $l((x_n y_n)) = l((x_n))l((y_n))$.

Theorem 6. *Non-standard models of first order arithmetic (models containing infinite integers and like creatures) exist.*

Theorem 7. *(Tychonoff's theorem) If* X_α *is a compact topological space for every* α *in some arbitrary index set* I*, then* $\prod_{\alpha\in I} X_\alpha$ *is compact in the product topology.*

3. The Stone-Čech compactification

Definition 3.1. Let X be a T_2 topological space. A *Stone-Čech compactification* of X is a compact T_2 topological space βX containing X so that:

(1) The topology induced on X as a subset of βX is the original topology of X.
(2) Whenever $f : X \to Y$ is a continuous map of X into some compact T_2 space Y, there exists a *unique* continuous map $\tilde{f} : \beta X \to Y$ whose restriction to X is f.

Remark 3.2. A rather non-trivial theorem (from our current perspective) says that if βX is a Stone-Čech compactification of X, then X is *dense* in βX, namely, the closure of X in βX is all of βX.

Theorem 8. *Any two Stone-Čech compactifications of the same topological space* X *are homeomorphic.*

class order:
1. Functions into X
2. βN
βN is compact Hausdorff
3 N's topology is the old one
$\bar{N} = \beta N$
4. βN is T2 compact
5. $N \to X \Downarrow \beta N \to X$

For simplicity, we will work below only with the space $X = \mathbf{N}$ — the natural numbers with the discrete topology. The results in this section all have analogues for an arbitrary completely regular (whatever that is) topological space, and in particular, for an arbitrary metric space.

Definition 3.3. Let $\beta\mathbf{N}$ be the set of all ultrafilters on $\mathbf{N}$. We will identify $\mathbf{N}$ as a subset of $\beta\mathbf{N}$ by identifying every integer n with the principal ultrafilter μ_n at n.

Theorem 9. *There is a (naturally defined) topology on* $\beta\mathbf{N}$ *for which it is a Stone-Čech compactification of* $\mathbf{N}$*. A basis for that topology is given by* $\mathcal{B} = \{U_A : A \subset \mathbf{N}\}$*, where for any set* $A \subset \mathbf{N}$*,*

$$U_A = \{\mu \in \beta\mathbf{N} : A \in \mu\}$$

Remark 3.4. Notice that all the sets U_A are actually clopen in $\beta\mathbf{N}$!

Proposition 3.5. $\mathbf{N}$ *is dense in* $\beta\mathbf{N}$.

Exercise 3.6. Prove that $\beta\mathbf{N}$ is limit point compact but not sequentially compact.

4. Hindman's theorem

Definition 4.1. For a set $A \subset \mathbf{N}$ and a number $n \in \mathbf{N}$ define $A - n = \{k \in \mathbf{N} : k + n \in A\}$. Let μ and ν be ultrafilters on $\mathbf{N}$. Define $\mu + \nu$ to be the collection

$$\mu + \nu = \{A \subset \mathbf{N} \ : \ \{n \in \mathbf{N} : A - n \in \mu\} \in \nu\}.$$

Proposition 4.2. *If μ and ν are ultrafilters on $\mathbf{N}$, then so is $\mu + \nu$.*

Proposition 4.3. *The operation $+ : \beta\mathbf{N} \times \beta\mathbf{N} \to \beta\mathbf{N}$ just defined has the following three properties:*

(1) *$+$ extends the usual addition of natural numbers. Namely, if $m, n \in \mathbf{N}$, then $\mu_m + \mu_n = \mu_{m+n}$.*
(2) *$+$ is associative: if $\mu, \nu, \rho \in \beta\mathbf{N}$, then $(\mu + \nu) + \rho = \mu + (\nu + \rho)$.*
(3) *$+$ is right-continuous. Namely, for each fixed $\mu \in \beta\mathbf{N}$, the function $\beta\mathbf{N} \to \beta\mathbf{N}$ defined by $\nu \mapsto \mu + \nu$ is continuous.*

Lemma 4.4. *If X is a non-empty compact space and $+ : X \times X \to X$ is associative and right continuous, then X contains (at least one)* ***idempotent*** *— an element ι of X for which $\iota + \iota = \iota$.*

Theorem 10. *(Hindman's theorem) Whenever the natural numbers are colored with finitely many colors (i.e., a function $f : \mathbf{N} \to \{$a finite set of colors$\}$ is specified), one can find an infinite subset $A \subset \mathbf{N}$ and a color c, so that whenever $F \subset A$ is finite, the color of the sum of the members of F is c.*

Remark 4.5. Hindman's theorem was proven by N. Hindman [4] in 1974. A simpler combinatorial proof was later found by Baumgartner [1]. The proof presented here was found by Glazer, and appears in print in [2]. A topological proof of a somewhat different flavor was found by H. Furstenberg [3].

References

1. J. Baumgartner, *A short proof of Hindman's theorem*, J. Combinatorial Theory Ser. A **17** (1974) 384–386.
2. W. W. Comfort, *Ultrafilters: some old and some new results*, Bull. Amer. Math. Soc. **83** (1977) 417–455.
3. H. Furstenberg, *Recurrence in ergodic theory and combinatorial number theory*, Princeton University Press, Princeton 1981.
4. N. Hindman, *Finite sums from sequences within cells of a partition of* $\mathbf{N}$, J. Combinatorial Theory Ser. A **17** (1974) 1–11.
5. S. Lang, *Algebra (second edition)*, Addison-Wesley, Menlo Park, California 1984.

Math 131, Mar 8 1993

$$\begin{tikzcd}[column sep=small]
 & & & \text{tot bndd} & + \xrightarrow[7]{M} \text{compact} \\
\text{Compact} \arrow[r] & \text{L.P.C} \arrow[r, "M"] & \text{seq com} \arrow[ur, "M"', "6"] \arrow[dr] & & \\
 & & & \text{max val thm} \arrow[r, "M"] & \text{Lebesgue cov lemma}
\end{tikzcd}$$

Do 6 & 7.

---

nices replaced by
* locally nice
* subspace of nice.

Def A space is called locally compact iff every pt in it has a nbd w/ a compact closure

example $\mathbb{R}$

Def X $T2$; $\alpha X = X \cup \{\infty\}$ w/ $\mathcal{T} = \mathcal{T}_X \cup \{\alpha X - C : C \text{ compact}\}$

example $\alpha \mathbb{N} = \{\frac{1}{n}\} \cup \{0\}$

$\alpha \mathbb{R} = S^1$

Thm this space is $T2$ iff X is locally compact.

Start w/ handout.

Math 131, Mar 10 1993

evals ; Did anyone succeed w/ chalange?

__Example__ $A \subset [0,1] \subset \mathbb{R} \to \mathbb{R}/\mathbb{Q}$; $A \xrightarrow{\text{1-1 \& onto}} \mathbb{R}/\mathbb{Q}$

A = in every equiv class choose one rep.

__Claim__ 1. $\forall\, q_1 \neq q_2 \in \mathbb{Q}$, $(q_1+A) \cap (q_2+A) = \emptyset$

2. $\bigcup_{q\in\mathbb{Q}} (q+A) = \mathbb{R}$

What is $m(A)$?

A similar but more complicated const. leads to Tarski's paradox.

Posets : refl, AS, trans. Zorn's lemma.

__Example__ Hamel basis.

__Thm__ Zorn's lemma $\Leftrightarrow$ AC.

Philosophize about convergence & filters (we want a better notion of conv.)

Do 2.1 – 2.7.

HW: 3.7.3–5(–e), 3.8.4,6,9 ;

0. $X = \{0,1\}^{\mathbb{R}}$ $A = \{a \in X : a(t) = 0$ for all but f. many t's$\}$

1. No seq in A conv. to $\overline{1}$ ($=1$)

2. $\exists$ Filter $\mathcal{F}$ on X s.t.

   a. $A \in \mathcal{F}$

   b. $\mathcal{F} \to \overline{1}$

Math 131 - HW due March 17:

1. 3.7.3, 3.7.4, 3.7.5a-d, 3.8.4, 3.8.6, 3.8.9 - all from the textbook.
2. Let R be the real numbers, and A be the positive reals. Show that there exists a filter on R which converges to 0 and which contains the set A.
3. Let X be the space of all functions (continuous or not) on the real numbers with values in the two-element set {0,1}. Let A be the subset made of the functions which are equal to zero in all but finitely many places. Let u be the function in X whose value is everywhere 1.
   a. Show that no sequence in A converges to u.
   b. Describe explicitly a filter on X which contains the set A and which converges to u.

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Mid-Semester Evaluation for Math 131

Lectures:	Very Poor				Very Good	Comments/Suggestions:
Organization:	1	2	3	4	5	
Content:	1	2	3	4	5	
Chance to participate:	1	2	3	4	5	
Overall:	1	2	3	4	5	
Pace:	Fast	2	3	4	Slow	

Jason's Section:	Very Poor				Very Good
Organization:	1	2	3	4	5
Content:	1	2	3	4	5
Chance to participate:	1	2	3	4	5
Relation with class:	1	2	3	4	5
Overall:	1	2	3	4	5
Pace:	Fast	2	3	4	Slow

Tom's Section:	Very Poor				Very Good
Organization:	1	2	3	4	5
Content:	1	2	3	4	5
Chance to participate:	1	2	3	4	5
Relation with class:	1	2	3	4	5
Overall:	1	2	3	4	5
Pace:	Fast	2	3	4	Slow

Homework (amount):	much	2	3	4	little
	Very Poor				Very Good
Homework (usefulness):	1	2	3	4	5
Reading:	1	2	3	4	5
Grading Policy:	1	2	3	4	5
Course overall:	1	2	3	4	5

Max A. Zorn, 86; Developed a Theory That Changed Math

By WOLFGANG SAXON

Prof. Max August Zorn, who developed a fundamental premise of algebraic structure theory, died Tuesday at Bloomington Hospital in Bloomington, Ind. He was 86 and lived in Bloomington.

The cause of death was congestive heart failure, said Indiana University, where Dr. Zorn taught mathematics from 1946 until his retirement in 1971.

While a Sterling Fellow at Yale University in the 1934-35 academic year, Dr. Zorn worked out an assumption about the basic structure of mathematics that came to be known as Zorn's lemma, a lemma being a proposition assumed to be true and used in proving a theorem. He posited it in his late 20's, and it remains a staple of mathematical study.

Zorn's lemma is the equivalent of the Axiom of Choice in set theory, which states that given an infinite number of sets it is possible to make a new set by choosing one item from each of the other sets.

"It sounds simple," said Professor John Ewing, head of the mathematics department at Indiana, "but it turns out that this cannot be derived from other axioms of mathematics."

"It is completely universal," Dr. Ewing added, "and must be invoked frequently in all areas of mathematics."

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After his stay at Yale, Dr. Zorn taught at the University of California at Los Angeles for 10 years and then joined the Indiana faculty. Even after his formal retirement 22 years ago, he remained a familiar figure on the Bloomington campus until the end of his life.

Dr. Zorn is survived by his wife, the former Alice Schlottau; a son, Jens C., of Ann Arbor, Mich.; a daughter, Liz A. Zorn of Bloomington; two grandchildren and one great-grandchild.

THE NEW YORK TIMES OBITUARIES THURSDAY, MARCH 11, 1993

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Math 131, Mar 12 1993

Distribute Zorn's Obituary

Finish Zorn & Hamel, Sketch proof.

<u>Example</u> Define filter.
Define ultrafilter.
Show that ultrafilter.

Phylosophize about filters.

Do handout in sequence.

Collect Evaluations !!

Math 131, Mar 15 1993

Jason's & Tom's announcement.

Midterm: Mon Mar 22, 7^{30} PM, Emersson 105
$\boxed{\text{Not open book}}$ -9^{30}
Some blend of "repeat class material", HW, exercises.
~200pts, Last year's midterm is not a bad model

Pre-midterm study party: Sun Mar 21, 7^{PM},
math department lounge.
Come prepared!

No office hours!

Continue from 2.9, hope to get to Tychonoff's. Do it before Thm's 5 & 6.

MATH 131 — HOMEWORK DUE MARCH 24, 1993

DROR BAR-NATAN

March 16, 1993

(1) Use Zorn's lemma to prove that *any* set can be simply-ordered. Can you find explicitly a simple order on the set of all real valued functions on the reals? (Reminder — a *simple order* is an order relation in which every two elements are comparable).

(2) Is there a filter $\mathcal{F}$ on the integers $\mathbf{Z}$ for which $A \in \mathcal{F}$ iff $(A+1) \in \mathcal{F}$, where for a set $A \subset \mathbf{Z}$ we define $(A+1) = \{a+1 : a \in A\}$?

(3) Prove that a subset A of a topological space X is closed iff whenever a filter $\mathcal{F}$ on X contains A and converges to a point $x \in X$, the point x is actually in A.

(4) Let X and Y be topological spaces. Prove that a function $f : X \to Y$ is continuous iff whenever a filter $\mathcal{F}$ on X converges to a point $x \in X$, the filter $f_*\mathcal{F}$ on Y converges to $f(x)$.

(5) Show that $\beta\mathbf{N}$ is not countable. (Hint: if you could find a dense sequence in $\mathbf{R}$, ...)

(6) Let $X = \{0,1\}^{\mathbf{R}}$ be the space of all $\{0,1\}$-valued functions on the reals, and let $A \subset X$ be the subset

$$A = \left\{ f \in X \ : \ \begin{array}{l} \{x \in \mathbf{R} : f(x) = 1\} \text{ is a finite union of} \\ \text{intervals, all having rational endpoints.} \end{array} \right\}.$$

   (a) Prove that A is countable.
   (b) Prove that A is dense in X.
   (c) $(*)$ Deduce that the cardinality of $\beta\mathbf{N}$ is at least as big as the cardinality of the set of all subsets of the real numbers. Could it be bigger?

Math 131, Mar 17 1993

Continue from Thm 3, hoping to get to the compactness of βN

$A \supset B \Rightarrow U_A \supset U_B$

$U_N = \beta N$

$U_\phi = \phi$

$U_{A \cap B} = U_A \cap U_B$

$U_{A \cup B} = U_A \cup U_B$

$U_{A^c} = (U_A)^c$

Math 131, Mar 19 1993

Do page 3 of Handout.

$\mathcal{U}$ $\quad \mu = \{A \subset N : \{\nu : A \in \nu\} \in \mathcal{U}\}$

$= \{A \subset N : U_A \in \mathcal{U}\}$

μ ultra? Yes $\quad M \in U_B$ for some B

$\Rightarrow B \in \mu \Rightarrow U_B \in \mathcal{U}$

$F : N \longrightarrow \underset{\text{compact, Hausdorff}}{X}$ $\quad \tilde{F} : \beta N \longrightarrow X$

~~Thm~~ $\tilde{F}$ is cont.

Lemma: $A \in \mu \Rightarrow \lim \mu \in \overline{A}$

Lemma: (X compact, T_2) U nbd of $x \Rightarrow \exists$ nbd V of x s.t. $x \in V \subset \overline{V} \subset U$

$\lim \mu =$ $\tilde{F}(\mu) \in U$ nbd pick

$\tilde{F}(\mu) \in V \subset \overline{V} \subset U$,

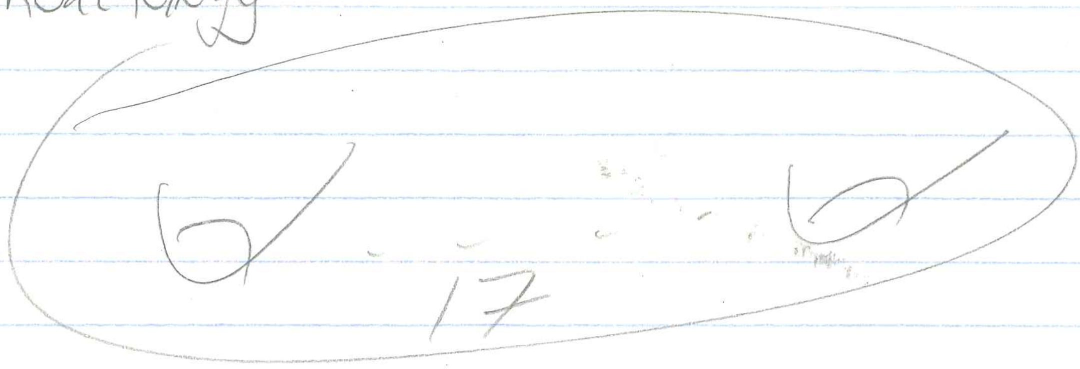
Let $W = U_{F^{-1}(V)} = \{\nu : F^{-1}(V) \in \nu\}$

claim W is a nbd of μ

claim $\tilde{F}(W) \subset \overline{V} \subset U$ QED

Math 131, Mar 22 1993

Roachology



Midterm Examination

Math 131, March 22 1993
Dror Bar-Natan

You have 120 minutes to answer the following 8 questions, each worth 25 points. It is a good idea to read the entire exam before starting to solve it. Notice that the maximal possible score is 200, but for the purpose of the final grade grades higher than 150 will count as 150. You may not use any material other than your pen or pencil. You may use any lemmas proven in class, provided that you quote them in full. Don't forget to sign your name on anything you submit.

(1) Give a precise definition of each of the following:
   (a) A *basis* for a topology.
   (b) An *embedding* of a topological space Y in a topological space X.
   (c) The *order topology* on a simply ordered set X.
   (d) A *locally compact* topological space.
   (e) The *pushforward* of a filter $\mathcal{F}$.

(2) Prove that if for every $\alpha \in I$ a connected topological space X_α is given, then their product $\prod_{\alpha\in I} X_\alpha$ is also connected, in the case when I is a finite set.

(3) Prove that if $f : X \to Y$ is a continuous function defined on a compact metric space X with values in a metric space Y, then f is uniformly continuous.

(4) (a) For a natural number $k \in \mathbf{N}$ define $k\mathbf{N} = \{kn : n \in \mathbf{N}\}$. Show that the collection $\mathcal{F} = \{A \subset \mathbf{N} : \text{for some } k \in \mathbf{N},\ k\mathbf{N} \subset A\}$ is a filter on $\mathbf{N}$.
   (b) Define $f : \mathbf{N} \to [0,1]$ by
$$f(n) = \frac{1}{\text{no. of prime factors of } n}.$$
   Does $\lim f_*\mathcal{F}$ exist? What is it?
   (c) Does the sequence $f(n)$ converge? What is its limit?

(5) A topological space X is called *regular* if whenever F is a closed subset of X and y is a point not in F, there exist disjoint open subsets U and V of X such that $F \subset U$ and $y \in V$.
   (a) Prove that a compact Hausdorff space is always regular,
   (b) Prove that all metric spaces are regular.

(6) Let X be an arbitrary topological space. Show that the diagonal $\{(x,x) : x \in X\}$, in the topology induced from $X \times X$, is homeomorphic to X.

(7) Let $\mathbf{R}^\infty$ be the subset of $\mathbf{R}^\mathbf{N}$ consisting of all sequences that are "eventually zero", that is, all $(x_1, x_2, \dots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbf{R}^\infty$ in $\mathbf{R}^\mathbf{N}$ in the box and product topologies? Justify your answer.

(8) Let $A \subset X$. Show that if C is a connected subset of X that intersects both A and $X - A$, then C intersects Bd A. (Recall that Bd $A = \overline{A} \cap \overline{X - A}$).

GOOD LUCK!!

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March 24 1993

MRS Gaines,
Springer Verlag
New-York. (FAX # 212-473-6272)
212-505-6528

Dear Mrs Gaines,

In Continuation of our phone conversation from a few minutes ago - I need copyright permission to make 50 copies of a book I need for a course I am teaching. The book is "A basic course in algebric topology" by William S. Massey, ISBN 0-387-97430-X, and I will need it beginning April 5th.

Thanks,

Dror Bar-Natan
Department of Mathematics
Harvard University
(617) 495-8797 (office)
(617) 495-5132 (Fax)

Taken from:
"Ultrafilters, some old and some new results",
by W. W. Comfort,
Bull. Amer. Math. Soc. vol 83 no. 4
pp. 417-455,
July 1977.

10. **Glazer's proof of Hindman's theorem.** The scene changes, from combinatorial topology to number theory. As with many difficult problems in this branch of mathematics, the statement of the question is quite easily understood. The theorem of Hindman proved below serves to establish the following statement, known for some years as the Graham-Rothschild conjecture.

If the natural numbers are divided into two sets then there is a sequence drawn from one of these sets such that all finite sums of distinct numbers of this sequence remain in the same set.

To prove this, we begin with a simple result from the theory of mobs.

Definition. Let X be a space and $+$ a function from $X \times X$ to X. Then $+$ is *right-continuous* if for all $p \in X$ the function $q \to p + q$ is a continuous function of q.

10.1. Lemma. *If X is a nonempty compact space and $+$ is an associative, right-continuous function from $X \times X$ to X, then there is a $+$-idempotent in X (i.e., an element p of X such that $p = p + p$).*

Proof. We define

$$\mathfrak{Z} = \{A \subset X: A \neq \emptyset, A \text{ is closed, and } A + A \subset A\},$$

and we note that $\mathfrak{Z} \neq \emptyset$ since $X \in \mathfrak{Z}$. Ordered by reverse containment, the set $\mathfrak{Z}$ satisfies the hypotheses of Zorn's lemma: If $\{A_i: i \in I\}$ is a chain in $\mathfrak{Z}$, then with $A = \bigcap_{i \in I} A_i$ we have

$$A + A \subset A_i + A_i \subset A_i \quad \text{for all } i \in I$$

and hence $A + A \subset A$; it follows readily that $A \in \mathfrak{Z}$. Hence there is a minimal element of $\mathfrak{Z}$. Let B be a minimal element of $\mathfrak{Z}$ and choose $p \in B$.

We note that $p + B \neq \emptyset$, that $p + B$ is the image of B under a continuous function and is therefore closed in X, and that

MATH 131 — HOMEWORK DUE APRIL 7, 1993

DROR BAR-NATAN

March 25, 1993

Notice: The questions in this HW assignment are hard. Therefore you will get full credit for solving any 3 of them, and extra credit for anything you do beyond that. However, I urge you to try solving *all* question - all of them are *fun* (at least when you know the answer). In solving any of these questions you are allowed to use the results of the other questions, even if you could not prove them.

(1) Regarding $\mathbf{N}$ as a subset of $\beta\mathbf{N}$, show that
  (a) its subspace topology is the discrete topology,
  (b) and its closure $\overline{\mathbf{N}}$ is the whole space $\beta\mathbf{N}$. (In other words, show that $\mathbf{N}$ is dense in $\beta\mathbf{N}$. In general, a subset A of a topological space X is called *dense* if $\overline{A} = X$).

(2) Let $f : \mathbf{N} \to X$ be an arbitrary sequence in a compact Hausdorff space. Let $\tilde{f} : \beta\mathbf{N} \to X$ be the natural extension of f to $\beta\mathbf{N}$, defined by setting $\tilde{f}(\mu) = \lim f_*\mu$. Prove that $\tilde{f}$ is continuous in the topology of $\beta\mathbf{N}$ defined in class.

(3) Show that $\beta\mathbf{N}$ is not countable. (Hint: if you could find a dense sequence in $[0,1]$, you'd have a map $\mathbf{N} \to [0,1]$ whose image is dense. What would the image of its extension to $\beta\mathbf{N}$ be?)

(4) Let $X = \{0,1\}^{\mathbf{R}}$ be the space of all $\{0,1\}$-valued functions on the reals, and let $A \subset X$ be the subset

$$A = \left\{ f \in X \ : \ \begin{array}{l} \{x \in \mathbf{R} : f(x) = 1\} \text{ is a finite union of} \\ \text{intervals, all having rational endpoints.} \end{array} \right\}.$$

  (a) Prove that A is countable.
  (b) Prove that A is dense in X.
  (c) Deduce that the cardinality of $\beta\mathbf{N}$ is at least as big as the cardinality of the set of all subsets of the real numbers. Could it be bigger?

(5) Prove that $\beta\mathbf{N}$ is limit point compact but not sequentially compact. (Hint: there is only one really good sequence in $\beta\mathbf{N}$ to start with).

(6) Prove that our function $+ : \beta\mathbf{N} \times \beta\mathbf{N} \to \beta\mathbf{N}$ can be obtained by extending from $\mathbf{N}$ to $\beta\mathbf{N}$ the usual function $+ : \mathbf{N} \times \mathbf{N} \to \mathbf{N} \subset \beta\mathbf{N}$, first on the left and then on the right. To be more precise, let $+_1 : \mathbf{N} \times \mathbf{N} \to \beta\mathbf{N}$ be the usual addition, composed with the standard inclusion $\mathbf{N} \subset \beta\mathbf{N}$. For each fixed integer n one gets a function $\mathbf{N} \to \beta\mathbf{N}$ defined by $m \mapsto m +_1 n$. This is a function with values in a compact space, and therefore, by question 2, it can be extended to $\beta\mathbf{N}$, giving a function $+_2 : \beta\mathbf{N} \times \mathbf{N} \to \beta\mathbf{N}$. Now for each fixed $\mu \in \beta\mathbf{N}$ one gets a new function $\mathbf{N} \to \beta\mathbf{N}$ defined by $n \mapsto \mu +_2 n$. Once again, by question 2, it can be extended to $\beta\mathbf{N}$, giving a function $+_3 : \beta\mathbf{N} \times \beta\mathbf{N} \to \beta\mathbf{N}$. Does this $+_3$ coincide with ultrafilter addition, as defined in class?

(7) Let Y_n be the state space of the two-dimensional machine displayed below, which is made of chain of $n+2$ rods connected using $n+1$ joints, and whose ends are attached to fixed points and are not allowed to move.

Assume that the chain is "almost tight", so that none of the rods in it may make a full turn. Can you find a simpler description for the space Y_n? It is a good idea to start from $n = 0$, proceed to $n = 1$, $n = 2$, $n = 3$, and then to try to generalize.

Math 131, Mar 26 1993

apologize for the incompleteness of our treatment of $\beta\mathbb{N}$

Hindman's theorem: whenever the natural numbers are colored by finitely many colors, one can find a set $A \subset \mathbb{N}$ all of whose finite subsums are of the same color.

(Philosophize about Ramsy theory)

Def $A \subset \mathbb{N}$, $A - n = \{k - n : k \in A\} \cap \mathbb{N}$

Lemma 1 $\exists \tau \in \beta\mathbb{N}$ s.t. $A \in \tau \Rightarrow \exists^{\infty \text{ many}} k \in A$ s.t. $A - k \in \tau$

Hindman from lemma: set $A = A_1 = \{n : n \text{ is colored by the } \tau\text{-dominant color (say blue)}\} \in \tau$

choose $0 < k_1 \in A_1$ s.t. $A_1 - k_1 \in \tau$ (so far all is well) set $A_2 = A_1 \cap (A_1 - k_1) \quad = (A - (A - k_1))$

choose $k_1 < k_2 \in A_2$ s.t. $A_2 - k_2 \in \tau$ (so far all is well) set $A_3 = A_2 \cap (A_2 - k_2)$ $\left(= A_1 \cap (A_1 - k_1) \cap (A_1 - k_2) \cap A_1 - (k_1 + k_2) \right)$

choose $k_2 < k_3 \in A_3$ s.t. $A_3 - k_3 \in \tau$ (all is well) set $A_4 = A_3 \cap (A_3 - k_3)$ $\left(= A_1 \cap (A_1 - k_1) \cap (A_1 - k_2) \cap (A_1 - (k_1 + k_2)) \cap A_1 - k_3 \cap A_1 - (k_2 + k_3) \cap \dots \right.$

Pf of lemma 1 Define (for $\mu, \nu \in \beta\mathbb{N}$) $\mu + \nu \subset 2^{\mathbb{N}}$, if $\tau + \tau = \tau$, we are done.

$A \in \mu + \nu \Leftrightarrow \{n : A - n \in \mu\} \in \nu$

Lemma 2 1. $\mu + \nu \in \beta\mathbb{N}$
2. $+$ is associative
3. $+$ is right-cont.

Lemma 3 if X is a non-empty compact space and $+ : X \times X \to X$ is associative and right cont, then X contains an idempotent – $\tau \in X$ s.t. $\tau + \tau = \tau$.

Pf: A: minimal non-empty closed set s.t. $A + A \subset A$ (Zorn's lemma)

chose $\tau \in A$; $(\tau + A) + (\tau + A) \subset \tau + A + A \subset \tau + A$, $\tau + A \subset A \Rightarrow$

$\tau + A = A \Rightarrow \exists \rho$ s.t. $\tau + \rho = \tau \Rightarrow C = \{\rho \in A : \tau + \rho = \rho\} \neq \emptyset$ closed, $C + C \subset C$

$\Rightarrow C = A \Rightarrow \tau + \tau = \tau$

TeX

Midterm Examination

Math 131, March 22 1993

Dror Bar-Natan

You have 120 minutes to answer the following 8 questions, each worth 25 points. It is a good idea to read the entire exam before starting to solve it. Notice that the maximal possible score is 200, but for the purpose of the final grade grades higher than 150 will count as 150. You may not use any material other than your pen or pencil. You may use any lemmas proven in class, provided that you quote them in full. Don't forget to sign your name on anything you submit.

(-1) no quoting connectedness lemma.

(1) Give a precise definition of each of the following:
   (a) A *basis* for a topology. $\forall x\ \exists B;\ x\in B_1\cap B_2 \subset B_3$
   (b) An *embedding* of a topological space Y in a topological space X.
   (c) The *order topology* on a simply ordered set X. -1 rays
   (d) A *locally compact* topological space.
   (e) The *pushforward* of a filter $\mathcal{F}$.

+#4 $U=\prod U_\alpha$ $V=\prod V_\alpha$ +5 π_α(closed) = closed.

(2) Prove that if for every $\alpha \in I$ a connected topological space X_α is given, then their product $\prod_{\alpha\in I} X_\alpha$ is also connected, in the case when I is a finite set. +5 got defs right.

(3) Prove that if $f : X \to Y$ is a continuous function defined on a compact metric space X with values in a metric space Y, then f is uniformly continuous.

(4) 10 (a) For a natural number $k \in \mathbf{N}$ define $k\mathbf{N} = \{kn : n \in \mathbf{N}\}$. Show that the collection $\mathcal{F} = \{A \subset \mathbf{N} : \text{for some } k \in \mathbf{N},\ k\mathbf{N} \subset A\}$ is a filter on $\mathbf{N}$.

   03 lim 70 proof, 10 (b) Define $f : \mathbf{N} \to [0,1]$ by

$$f(n) = \frac{1}{\text{no. of prime factors of } n}.$$

   Does $\lim f_*\mathcal{F}$ exist? What is it?

   5 (c) Does the sequence $f(n)$ converge? What is its limit?

(5) A topological space X is called *regular* if whenever F is a closed subset of X and y is a point not in F, there exist disjoint open subsets U and V of X such that $F \subset U$ and $y \in V$.
   13 (a) Prove that a compact Hausdorff space is always regular, only 12 if using class
   12 (b) Prove that all metric spaces are regular. -3 no mention of T₂ness or no exp why $d(F,y)>0$.

-10 order top (6) Let X be an arbitrary topological space. Show that the diagonal $\{(x,x) : x \in X\}$, in the topology induced from $X \times X$, is homeomorphic to X. $U\times V$ -5

3: writing the homo.

Ans 3

(7) Let $\mathbf{R}^\infty$ be the subset of $\mathbf{R}^\mathbf{N}$ consisting of all sequences that are "eventually zero", that is, all $(x_1, x_2, \ldots)$ such that $x_i \neq 0$ for only finitely many values of i. What is the closure of $\mathbf{R}^\infty$ in $\mathbf{R}^\mathbf{N}$ in the box and product topologies? Justify your answer. (-3) insufficient care at forwlid $x_i=0$ (-6) $\|\cdot\|_\infty$

box 12 Prod 13 $-\frac{1}{2}\cdot\frac{1}{2}$ insufficient exp.

(8) Let $A \subset X$. Show that if C is a connected subset of X that intersects both A and $X - A$, then C intersects Bd A. (Recall that Bd $A = \overline{A} \cap \overline{X - A}$).

(-4) $C\cap \operatorname{Int} A \overset{?}{=} \emptyset$

Correct, but no mention of where the connectivity of C is used.

GOOD LUCK!!

Math 131, Apr 5 1993

Midterm came out well, 22 got 150, only about 10 got less than 100. Those should be worried!

Announce books, No office hours, 4 missing studs.

Finish Hindman as in previous class.

<u>Definition</u> A T_2 X is called an n-manifold if ...

<u>Exa</u>  $= S^2$,  $= T^2$, RP^2, $RP^2 \times T^2$ 4d, ...

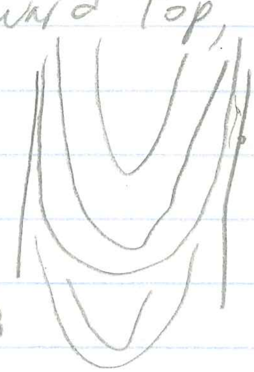
One of the main goals of topology ...

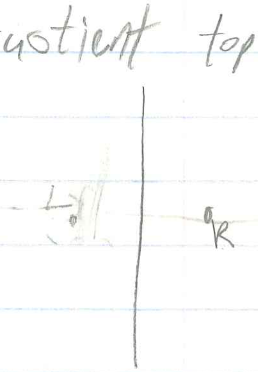
Surfaces All come via some gluings.

$p : X \longrightarrow Y$, the pushforward top, quotient top.

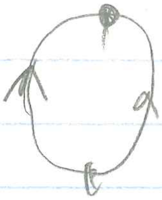
$(X, \sim)$

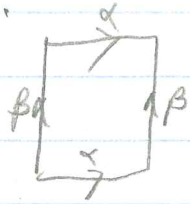
quotient top is ill-behaved:

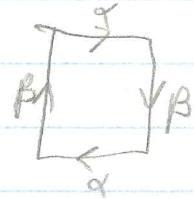




well behaved:







Connected sum: (on path-connected surfaces)

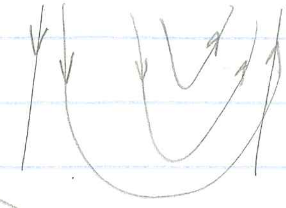
<u>Thm</u> Any cmpct surface is either S^2 or $\#^n T^2$ or $\#^n RP^2$.

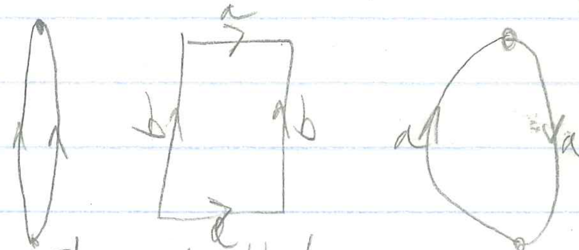
HW Read Munkres 2-11
Massey I-1-4.

Math 131, Apr 7 1993

Remind $X, \sim$

Aside quotient top is ill behaved





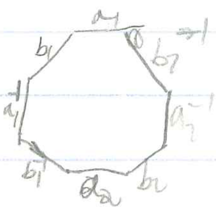
aa^{-1} $aba^{-1}b^{-1}$ aa

Connected sum, well definededness

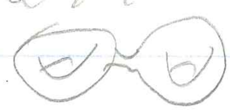
Thm compact S is $\underbrace{S^2, \#T^2}_{\text{orientable}}$, $\underbrace{\#^n RP^2}_{\text{non-orientable}}$

orientation fuzzy def

Presentations:



$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}$ = ? = maybe



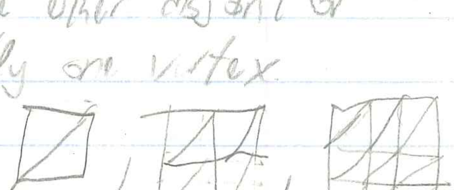
?

Yes!

$S^2 = aa^{-1}$ $\#^n T^2 =$ $\#^n RP^2 = a_1 a_1 a_2 a_2 a_3 a_3 \ldots a_n a_n$

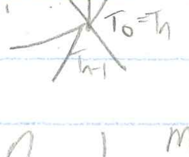
A triangulation of a cmpct surface:
Subdivision into triangles s.t. any two Δ's are either disjoint or have exactly one ~~vertex~~ edge in common or exactly one vertex.

E.g.



Props: 1. Each edge is the edge of exactly two triangles

2.



HW: Read Mun: 2.11, Mass: 1.1-6. Do Mun: 2.11.1,4,5,6; Mas 5.1, Triangulate S^2, $\#^2T^2$, RP^2

EULER CHARACTERISTIC

THM ...

PROBLEM ...

SOLUTION:

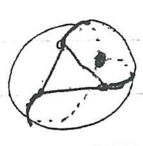
GIVEN S a COMPACT surface and $\{T_1, \dots, T_m\}$ A TRIANGULATION: Let

$$\begin{aligned} v &= \# \text{ of distinct vertices of } S \\ e &= \# \text{ of } \quad '' \quad \text{edge} \quad '' \quad '' \\ t &= m = \# \text{ of } \quad '' \quad \text{triangles} \quad '' \quad '' \end{aligned}$$

$\chi(S) = v - e + t$ EULER CHARACTERISTIC

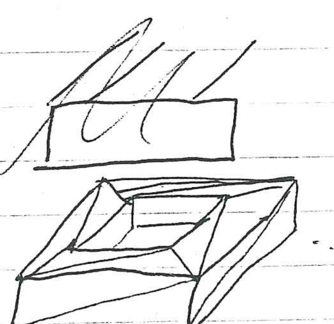
(WE will "see" that $\chi(S)$ is independent of $\{T\}$)

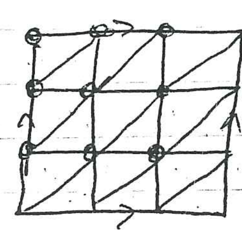
1° sphere



$\chi(S^2) = 3 - 3 + 2 = 2$

2° torus



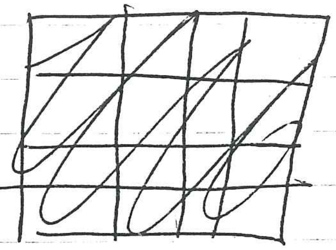


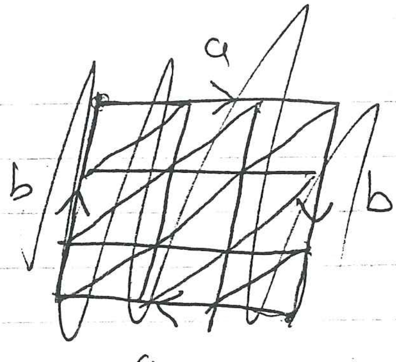
18

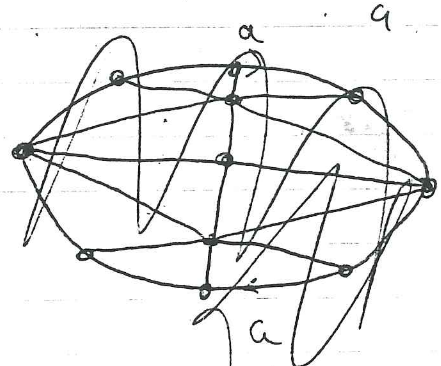
~~26~~ ~~18~~ 9

$\chi(T^2) = 9 - \cancel{36} + 18 = 0$

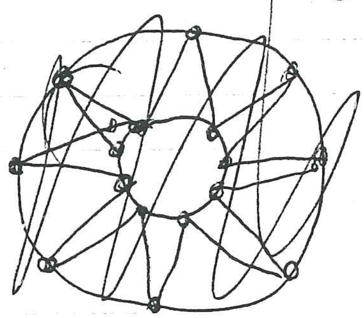
27

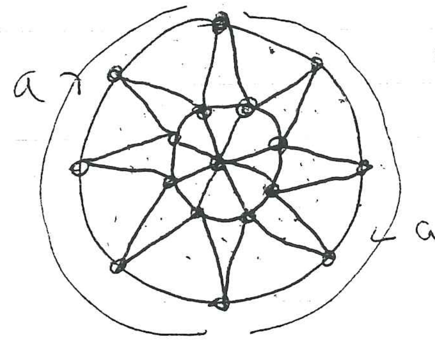






$\chi(\mathbb{RP}^2) = \cancel{4-6+5}$





$13 - 36 + 24 = 1$

$4+16+8+8$ $\quad$ 20 $\quad$ $16+8$

$36 - 24 = 12$

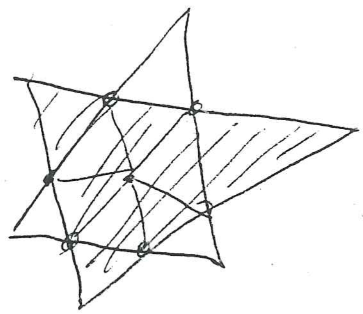
ThM || EULER CHARACTERISTIC IS INDEPENDENT OF TRIANGULATION

GIVEN S A COMPACT SURFACE LET

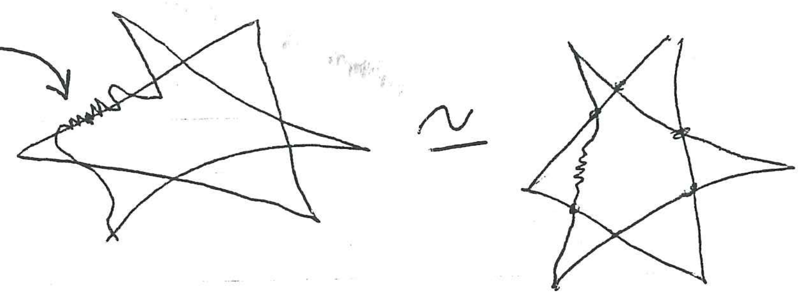
$\{T_1, \ldots, T_n\}$ and $\{T_1' \ldots T_m'\}$

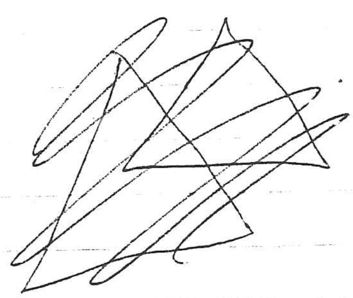
BE two TRIANGULATION

ASSUME $(\partial T_i \cap \partial T_j')$ "is FINITE"

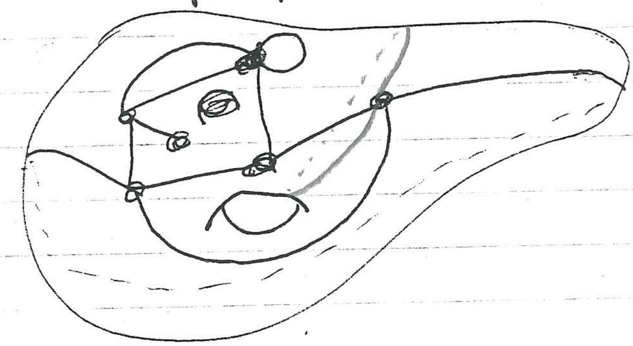


problem



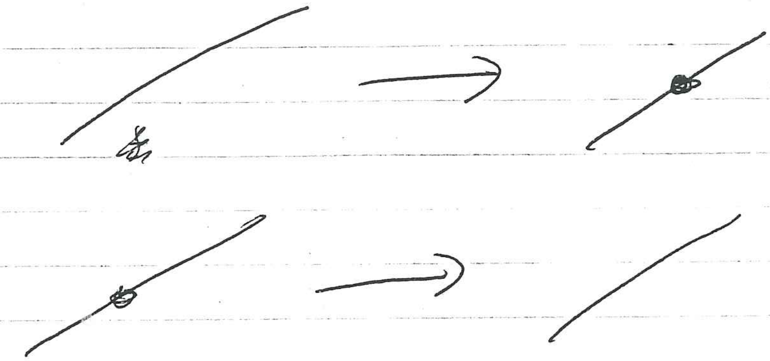


Consider generalized $\chi(S)$ for polygons (graph)



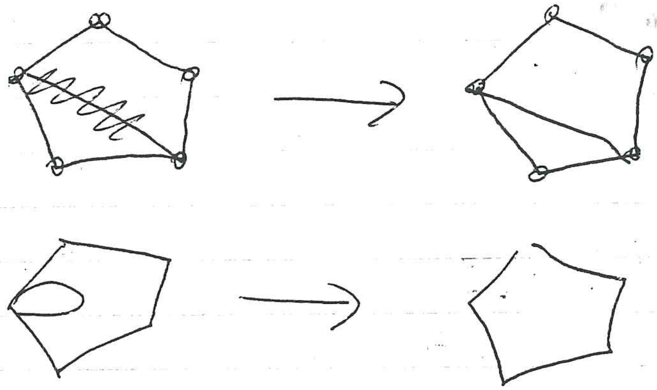
$\chi(S) = \# \, V - E + f$

(a)



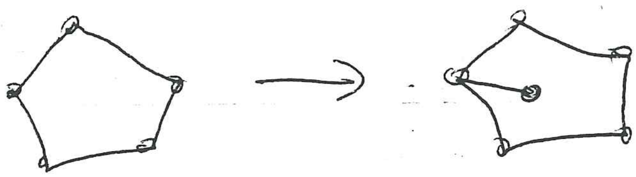
$f = f'$
$e = e' - 1$
$v = v' + 1$

(b)



$f = f' - 1$
$e = e' + 1$
$v = v'$

(c)



$f = f'$
$e = e' - 1$
$v = v' - 1$

Go from $\{T\}$ to $\{T'\}$ by move of kind (a) (b) and (c).

Now what is $\chi(S_1 \# S_2) =$

$$\boxed{\chi(S_1) + \chi(S_2) - 2}$$

Proof ...

$\chi(S^2) = 2$

$\chi(\underbrace{T^2 \# T^2 \# \cdots \# T^2}_{n}) = 2 - 2n$ } n is the genus

→ classification for n-manifold $(n \geq 3)$!?!

FUNDAMENTAL GROUP.

- LEXIC: ARC ⟷ PATH

  ARCWISE CONNECTED ⟷ PATH CONNECTED

  ⋮

GIVEN two INTERVAL

$$[a,b] \longrightarrow [c,d]$$

$$t \longmapsto \frac{t-a+c}{b-a+c}\, d \qquad \text{orientation preserving homeomorphism}$$

$$t \longmapsto \frac{\cancel{a}-t+d}{a-b+d}\, c \qquad \text{orientation reversing homeomorphism}$$

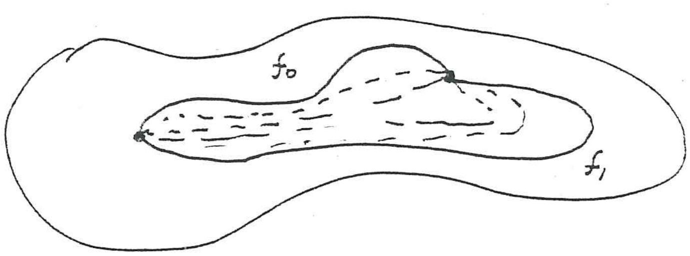
Most of the time we use $[0,1] = I$

two paths $f_0, f_1 : [0,1] \longrightarrow X$ ~~are~~ such that $f_0(0) = f_1(0)$ and $f_0(1) = f_1(1)$ are "<u>equivalent</u>" (homotopic) $\underline{f_0 \sim f_1}$ iF there is a continuous map

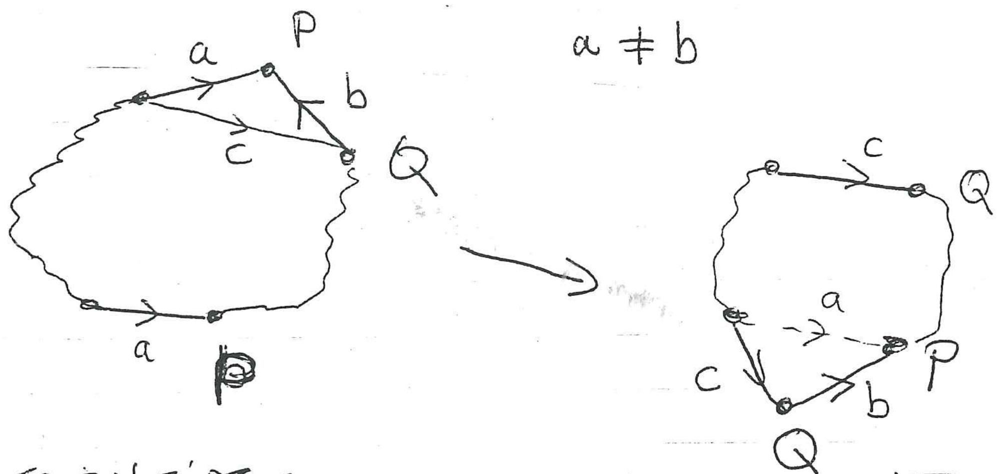
$$f : I \times I \longrightarrow X$$

such that

$$f(t,0) = f_0(t) \qquad \left(f(0,s) = f_0(0) \right)$$
$$f(1,1) = f_1(t) \qquad \left(f(1,s) = f_1(1) \right)$$



3° (In all stadard presentation, the vertex are all identify

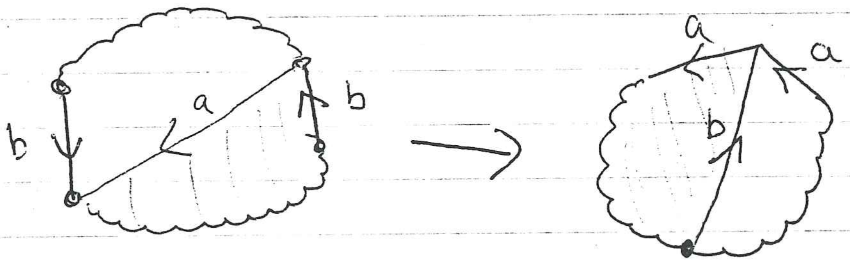


~~2 POINT SIDES~~

1 - less point identify to p

allway using 2° proceed to eleminate all point identify with P.

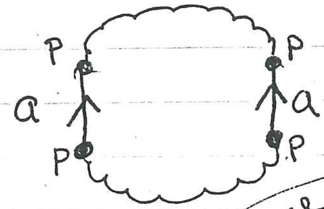
4°



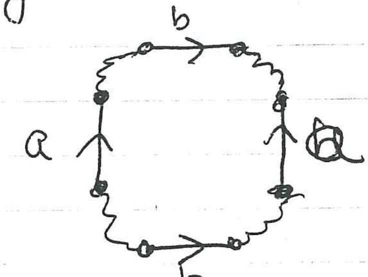
continue for all pair of this sort

$\boxed{\text{this change orientation of } \mathcal{E} \text{ So we do this step as much as we can}}$

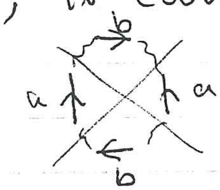
5° a) we are donne $a_1a_1a_2a_2\cdots a_ka_k$

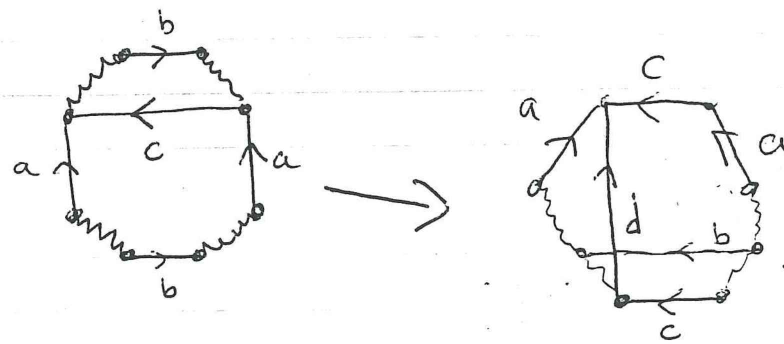
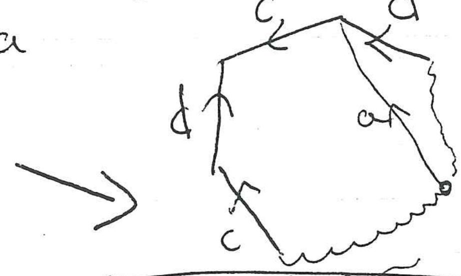
b) 

but theire must be an edge b such that



because P is the same point

(by 4°, it cannot be )

Do NOT CHANGE orientation of other edges in 

repeat ...

one get a connected sum of $\mathbb{P}^2$'s an $\mathbb{T}^2$'s

IF there is only $\mathbb{T}^2$'s we are donne / $a_1b_1a_1^{-1}b_1^{-1}\cdots a_kb_ka_k^{-1}b_k^{-1}$

IF it is mix change each occurence of $\mathbb{P}^2\#\mathbb{T}^2$ by $\mathbb{P}^2\#\mathbb{P}^2\#\mathbb{P}^2$ ∎

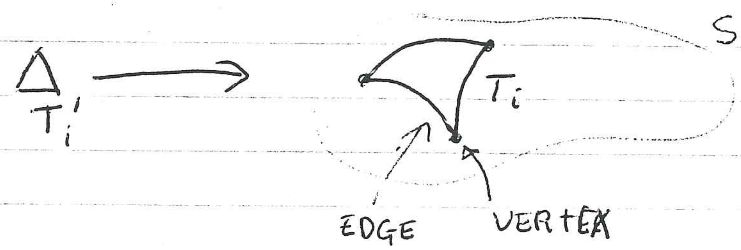
TRIANGULATION OF A COMPACT SURFACE S.

FINITE FAMILY OF CLOSED SUBSET

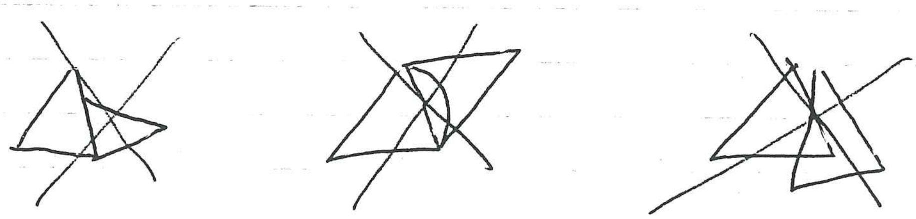
$$\{T_1, T_2, \ldots, T_n\}$$

THAT COVER S SUCH THAT

1) WE HAVE $\varphi_i : T_i' \longrightarrow T_i$ AN HOMEOMORPHISM
with ~~or~~ T_i' A TRIANGLE IN $\mathbb{R}^2$



2) ~~If $T_i \cap T_j \neq \emptyset$ then it is~~
EITHER $T_i \cap T_j \neq \emptyset$
or $T_i \cap T_j$ is A 1 COMMON VERTEX
or $T_i \cap T_j$ is ONE COMMON EDGE



$\boxed{\text{THM}}$ (RADÓ) EVERY COMPACT SURFACE CAN BE TRIANGULATED

SO GIVEN A COMPACT SURFACE S, FIND A TRIANGULATION

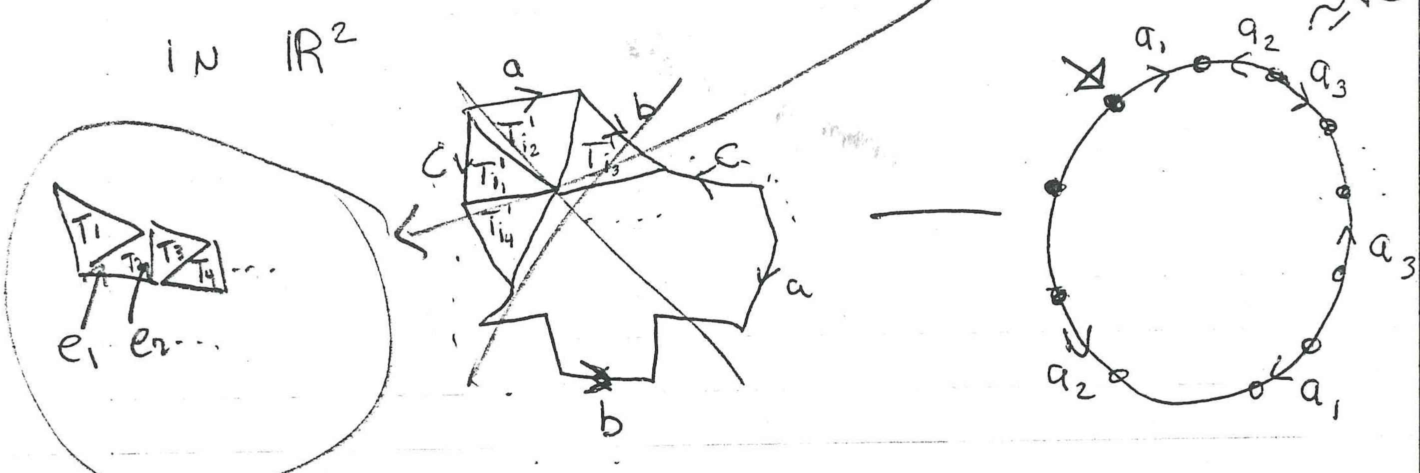
$\{T_1, T_2, \dots, T_n\}$

$\boxed{T_i \cap \left(\bigcup_{j=1}^{i-1} T_i\right) \ni e_i}$

recall that $\varphi_i : T_i' \longrightarrow T_i$

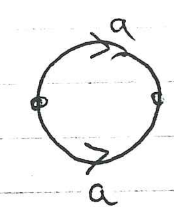
IN $\mathbb{R}^2$

$\cong S$

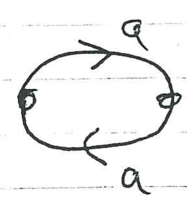


ENCODE S by WORD $a_1 a_2^{-1} a_3 \cdots a_3^{-1} \cdots a_1 \cdots a$

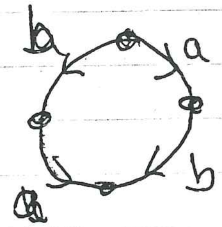
Standard coding



aa^{-1} SPHERE

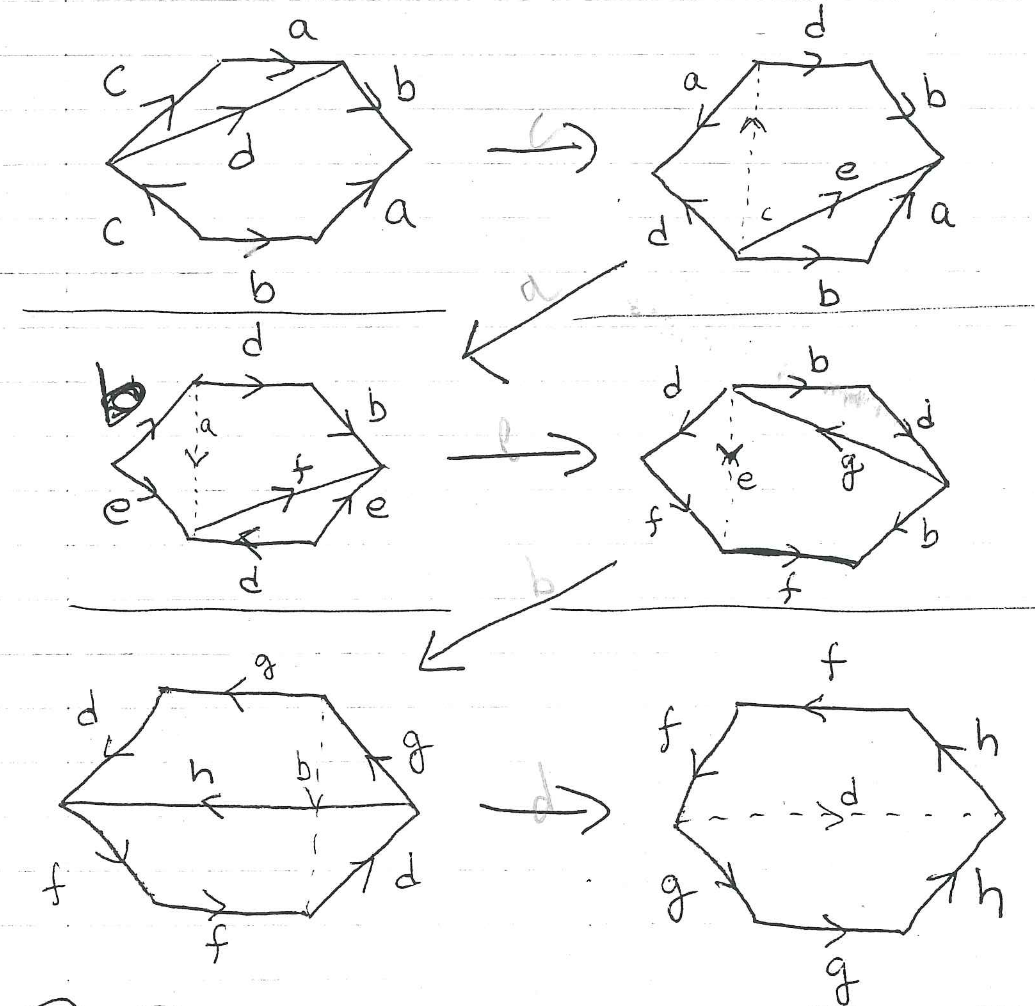


aa Projective plane



$aba^{-1}b^{-1}$ Torus

$T^2 \# \mathbb{P}^2$

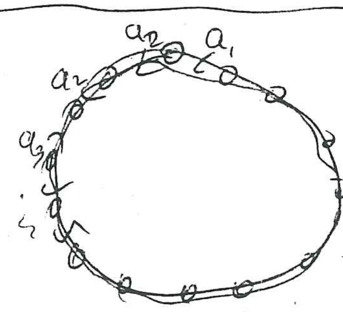


$\simeq \mathbb{P}^2 \# \mathbb{P}^2 \# \mathbb{P}^2$

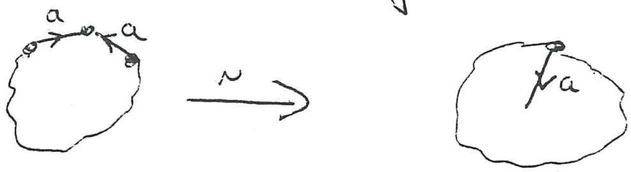
TH: A COMPACT SURFACE $S \simeq S^2$ or $\mathbb{P}^2$ or T^2 or $\mathbb{P}^2 \# \cdots \# \mathbb{P}^2$ or $T^2 \# \cdots \# T^2$

Proof

1° $S \xrightarrow{\text{triangulation}}$



2° ELIMINATE every $\cdots a a^{-1} \cdots$ (Except if aa^{-1})



WED APRIL 14

Math 131 - Homework due April 21.

1. In W.S. Massey 7.1, 7.3, (Page 21), 7.6 (with 7.1, 7.3) (Page 28), 8.1, 8.3, 8.5 (Page 34).
2. Find the orientable compact surface of smallest genus on which you can draw the complete graph on 5 points without intersecting edges. Draw it .
3. Describe how you can figure out the type of any connected compact surface you are considering with a glance at any of its polygonal representation with edges identified in pairs. (No cut-and-paste.)

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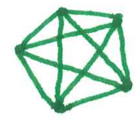

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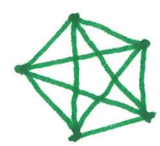

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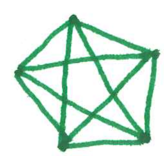
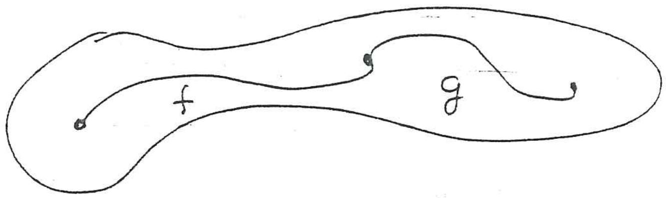


Product of path

iF $f, g : [0,1] \longrightarrow X$ are such that $f(1) = g(0)$



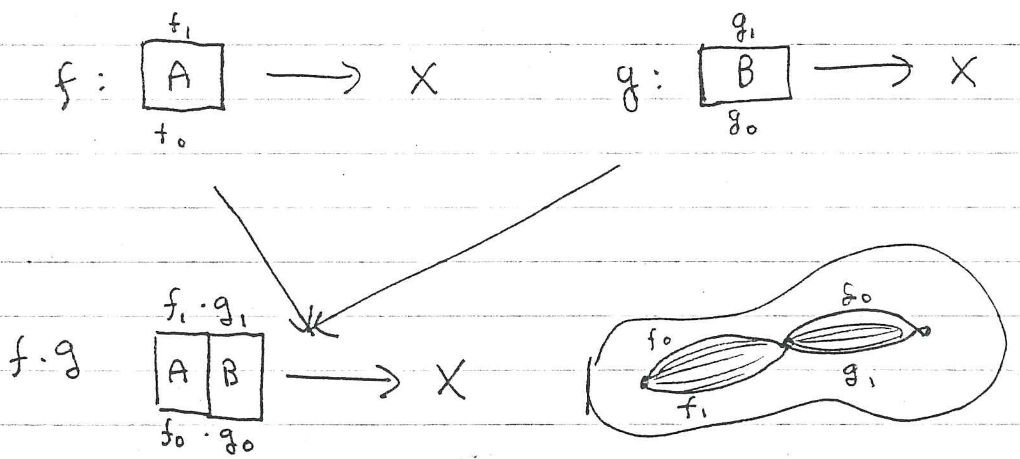
$f \cdot g : [0,1] \longrightarrow X$

pasting lemma

$$t \longmapsto \begin{cases} f(2t) & t \in [0, \frac{1}{2}] \\ g(2t-1) & t \in [\frac{1}{2}, 1] \end{cases}$$

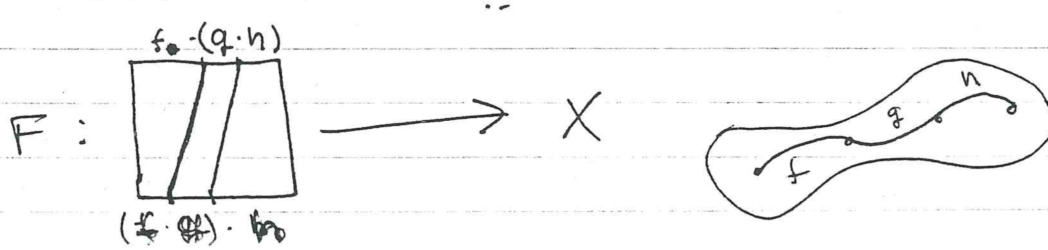
- iF $f_0 \sim f_1$ and $g_0 \sim g_0$ then $f_0 \cdot g_0 \sim f_1 \cdot g_1$

[PRODUCT is WEll DEFINED ON HOMOTOPY ClASSES] $[f] \cdot [g]$



- ON HOMOTOPY ClASSES the product is associative !!

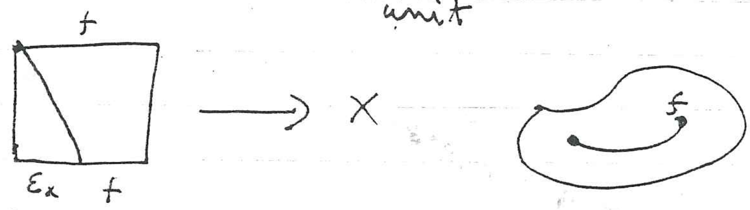
$([f][g])[h] = [f]([g][h])$



$$\varepsilon_x : [0,1] \longrightarrow X$$
$$t \longmapsto x \qquad \text{CONSTANT MAP}$$

- <u>ON HOMOTOPY CLASSES</u> $[\varepsilon_x]\cdot[f] = [f]$

unit



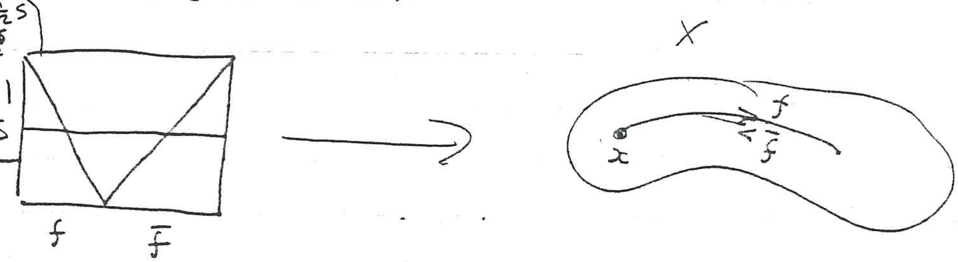
- <u>INVERSE</u>

$$\bar{f}(t) = f(1-t) \qquad (\text{reverse orientation})$$

$$[f]\cdot[\bar{f}] = [\varepsilon_x]$$

$$F(t,s) \begin{cases} f(2t) & 0 \le t \le \tfrac{1}{2}s \\ f(s) & \tfrac{s}{2} \le t \le 1-\tfrac{s}{2} \\ \bar{f}(2-2t) & 1-\tfrac{s}{2} \le t \le 1 \end{cases}$$

"$\bar{f}(2t)$"



- <u>closed path</u> $\quad f : [0,1] \longrightarrow X \;,\; f(0) = f(1)$

★ <u>FUNDAMENTAL GROUP</u> (FIXE A POINT x IN X

$$\pi(X, x) = \{ [f] : f : [0,1] \to X \,,\, f(0) = f(1) \}$$

<u>is A GROUP</u>

- iF $\alpha : [0,1] \longrightarrow X$ ~~such that~~ $\alpha(0) = x \quad \alpha(1) = y$

THEN $\pi(X, x) \simeq \pi(X, y)$

$$[f] \longmapsto [\alpha f \bar{\alpha}]$$

THM | IF X IS PATH CONNECTED THEN $\pi(X) = \pi(\cancel{X}, x)$ IS IND. OF THE CHOICE OF $x \in X$

$$\begin{tikzcd} & X \arrow[r, "\varphi"] & Y \\ {[0,1]} \arrow[ur, "f"] \arrow[urr, dashed, "\varphi\circ f"'] & & \end{tikzcd}$$

1° $f_0 \sim f_1 \Rightarrow \varphi\circ f_0 \sim \varphi\circ f_1$

$F : I\times I \to X$

$\varphi\cdot F : I\times I \to Y$

So $\varphi_*[f] = [\varphi\circ f]$ WELL DEFINE

2° $\varphi_*([f][g]) = \varphi_*[f\cdot g] = [\varphi\circ(f\cdot g)] = [(\varphi\circ f)\cdot(\varphi\circ g)] = \varphi_*[f]\cdot\varphi_*[g]$

3° $\varphi_*[\varepsilon_x] = [\varepsilon_{\varphi(x)}]$

4° $\varphi_*[f]^{-1} = (\varphi_*[f])^{-1}$

So GIVEN $\varphi : X \to Y$ CONT

WE HAVE

$$\varphi_* : \pi(X, x) \longrightarrow \pi(Y, \varphi(x))$$ <u>HOMOMORPHISM</u>

ALSO

5°

$$\begin{tikzcd} & X \arrow[r, "\varphi"] & Y \arrow[r, "\psi"] & Z \\ {[0,1]} \arrow[ur, "f"] & & & \end{tikzcd}$$

$(\psi\circ\varphi)_*[f] = \psi_*\circ\varphi_*$

$(\psi\circ\varphi)\circ f = \psi\circ(\varphi\circ f)$

6° $Id : X \to X$ $\quad Id_*[f] = [f]$

also identity

Monday April 19

1

$$\text{top.} \xrightarrow{\ \pi\ } \text{GROUP}$$

$(X, x) \xrightarrow{\varphi} (Y, y)$ $\quad \varphi(x) = y$

$G_1 \xrightarrow{h} G_2$

: $1^\circ - 6^\circ$
$\pi(X; x)$ is a group } π is A FUNCTOR.

"TRANSLATE top. PROBLEM INTO GROUP PROBLEM"

REMARK

IF φ is HOMEOMORPHISM
THEN φ_* is ~~AN~~ ISOMORPHISM

(~~But~~ φ can be one-to-one or onto ~~both~~ and φ_* Not)

MORE GENERAL NOTION OF HOMOTOPY

$\varphi_0 ; \varphi_1 : X \longrightarrow Y$ CONT FUNC $\quad \varphi_0|_A = \varphi_1|_A$

$A \subseteq X$

ARE HOMOTOPIC ($\varphi_0 \sim \varphi_1$)
(relative to A)

IF THERE IS $\Phi : X \times I \longrightarrow Y$ CONT.

SUCH THAT
$$\begin{cases} \Phi(x, t) = \varphi_t(x) \\ \Phi(a, t) = \varphi_0(a) \quad \text{for } a \in A \end{cases}$$

THM IF $\varphi_0, \varphi_1 : X \longrightarrow Y$ ARE HOMOTOPIC relative to $\{x\}$

THEN $\varphi_{0*} = \varphi_{1*}$

Proof

Let $f:[0,1] \longrightarrow X \overset{\varphi_0}{\underset{\varphi_1}{\rightrightarrows}} Y$

$\varphi_{0*}[f] \overset{?}{=} \varphi_{1*}[f]$ i.e. $\varphi_{0*}[f] \sim \varphi_{1*}[f]$

$F = \Phi \circ f$

RETRACTION

$A \subseteq X$

$r: X \longrightarrow A$ cont. s.t. $r(a) = a$ for $a \in A$

Ex

$X = \mathbb{R}^2 \setminus \{(0,0)\}$ $A = S^1$

$$r(x,y) = \frac{(x,y)}{\sqrt{x^2+y^2}}$$

$i: A \hookrightarrow X$

So $r \circ i = Id_A \Rightarrow r_* i_* = Id_{\pi(A,a)}$

hence i_* is AN EPIMORPHISM
r_* is A MONOMORPHISM.

When is i_* AN ISOMORPHISM.?

DEFORMATION RETRACT

$r: X \rightarrow A$ is A RETRACTION

AND

$i \circ r \sim Id_X$ RELATIVELY to A.

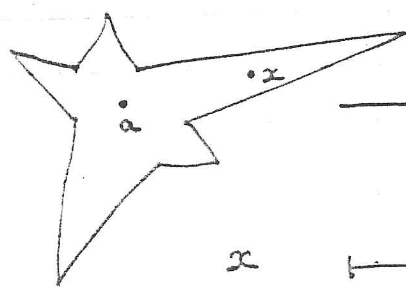
<u>THM</u> IF $r : X \to A$ IS A DEFORMATION RETRACT
THEN $i_* : \pi(A,a) \longrightarrow \pi(X,a)$ IS AN ISOMORPHISM.

<u>Proof</u> $r_* i_* = Id_{\pi(A,a)}$ $\qquad i_* r_* = Id_{\pi(X,a)}$ |

<u>Ex1</u> $r : \mathbb{R}^2 / \{(0,0)\} \longrightarrow S^1$ IS A DEFORMATION RETRACT.

$$F(x,t) = (x,y)\left[\frac{t}{\sqrt{x^2+y^2}} + (1-t)\right]$$

<u>Ex 2</u>

$r :$  $\longrightarrow \{a\}$

$x \longmapsto a$

$$F(x,t) = tx + (1-t)a \quad |$$

$$\boxed{\begin{array}{c} \pi(S^1) = \mathbb{Z} \\ \pi(\{a\}) = 0 \end{array}}$$

last Home Work

- P63 4.1
- P65 4.2, 4.5
- P67 4.9
- P74 5.5, 5.6
- P77 7.1, 7.3, ~~7.5~~
- P82 8.1 ~~8.2~~

NEXT MONDAY : PERSI

$$\begin{array}{ccc} (X, x) & \longrightarrow & \pi(X, x) \\ \downarrow \varphi & \dashrightarrow & \downarrow \varphi_* \\ (Y, \varphi(x)) & \longrightarrow & \pi(Y, \varphi(x)) \end{array}$$

THM $\varphi_0 \sim \varphi_1$ rel $\{x\}$ THEN $\varphi_{0*} = \varphi_{1*}$

deformation retract

- $r : \cancel{X} \longrightarrow A$ cont. $r(a) = a$ FOR $a \in A$
- $i \circ r \sim Id_X$ rel to A

then $i_* : \pi(A, a) \xrightarrow{\simeq} \pi(X, a)$

DEF : <u>Simply connected</u> : - PATH CONNECTED
- $\pi(X) = 1$

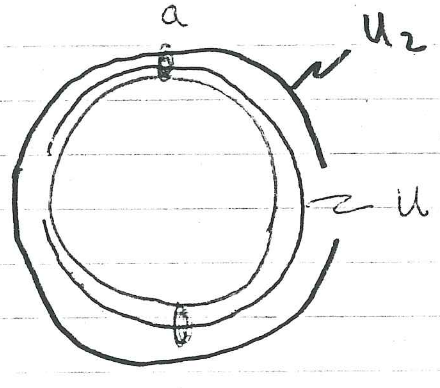
(No Holes)

DEF : <u>CONTRACTIBLE</u> : if ~~...~~ THERE is $a \in X$ SUCH THAT $\{a\}$ is A DEFORMATION RETRACT of X

[π (CONTRACTIBLE) = 1]

$$\boxed{\pi(S^1) \simeq \mathbb{Z}}$$

IdeA :



U_1 (AND U_2) are contractible (simply connected)

So every close path

$$f : [0,1] \longrightarrow S^1 \quad \sim \quad \alpha^m \qquad (m \in \mathbb{Z})$$
$$0 \longmapsto a$$
$$1 \longmapsto a$$

where $\alpha = (\cos 2\pi t, \sin 2\pi t)$

Now we need to see that $\alpha^m \not\sim \alpha^n$ if $m \neq n$.

$\therefore$ USE COMPLEX ANALYSIS AND FIND A CONTINUOUS DETERMINATION OF THE ARGUMENT of

$z = e^{i\theta} \qquad a(z) = \theta + 2n\pi$

(winding number)

⋮

$\pi(\infty) =$ FREE GROUP $\{\alpha, \beta\}$

$= \{ \underbrace{\alpha\beta\alpha\alpha\beta^{-1}\alpha^{-1}\alpha\beta \cdots \beta}_{\text{FINITE WORD}} \}$

$\pi(\bigcirc) =$ FREE GROUP $\{\alpha\}$

$\pi(\text{Ꮻ})$ = Free Group $\langle \alpha, \beta, \gamma \rangle$

shell

⋮

Math 131, April 21 1993

Nantel's material: triangulations, disks w/ identifications, $T^2 \# \mathbb{RP}^2 \cong \#^3 \mathbb{RP}^3$; eliminate aa^{-1}; one vertex; No ..b.., b..; No ...a..a^{-1}...; Euler char, invariance, ...

$\chi(\#^n T^2)$? $\chi(\#^n \mathbb{RP}^2)$?

Groupness of π_1; pushforwards φ_*; functoriality; homotopy of maps; retracts; def ret; simply conn; contractible; $\pi_1(S^1)$.

Brouwer fixed pt.: 1. state 2. why a. eq'ns b. flows
3. No $\Rightarrow$ retract
4. $S^1 \xrightarrow{i} B^2 \xrightarrow{r} S^1$

Def $p: E \to B$ cont onto. U (open) evenly covered $\Leftrightarrow p^{-1}(U) = \bigcup V_\alpha$, V_α open, $p|_{V_\alpha}$ is homeo

V_α's are "slices"

Def Covering map $p: E \to B$ E "covering space".

Eg: $\mathbb{R} \to S^1$, $S^1 \xrightarrow{n} S^1$

Def lifting

$$\begin{tikzcd} & E \arrow[d, "p"] \\ X \arrow[ur, "\tilde{F}"] \arrow[r] & B \end{tikzcd}$$

The path lifting prop: $p: E \to B$ covering map $p(e_0) = b_0$, $f: [0,1] \to B$, $f(0) = b_0 \Rightarrow \exists! \tilde{\gamma}$

The homotopy lifting prop $\;\; b_0\,\boxed{F}\,b_1 \longrightarrow b_0\,\boxed{\tilde{F}}\,b_1$

Thm $\pi_1(S^1) = \mathbb{Z}$

$\boxed{\text{HW}}$

Math 131 - HW due April 28:

1. Read Massey II.6 and Munkres 8.3 and 8.4.
2. Do Massey 3.1, 3.3, 4.2, 4.4, 4.8, and Munkres 8.3.5, 8.4.9.
3. Recall the "roach surface" X of March 22nd.
   a. Compute the Euler characteristic of X by first computing v, e, and f.
   b* Show that X is orientable.
   c. Deduce that X is the connected sum of 17 copies of the torus.

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MATH 131 — PRIZES! PRIZES! PRIZES!

DROR BAR-NATAN

April 25, 1993

Remember, during term each student may accumulate up to 150 points by solving various (normally rather hard) problems and collecting prizes.

To preserve my own sanity (or at least a part of it), I will only read solutions which are mathematically crystal clear that are either typed or written with a perfect handwriting.

You may drop solutions in my math department mailbox any time before the final examination.

Prize 1. 50 points to be awarded to the first few who solve this problem, provided all the solutions are original. When (and if) the solution will become general knowledge, I will remove this problem from the list.

Let R be a finite closed rectangle in the plane, and let $f : R \to \mathbf{R}^2$ be a distance non-increasing map — a function satisfying $d(f(x), f(y)) \le d(x, y)$ for every $x, y \in R$, where d is the standard Euclidean distance function. Is it always the case that the length of the boundary of the image $f(R)$ of R under f is smaller than or equal to the length of the boundary of R? Prove or give a counterexample.

You can win an extra 30 points by deciding whether or not it is possible to take a rectangular envelope and fold it a few times in such a way that when you put the folded envelope back in the plane, the length of the boundary of the resulting planar domain is more than the original boundary of the envelope.

You can win an extra 25 points by deciding whether or not a map f as above can be found for which the length of the boundary of $f(R)$ is infinite.

Prize 2. 150 points to anyone who convinces me that she/he *fully* understands the proof of the following theorem, which I think is one of the most amazing theorems in mathematics:

Any continuous function $f : \mathbf{R}^2 \to \mathbf{R}$ is a finite composition of functions $\phi_i : \mathbf{R} \to \mathbf{R}$ and the function $+ : \mathbf{R}^2 \to \mathbf{R}$. (For example, $xy = e^{\log x + \log y}$ and $x/y = e^{\log x + (-\log y)}$).

This theorem is the solution of one of the famous 23 problems Hilbert posed in the 1900 conference. It is due to Kolmogorov, and you may find a proof as well as some further references in Lorentz, *approximation of functions*, which is on reserve in the Cabot library. The proof is beautiful and uses no more than what we've studied, but it's not quite obvious.

Prize 3. The notion "topology" was partially invented in order to properly treat convergence. There is an analogue notion, called "uniform structure", whose purpose is to serve as the abstract foundation for uniform convergence. One of the theorems we've proven in class lends itself naturally to be treated in that language. Which one is it? You can win up to 150 points by writing an essay on uniform structures, which should be comparable in size, level and elegance to an average chapter in Munkres' book. In particular, it should contain all the necessary definitions, some motivating discussions and some examples, at least one non-trivial theorem and it should be easily readable.

Prize 4. Hindman's theorem says that whenever the natural numbers are colored with finitely many colors (i.e., a function $f : \mathbf{N} \to \{\text{a finite set of colors}\}$ is specified), one can find an infinite subset $A \subset \mathbf{N}$ and a color c, so that whenever $F \subset A$ is finite, the color of the sum of the members of F is c. You may win up to 100 points by finding a non-topological proof of this theorem. ("Finding" means either on your own or in the library).

Prize 5. (50 points) Let $\mathcal{F}$ be a non-principal ultrafilter on $\mathbf{N}$. Determine if the set

$$A_{\mathcal{F}} = \left\{ \sum_{n \in F} \frac{1}{2^n} \; : \; F \in \mathcal{F} \right\}$$

is Lebesgue measurable and if it is measurable, determine its Lebesgue measure. (Said differently, $A_{\mathcal{F}}$ is the collection of all numbers $x \in [0,1]$ for which the set of 1s in the binary expansion of x is in $\mathcal{F}$).

Prize 6. In what sense is the dimension of the Cantor set C equal to $\log_3 2$? 30 points for a *complete* and *written* answer. A complete answer should also include some justification for the notion of dimension that you choose to use.

Prize 7. (50 points) In class we proved that the existence of ultrafilters implies Tychonoff's theorem. Can you show the converse (at least for $X = \mathbf{N}$)? Namely, prove that there exist non-principal ultrafilters on $\mathbf{N}$ using Tychonoff's theorem, and without using the axiom of choice or Zorn's lemma in any other way.

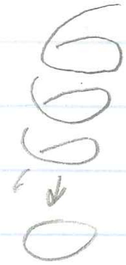
Prize 8. Can there be a continuous *onto* map $[0,1] \to [0,1]^2$? The surprising answer is *yes*. 30 points if you convince me (in writing and in detail) that you've understood why, 30 points more if you write a computer program that graphs such a map, an extra 20 if you're program is shorter than mine, and 10 extra points if you find such a map for which the inverse image of every point in $[0,1]^2$ is *uncountable*.

Prize 9. (80 points) Convince me (in writing and in detail) that you understand that there exist continuous but nowhere differentiable functions $\mathbf{R} \to \mathbf{R}$.

Prize 10. (120 points) What are $SU(2)$ and $SO(3)$? Show that $SU(2)$ is a covering of $SO(3)$ in some natural way, and use that to compute the fundamental group of $SO(3)$.

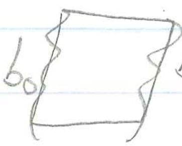
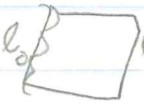
Math 131, Apr 23 1993

Remind covering spaces, $\mathbb{R} \to S^1$



Def lifting $X \xrightarrow{\tilde{f}} E$, $E \downarrow$, $X \xrightarrow{f} B$

The path lifting prop: $p(e_0) = b_0$, $f : [0,1] \to B$, $f(0) = b_0 \Rightarrow \exists! \tilde{f}$

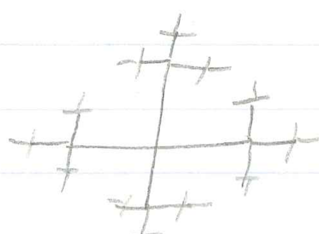
The homotopy lifting prop: b_0  b_1 $\rightsquigarrow$ e_0  e_1 e_1 is det.

Thm $\pi_1(S^1) = \mathbb{Z}$; $\pi_1(T^2) = ?$

Thm $\pi_1(X \times Y) = \pi_1(X) \times \pi_1(Y)$; $\pi_1($  $) = ?$

Van Kampen thm:

Free groups



gen & rels: $\langle a, b : ab = ba \rangle = \mathbb{Z}^2$

$G = \langle a, b : a^2 = b^2 = 1, aba = bab \rangle \cong S_3$

$\left(\begin{array}{l} a \to (12) \quad b \to (23) \\ G = \{1, a, b, ab, ba, aba\} \end{array} \right)$

$i_i : H \to G_i$

$G_1 \times_H G_2$; Van Kampen: $X = U \cup V$, U, V open + pathwis connected, $x_0 \in U \cap V \neq \emptyset$, pathwis connected

$\Rightarrow \pi_1(X) = \pi_1(U) \times_{\pi_1(U \cap V)} \pi_1(V)$

example $\pi_1($  $)$

Example $\pi_1($  $)$

Example $\pi_1($  $)$

Math 131, Apr 28 1993

Continue Van-kampen.

~~Mention Homotopy type~~

pg 50 ex 5.3

Massey pg 53 ex 7.2

HW: Read Gray & Do Sem B

+ Ex 1*, 7,

Reading/Examination Period Time Table

Math 131, May 3 1993

Dror Bar-Natan

- Dror's office hours — as usual until Wednesday May 12, then none.
- **Monday May 3:** Last class meeting, time and place as usual.
- **Monday May 3:** Tom's review session, 3PM Science Center 112A.
- **Tuesday May 4:** Jason's review session, 8PM Science Center 309.
- **Wednesday May 5:** One extra class, only for those interested, about how to use the fundamental group in order to distinguish between knots.
- **Friday May 7:** All HW is due today! 5PM at Jason's and/or Tom's mailboxes.
- **Saturday May 22:** Jason's review session, 3PM Science Center 309.
- **Sunday May 23:** Tom's review session, 3PM Science Center 309.
- **Monday May 24:** Dror's Prefinal party. 7PM in the math lounge. Bring questions!
- **Tuesday May 25:** Tom's office hours, 7PM at the Greenhouse.
- **Wednesday May 26:** The final examination. 9AM, Boylston Aud. All prizes are due before the beginning of the exam!

Math 131 - May 3rd 1993

Proof of Van-kampen's thm

statement $X = U \cup U'$; $U, U', U \cap U'$ open & path connected

$\Rightarrow \pi_1(X) = \pi_1(U) \times_{\pi_1(U \cap U')} \pi_1(U') = \left\langle \begin{matrix} g \in \pi_1(U) \\ g' \in \pi_1(U') \end{matrix} \;\middle|\; \begin{matrix} \text{rels of } \pi_1(U) \\ \text{rels of } \pi_1(U') \\ i_*(\gamma) = i'_*(\gamma) \; \gamma \in \pi_1(U \cap U') \end{matrix} \right\rangle$

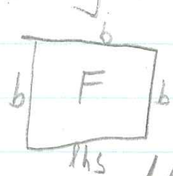
where $i : U \hookrightarrow X$, $i' : U' \hookrightarrow X$

Proof

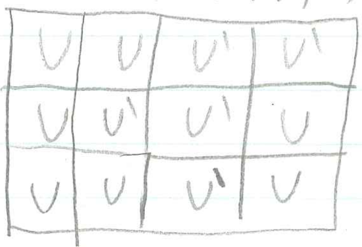
1. Define $\sigma : \overset{G}{\overset{\shortparallel}{\pi_1(U) \times_{\pi_1(U \cap U')} \pi_1(U')}} \longrightarrow \pi_1(X)$;
   well definedness:
   - a. rels of $\pi_1(U)$
   - b. rels of $\pi_1(U')$
   - c. $i_*(\gamma) = i'_*(\gamma)$

2. ontoness
   - a. Lebesgue's lemma
   - b. might as well assume $\gamma(\frac{j}{N}) = b \; \forall j$
   - c. ontoness

3. 1-1ness:
   - a. need to show $i_*(g_1) i'_*(g'_1) i_*(g_2) \cdots i'_*(g_n) = 1$
     $\Rightarrow$ l.h.s was already 1 mod rels.
   - b. $F : [0,1]^2 \to X$ s.t.



   - c. Lebesgue's lemma, N a multiple of $2n$
   - d. might as well assume $F(\frac{i}{N}, \frac{j}{N}) = b \;\; \forall \, i, j$
   - e. Finish proof.



map every segment to the corres. gen of G, regarding it either as an element of $\pi_1(U)$ or as an element of $\pi_1(U')$ as written below/left to it

Math 131 Final

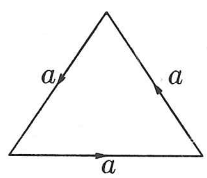
May 26 1993

Dror Bar-Natan

You have 180 minutes to solve the following 8 questions, whose total value is 100 points. Do as much as you can, but keep in mind that it is unlikely that anybody will be able to do *all* 8 questions. Plan your time wisely! It is a good idea to read the entire exam before answering any question. You may not use any material other than your pen or pencil. Don't forget to write your name on anything you submit.

(1) (18 points) Write the precise definition of each of the following notions:
   (a) A *path-connected* topological space.
   (b) The *box topology* on an arbitrary product of topological spaces.
   (c) The *sum* $\mu+\nu$ of two ultrafilters $\mu,\nu\in\beta\mathbf{N}$.
   (d) A *triangulation* of a two-dimensional manifold M.
   (e) f and g are *homotopic*, where both f and g carry (X,b) to (X',b').
   (f) A *covering* map $\pi: E\to B$.

(2) (12 points) Compute (and justify your computation) the fundamental group of the space Y obtained from a triangle by identifying its sides in the following manner:



$\left(\begin{array}{c}\text{all sides identified}\\ \text{to each other}\end{array}\right)$

Can you generalize your result to the case of a square with similar identifications? A pentagon? An n-gon?

(3) (12 points) Recall that the Stone-Čech compactification $\beta\mathbf{N}$ of $\mathbf{N}$ is the set of all ultrafilters on $\mathbf{N}$, with the topology generated by the basis $\mathcal{B}=\{U_A : A\subset\mathbf{N}\}$, where for any set $A\subset\mathbf{N}$,

$$U_A=\{\mu\in\beta\mathbf{N} : A\in\mu\}.$$

Prove that $\beta\mathbf{N}$ is compact.

(4) (12 points) Let S be the surface whose symbol is $abcdc^{-1}bdea^{-1}fe^{-1}gg^{-1}f^{-1}$. Can you describe S in terms of the classification theorem for two-dimensional manifolds?

(5) (12 points) A *contraction* on a metric space (X,d) is a transformation $T: X\to X$ for which $d(Tx,Ty)<d(x,y)$ for every $x,y\in X$.
   (a) Show that if (X,d) is compact and T is a contraction on X, then T has a unique *fixed point* — there exists a point $x\in X$ for which $Tx=x$.
   (b) Find a metric space X and a contraction T on X that has no fixed points.

(6) (12 points) Sketch the proof (namely, prove but omit technical details) of the Brouwer fixed point theorem. You may use the fact that $\pi_1(S^1)=\mathbf{Z}$, if you so wish.

(7) (12 points) Let X_n, $n\in\mathbf{N}$ be a sequence of non-empty topological spaces. Prove that $\prod_{n\in\mathbf{N}}X_n$ is metrizable iff X_n is metrizable for every $n\in\mathbf{N}$.

(8) (10 points) Show that a topological space X is Hausdorff iff the diagonal $\{(x,x) : x\in X\}$ is closed in $X\times X$.

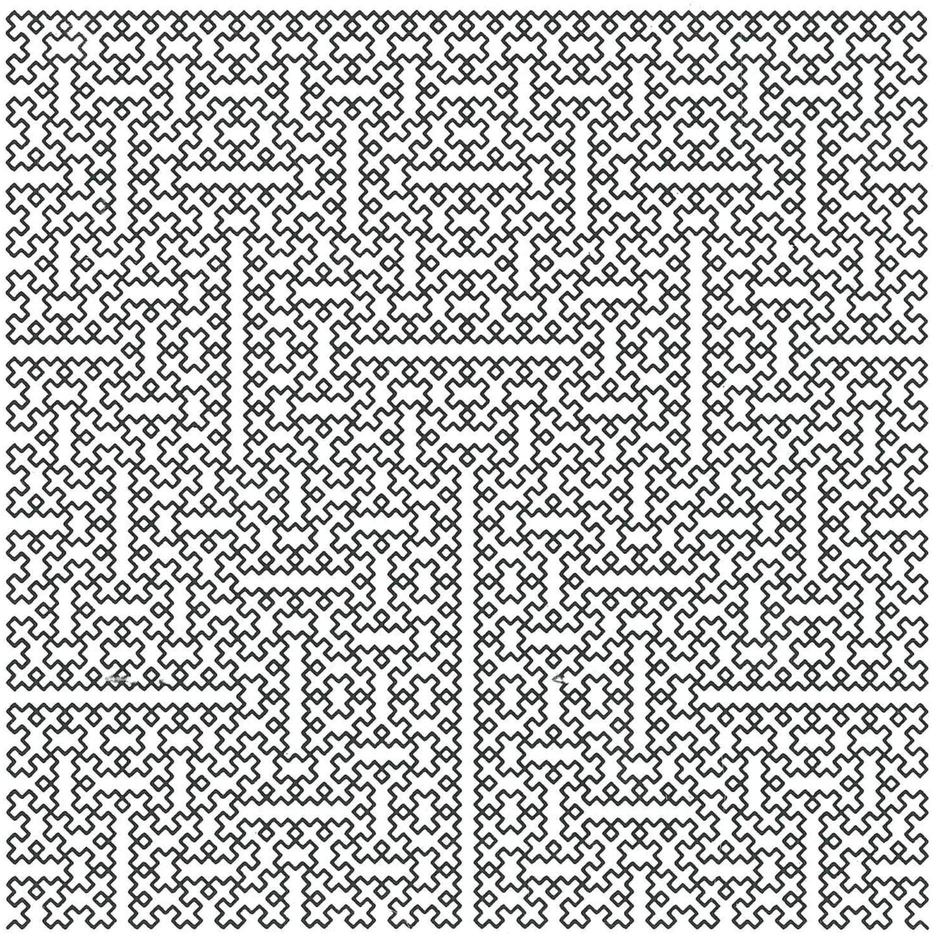
The 3-line Mathematica program below, as well as its output, is completely irrelevant for this exam:

```
Mathematica 2.0 for SPARC
Copyright 1988-91 Wolfram Research, Inc.
 -- X11 windows graphics initialized --

In[1]:=
(b[t[x_,y_]]:={t[x,-y],t[x,y],t[x,y],t[-x,y]}; b[l_List]:=Flatten[b /@ l];
 Peano[n_]:=Show[Graphics[{Thickness[0.002],Line[FoldList[Plus,{0,0},
   List @@ Join @@ Nest[b,t[{1,1},{1,-1}],n]]]},AspectRatio->1]])

In[2]:= Peano[6]

Out[2]= -Graphics-
```



— GOOD LUCK —

Problems 4, 5, 6, 8

(4)

Calculation of Euler characteristic +4 (-1 for miscounting v)

" of orientability +4

Identification of surface +4 (-1 for not wording answer as sum tori or sum projec. planes)

Total 12 points.

Alternately, cut and pasting.

Steps 2-5 of Pf. Thm 5.1 (each 3 points) Total 12 points

If messed up:

Remembering class thm. Torus + Proj. Plane = 3 Proj. Planes +2

Statement of classific. thm. +2

(5) 7 points for part 1, 5 points for part 2

(a) Show T is continuous 2 points

uniqueness of fixed point 2 points

remainder of proof 3 points

(if say problem is messed up 2 points)

(b) 5 points $\leftrightarrow$ example works

(-1 or -2 for small mistake)

(over)

(-1 if arrow goes wrong way)

6) Step 1 No fixed point → retraction to boundary 4 points

Step 2 r retract ⇒ r* onto (def. retract also OK if say why r is and that r* is iso.) 4 points

Step 3 Use $\pi_1(B^2) = id$, $\pi_1(S^1) \cong \mathbb{Z}$ 4 points

(1 point for correct statement of theorem if rest messed up)

(n=1 case 4 points)

8) X Haus ⇔ diag closed in X × X

→ 5 points total

← 5 points total ~~[illegible]~~

(-2 if use fact projections are open maps w/out proving) ← this approach avoids need of basis elements of product to[illegible]

Final grading key (1, 2, 3, 7)

$-\frac{1}{2}$ means $-(\frac{1}{2}+\epsilon)$

1. a $-\frac{1}{2}$ no def path.

b $-2\frac{1}{2}$ U open $\Leftrightarrow$ $\forall\alpha$ $\pi_\alpha(U)$ open

c -1 used the "extension" def -1 first right then left $-\frac{1}{2}$ $\mu \leftrightarrow \nu$

d $-\frac{1}{2}$ didn't define Δ. $-1\frac{1}{2}$ forgot intersection reqs.

e $-\frac{1}{2}$ not fixing basepoint. -1 path homotopies.

F $-\frac{1}{2}$ Forgot openness of V_α. $-\frac{1}{2}$ forgot "by π".

2. 9 - compute
3 - generalize
$+3$ correct guess. -1 writing ans. as $\langle a : a^3 = 1 \rangle$

-6 $\pi_1(\triangle) = \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}$ $+2$ used Van-Kampen but totally wrong.

3. 3 iden + quote 3 proof that it is indeed an UF
3 def of lim UF 3 proof of convergence.
$+2$ stated idea but did nothing about it.

7. 3 $\prod X_n$ met $\Rightarrow \forall n$ X_n met
3 def of met on $\prod X_n$
6 two inclusions. (3 each)
-2 $\rho = \sup(\rho_i)$ (no bounding -1, no rescaling -1) $+1$ met $= \sum \rho_i$
$+3$ proved $\Leftarrow$ only in finite case.
$-\frac{1}{2}$ max instead of sup

6/3/93

- Math 131 makeup final:

1. Define: a. The quotient topology. (Given a space X and an equiv. rel. ~)

18

b. A retract.

c. Homotopic paths.

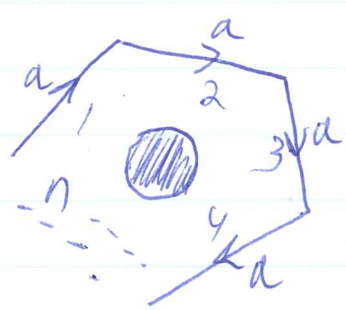
d. A non-pricipal ultrafilter on X

e. A manifold.

f. Limit-point-compactness

2. What is the fundamental group of the space $Y_n \# Y_m$, where Y_n is defined by

15



and the "connect-sum" operation is performed along the marked blob (●).

3. Prove that addition of ultrafilters is continuous from the right.

12

4. Which surface (in terms of our classification) is:

12

$$aba^{-1}cdeb^{-1}ffgg^{-1}e^{-1}d^{-1}c \quad ?$$

5. Sketch the proof of $\pi_1(S^1) = \mathbb{Z}$.

13

.../cont.

6. Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of non-empty topological spaces. Prove that

12

$$\prod_{\alpha \in I} X_\alpha \text{ is connected} \iff \forall \alpha \in I \ X_\alpha \text{ is connected}$$

You may use the corresponding fact for finite products without proof, if you so wish.

7. Let $f : X \longrightarrow Y$, let Y be compact Hausdorff. Then f is continuous if and only if the graph of f,

18

$$G_f = \{(x, f(x)) : x \in X\} \subset X \times Y$$

is closed. Prove that!

Hint: If G_f is closed and V is a neighborhood of $f(x_0)$, find a tube about $x_0 \times (Y - V)$, not intersecting G_f.

GOOD LUCK!

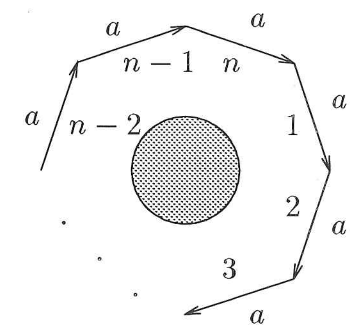
& remember, get as much as you can but getting everything is probably more than anyone can.

Dror Bar-Natan

You have 180 minutes to solve the following 7 questions, whose total value is 100 points. Do as much as you can, but keep in mind that the exam is probably too hard to be completely solved. Plan your time wisely! It is a good idea to read the entire exam before answering any question. You may not use any material other than your pen or pencil.

(1) (18 points) Write the precise definition of each of the following notions:
   (a) The *quotient* topology of a space with an equivalence relation.
   (b) A *retract.*
   (c) *Homotopic paths* in a topological space.
   (d) A *non-principal ultrafilter* on a set X.
   (e) A *manifold.*
   (f) *Limit point compactness.*

(2) (15 points) What is the fundamental group of the space $Y_n \# Y_m$, where Y_n is defined by



(a polygon with n sides, all identified to each other)

and the "connect-sum" operation is performed along the marked blob?

(3) (12 points) Prove that addition of ultrafilters is continuous from the right.

(4) (12 points) Let S be the surface whose symbol is $aba^{-1}cdeb^{-1}ffgg^{-1}e^{-1}d^{-1}c$. Can you describe S in terms of the classification theorem for two-dimensional manifolds?

(5) (13 points) Sketch the proof of $\pi_1(S^1) = \mathbf{Z}$.

(6) (12 points) Let $\{X_\alpha\}_{\alpha \in I}$ be a collection of non-empty topological spaces. Prove that $\prod_{\alpha \in I} X_\alpha$ is connected iff for every $\alpha \in I$, X_α is connected. You may use the corresponding fact for finite products without proof, if you so wish.

(7) (18 points) Let $f : X \to Y$ be a function with X an arbitrary topological space and Y compact and Hausdorff. Prove that f is continuous iff the graph of f,

$$G_f = \{(x, f(x)) : x \in X\},$$

is closed.

Hint: If G_f is closed and V is a neighborhood of $f(x_0)$, find a tube about $x_0 \times (Y - V)$, not intersecting G_f.

— GOOD LUCK —