115 Week of Sep 23

M

23 CBI 33, 36–40

Cauchy's theorem; generalizations to non-simply connected domains, independence of path, anti-derivatives.

Cauchy's Formula, and same to all orders.

Morera's theorem.

W

25

F

27

115 Week of Sep 16

M 16 Introduction; and:
Brief review of the complex number system - one exercise on each of: multiplication, division, Geometric interpretation, muduli, Polar form, exponential form, Power, root.
Topological words: ε-neighborhood, interior pt, exterior pt, boundary pt, openness, closedness, closure, connectedness, domains, regions, boundedness, acm. pt, including ∞, stereographic proyection, functions, Polynomials, rationals, limits & properties of them, continuity.

W 18 CBI 2.14-20 Analitic functions - the Cauchy-Riemann Equations; the exponential trigonometric & hyperbolic functions

F 20 The logarithm, complex exponentials & inverse trigonometric functions.
Integration along contours.

115 Sep 16, 1991

BS: 3 Introduce myself & Jason
What is this course about? Generalizing notions that you already know

3 1. complex number
(because they were created by god.)
evaluation of integrals
potential theory
steady temperature
2-dim fluid flow

3 2. PDE Heat
Laplace
Wave
General methods to solve these.

3 3. Calculus of Variations
The cash problem

3 Reading policy
3 Home work policy
3 computer policy
3 midterms & grading

See next page { multiplication division conjugate (and rules) moduli (and triangle)
polar form exponential form, Geometric interp,.
Powers & roots

HW: Read #1 1-7
Exercises 2.1, 2.3, 2.10, 4.2, 4.4, 7.2, 7.18

$(2+3i)+(5-4i)$

$(2+3i)\cdot(5-4i)$

Is $\overline{z_1 \cdot z_2} = \overline{z_1}\cdot\overline{z_2}$

$(2+3i)/(5-4i)$

$|(2+3i)(5-4i)|$; $|(2+3i)+(5-4i)|$

(Geometric interp)

$(1+\sqrt{3}i)^5$

$32^{1/5}$

INFORMATION SHEET FOR MATH 115

Name: Class:

Dorm address:

Dorm phone number:

I want to major in:

I'm taking this class because:

I've taken the following math courses before: (List high school courses if you haven't taken math courses in Harvard yet)

I've taken the following physics courses before:

The other science courses that I'm taking this term are:

Please list your computer experience - which computer languages have you used, what computers have you used, what type of software applications have you used and to what extent. Have you ever written programs that used computer graphics?

I've used electronic mail / I intend to use electronic mail (Y/N)?

My electronic mail address is:

NOTE: Computer literacy IS NOT a prerequisite for taking this course!

MATH 115 COURSE DESCRIPTION

Methods of Analysis and Applications, Fall 1991

12 Noon MWF, Science center 119.
INSTRUCTOR: Dror Bar-Natan, Science Center 426G, 495-8797, dror@math.
OFFICE HOURS: 10am-11am MWF.
TEACHING FELLOW: Jason Fulman, 493-5794, fulman@husc4.

TEXTBOOKS:	#1	Churchill & Brown, Complex Variables and Applications.
	#2	Churchill & Brown, Fourier Series and Boundary Value Problems.
	#3	Gelfand & Fomin, The Calculus of Variations.

	M	W	F	Topics
Sep	16	18	20	Complex numbers and functions, Cauchy's theorem and
	23	25	27	its applications. Series, residues.
Oct	30	2	4	(#1, chapters 2-6)
	XX	9	11	Conformal mappings and their applications for electrostatic
	14	16	18	potentials, steady temperature and 2-dimensional flow.
	21	23	25	(#1, chapters 7-11)
Nov	T1	30	/1	Separation of variables to solve partial differential equations,
	4	6	8	Fourier series and boundary value problems (#2, chapters 2-4)
	XX	13	15	Fourier integrals, Bessel functions, Legendre polynomials.
	18	20	22/	(#2, chapters 5-9)
	T2	27	XX	First variations, Euler Lagrange equations, constrains and
Dec	2	4	6	boundary effects. (#3, chapters 1-3)
	9	11	13	The Hamiltonian approach. (#3, chapter 4)
	16	18	XX	Second variation. (#3, chapter 5)

T1,2 - Exams no. 1,2.

Homework will be assigned at the end of each lecture and due the next. There will be about two hours of homework for each lecture. If things will work out, some of the students will be asked to prepare demonstrations for parts of the material, and will receive extra credit for that.

Final Grade=(T1+T2+3*Final+Homework+(or -)Instructor's grade)/6

115 Sep 18, 1991

10 Do $\sin z + \sin 2z + \dots + \sin nz$

3 Define a complex function: $f(z) = e^z$ $g(x+iy) = (x - \sin y, y^2 - 3x)$

8 Define differentiation (at a point.), check on $f(z) = c, z$

3 All rules apply $(cf)' = cf'$ $(f+g)'$ ✓ $f(g(z))' =$ $f \cdot g$ f/g ✓

10 The Cauchy-Riman eq'ns

5 2 examples - f, g ✓

5 Thm: Cont. partial der. sat C-R $\Rightarrow$ diffability ✓

HW: Read up to 17 (and what I didn't prove you don't have to prove either)

EX 8.1, 12.3, 12.4, 15.2, 18.3, 18.7

115 Sep 18, 1991

5 Review. Definition of analyticity. in a domain.

5 why Laplace eqn?

C-R ⟷ Laplace

4 C-R $\Longrightarrow$ Laplace

2 "harmonic conjugate" (asymm!)

5 existance of harmonic conjugate?

5 Example: $u(x,y) = x^2 - y^2$

5 e^z : entire, $(e^z)' = e^z$, $e^{z+w} = e^z e^w$

1 $\sin z, \cos z$

1 $\sinh z, \cosh z$

8 log
1. difinition principl.
2. derivative
3. problematic

3 z^c example: i^i

3 $\cos^{-1} z$

Hw: read 20–28 PS: Things will get slower!

Ex: 20.9,10,11 ; 22.11,13
26.14 ; 28.10a

20.9.d : find harm. conj $\frac{y}{x^2+y^2}$

~~20.10 $\bar{u} = v$ $\bar{v} = u \Rightarrow u = v = c$~~

20.11 $\bar{f}$ anal $\rightarrow f = c$

22.11 $\overline{e^{z}} = e^{\bar{z}}$; Solve $e^{i\bar{z}} = \overline{e^{iz}}$

22.13 behav e^z as $z \to \infty$

26.14 use CR for anal of log

28.10a $tg^{-1}(2i)$

Next time say something about conformal maps!

Existance of harmonic conjugate:

Want to solve (given u w $u_{xx}+u_{yy}=0$)

$$u_x = v_y \qquad u_y = -v_x$$

$u_x = v_y$ Find $\phi(x,y)$ with $\phi_y = u_x$

set

$$v = \phi(x,y) + \psi(x)$$

$v_x = -u_y$

$$\phi_x(x,y) + \psi'(x) \overset{?}{=} -u_y$$

$$-\psi'(x) = (\phi_x(x,y) + u_y)$$

can be solved iff RHS is independent of y!

Möbius Transformations

Math 115, Sep 23, 1991

Finish reading chapter 4 of Churchill-Brown, and solve the following exercises:

1. (a) Describe the behavior of e^{x+iy} as $x \to -\infty$.
   (b) Describe the behavior of e^{2+iy} as $y \to \infty$.
2. (a) Show that $\log z = \frac{1}{2}\log(x^2+y^2) + i\arctan(y/x)$ for $z = x+iy$.
   (b) Verify that $\log z$ satisfies the C-R equations and prove explicitly that $(logz)' = \frac{1}{z}$.
3. Compute $\arctan 2i$.

The exercises below are a bit harder. Don't feel bad if they are too hard for you.

4. The map $M : \{2 \times 2 \text{ invertible complex matrices}\} \to \{\text{analytic functions}\}$ is defined by:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto M_A(z) = \frac{az+b}{cz+d}.$$

   Prove that $M_{A \cdot B} = M_A \circ M_B$. (Namely, $M_{AB}(z) = M_A(M_B(z))$)

5. Prove that M_A maps circles into circles. Namely, prove that the set $\{|z-c| = r\}$ is mapped by M_A to a set of the form $\{|z-c'| = r'\}$.
6. Prove that if $\theta \in \mathbf{R}$, a is a complex number satisfying $|a| < 1$ and the matrix A is defined by

$$A = \begin{pmatrix} e^{i\theta} & a \\ \bar{a} & e^{-i\theta} \end{pmatrix},$$

   then M_A maps the unit disk $\{|z| < 1\}$ onto itself, and is $1-1$ (one-to-one).

7. In the picture $\{\text{analytic functions}\} \rightleftharpoons \{\text{simply connected domains}\}$ described in class, why is the map going from right to left many-valued? Hint: use 2,3 (even if you couldn't solve them).

Good Luck!

Möbius Transformations

Hard Part:

The map $M: \left\{\begin{matrix}2\times2 \text{ invertible} \\ \text{complex matrices}\end{matrix}\right\} \longrightarrow \{\text{Analytic functions}\}$ is defined by:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \longmapsto M_A(z) = \frac{az+b}{cz+d}$$

1. Prove that $M_{A\cdot B} = M_A \circ M_B$. (Namely, $M_{AB}(z) = M_A(M_B(z))$)

2. Prove that M_A maps circles to circles.

3. Prove that if $\theta \in \mathbb{R}$ and $|a|<1$ where a is a complex number and the matrix A is defined by

$$A = \begin{pmatrix} e^{i\theta} & a \\ \bar{a} & e^{-i\theta} \end{pmatrix}$$

than M_A maps the unit disk $\{|z|<1\}$ 1-1 onto itself.

4. In the picture

$$\begin{tikzcd} \boxed{\text{analytic functions}} \arrow[r, bend left] & \boxed{\text{simply connected domains}} \arrow[l, bend left] \end{tikzcd}$$

described in class, why is the map going from left to right many-valued? Hint: use 2,3 (even if you couldn't solve them).

Easier Part:

5. a) Describe the behavior of e^{x+iy} as $x \to -\infty$

   b) Behavior of e^{2+iy} as $y \to \infty$

6. a) Show that $\text{Log}\, z = \frac{1}{2}\text{Log}(x^2+y^2) + i\, tg^{-1}\left(\frac{y}{x}\right)$ for $z = x+iy$

   b) Verify that $\log z$ satisfies C-R and prove explicitly $(\log z)' = \frac{1}{z}$

7. Compute $tg^{-1}(2i)$

Good luck!

115 Sep 23, 1991

5 Analytic functions $\Longleftrightarrow$ $\overset{\text{simply connected}}{\text{domains}}$

3 Easy side

5 What do small circles map to ? ↑

3 Analytic functions are conformal maps.

15  $\longrightarrow$ graph. $\longrightarrow$ Andrei's theorem

Story telling.

5 Theorem: (   ) in the limit, one gets a conformal map.

8 log : definition, principal
derivative
Problematic

3 z^c example i^i

3 $\cos^{-1} z$

real stuff.

H/W - See page

Project!

115 Sep 25, 1991

review

integrationg along a contour as a sum

using a parametrization

example: $\int_{\circlearrowleft} \frac{1}{z}$

as a differential form.

Remember Green's theorem: $\int_{\circlearrowleft} P dx + Q dy = \iint_{\bullet} (Q_x - P_y) dx dy$

Cauchy's theorem, weak form.

The strong form holds.

HW: read 30-35, and if you feel like it, glance at 36, 37

DO: 31,2 ; 31,3; 31,6; 33,6,7,8 ✓

int on the line ✓ contour integ.

Project; Jason.

115 Sep 27, 1991 6

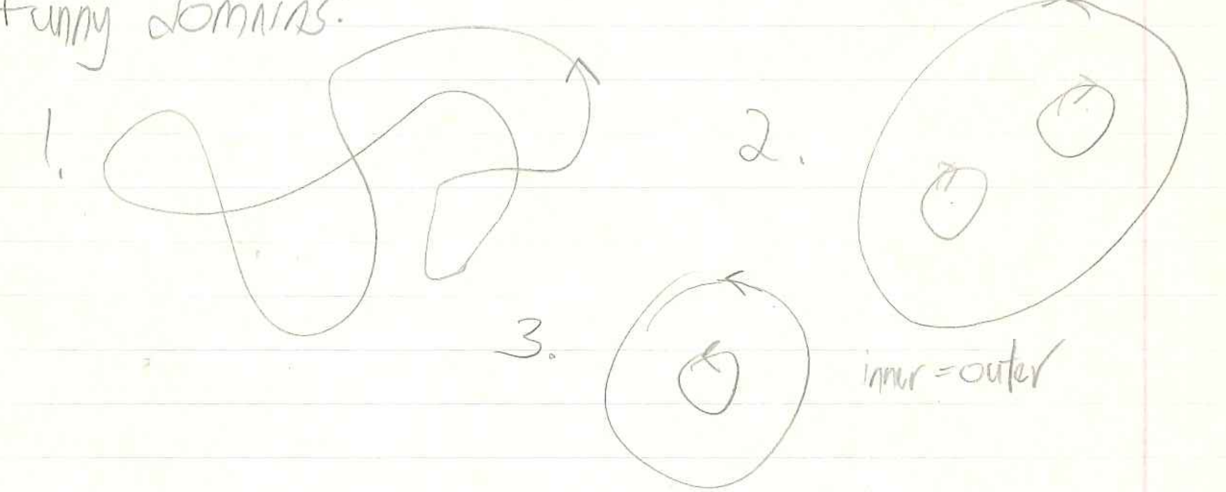
5 Review

10 Green's theorem $\int P\,dx+Q\,dy=\iint(Q_x-P_y)\,dx\,dy$

$$\int\left\langle\begin{pmatrix}dx\\dy\end{pmatrix},\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}Q\\P\end{pmatrix}\right\rangle=\iint(Q_x+P_y)\,dx\,dy$$

10 TFAE 1. f has an anti-derivative in Ω
$\Updownarrow$ 2. indep. of path
$\Updownarrow$ 3. 0 on circles
$\Updownarrow$ 4. f in analytic.

15 Funny domains:



5 Cauchy's integral formula:

$$f(z_0)=\frac{1}{2\pi i}\oint\frac{f(z)}{z-z_0}\,dz$$

5 The values of an analytic fcn. on the boundary determine the values inside!!

5 $f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_C\frac{f(z)\,dz}{(z-z_0)^{n+1}}$; Derivatives to all orders exist!

HW: read 34, 38-41 do: 38. 7, 8, 9, 12 ; 41. 1, 9, 10.

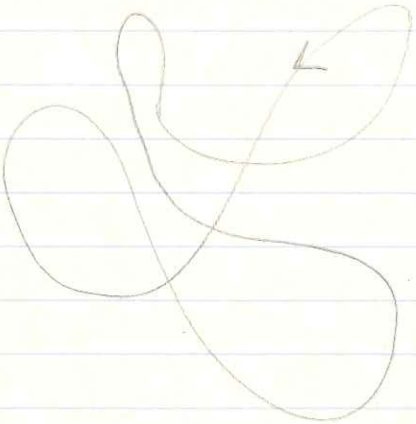
115 Sep 30, 1991

Office hours?

Review: TFAE 1. anti-derivative
2. indep of contour
3. 0 on circles
4 analytic
do?

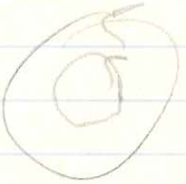
Funny domains

1.



2.

3.



inner = outer

Cauchy's integral formula $f(z_0)=\frac{1}{2\pi i}\int_{\partial}\frac{f(z)}{z-z_0}dz$

bndry values determine interior!

$f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_C\frac{f(z)dz}{(z-z_0)^{n+1}}$; deriv. to all orders exist.

Gausses' mean value theorem

The maximum principle

Liouville's Thm. (bnd. entire ⇒ constant)

The fundamental Thm of algebra.

HW: Finish reading chap 4, do 41.1,9,10, 43.4 (don't see 43.5,6,8)

Math 115, Oct 2, 1991

2 Review: $f(z_0) = \frac{1}{2\pi i}\int \frac{f(z)}{z-z_0}dz$

3 $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int \frac{f(z)\,dz}{(z-z_0)^{n+1}}$

2 Gauss' mean value theorem

2 The maximum principle.

5 Liouville's theorem (bndd entire ⇒ constant)

5 The Fundamental Thm of algebra.

12^{30} 5 Series $f(z)=\sum_{n=0}^{\infty} a_n z^n$ $\left(=\sum_{n=0}^{\infty} a_n (z-z_0)^n\right)$
(main point compare with $\frac{1}{1-x}$ $\frac{1}{1+x^2}$)

10 1. Analytic f's have such representations, convergent at least as far as the smallest non-analyticity of f. (main pt. $\frac{1}{s-z} = \frac{1}{s} + \frac{1}{s^2}z + \ldots + \frac{1}{s^N}z^{N-1} + z^N\frac{1}{(s-z)s^N}$)

5 2. $\sum_{n=0}^{\infty} a_n z^n$ convergent for $|z|=R \Rightarrow$ Uniformly convergent for $|z|<r$ where $r<R$

5 3. $\sum_{n=0}^{\infty} a_n z^n$ analytic where convergent.

5 4. Convergence precisely up to first non-analyticity

5. all operations legit.

(HW: read chap. 4,5 ignoring Laurent series
Do: 43.6,8 44.1,5,8,11)

Math 115, Oct 4 · 1991

Finish series.

Taught by Jason.

HW: 45.1,5,8,11   48.4,5

Math 115, Oct 7, 1991

Complex functions are one-variable functions of z, and not of $\bar{z}$!

Residues by examples:

Thm: $\int_C f(z)\,dz = 2\pi i \sum_{\text{poles } z_i} \mathrm{Res}_{z_i}(f)$

HW read	do

Example $\int_{-\infty}^{\infty} \frac{x^2-x+2}{x^4+10x^2+9}\,dx$

Example: $\int_0^{\infty} \frac{\cos x}{x^2+a^2}\,dx$

Example: $\int_0^{\pi/2} \frac{dx}{a+\sin^2 x} \quad |a|>1$

Example: $\int_0^{\infty} \frac{\log x}{x^2+a^2}\,dx$

HW: read 58.

Math 115, oct 7 1991 page 2

$$\int_{-\infty}^{\infty}\frac{x^2-x+2}{x^4+10x^2+9}dx = \int_{-\infty}^{\infty}\frac{\cancel{x}^2-\cancel{x}+2}{(\cancel{x}^2+1)(\cancel{z}^2+9)}dz$$

$$=\int\frac{z^2-z+2}{(z-i)(z+i)(z-3i)(z+3i)}dz = \#$$

$$\mathrm{Res}_i(f) = \frac{-1-i+2}{2i\cdot 8} = \frac{1-i}{16i} = -\frac{1}{16}-\frac{i}{16}$$

$$\mathrm{Res}_{3i}(f) = \frac{-9-3i+2}{-8\cdot 6i} = \frac{7+3i}{48i}$$

$$\# = 2\pi i\left(\frac{1-i}{16i}+\frac{7+3i}{48i}\right)$$

$$\int_0^{\infty}\frac{\cos x}{x^2+a^2}dx = \pi i\left(\frac{e^{-a}}{2ia}\right) = \frac{\pi}{2}e^{-a}$$

$$\int_0^{\frac{\pi}{2}}\frac{dx}{a+\sin x} = \frac{1}{4}\int_0^{2\pi}\frac{d\theta}{a+\sin\theta}$$

$$z=e^{i\theta} \quad dz = ie^{i\theta}d\theta \quad d\theta = \frac{dz}{iz} \quad \sin\theta = \frac{z-z^{-1}}{2i}$$

$$= \frac{1}{4}\int_{a}\frac{dz}{iz}\frac{1}{a+\left(\frac{z+z^{-1}}{2i}\right)} = \frac{1}{2}\int\frac{dz}{z}\frac{1}{2ia+z+z^{-1}} =$$

$$= \frac{1}{2}\int\frac{dz}{z^2+2iaz+1} = \pi i\frac{1}{\sqrt{-4a^2-4}} = \frac{\pi}{2}\frac{1}{\sqrt{1+a^2}}$$

EXERCISES

Math 115, Oct 7 1991

1. Find the poles and residues of the following functions:

(a) $\frac{1}{z^2 + 5z + 6}$, (b) $\frac{1}{(z^2 - 1)^2}$, (c) $\frac{1}{\sin z}$, (d) $\cot z$,

(e) $\frac{1}{\sin^2 z}$, (f) $\frac{1}{z^m(1 - z)^n}$ (m, n positive integers).

3. Evaluate the following integrals by the method of residues:

(a) $\int_0^{\pi/2} \frac{dx}{a + \sin^2 x}$, $|a| > 1$, (b) $\int_0^{\infty} \frac{x^2\,dx}{x^4 + 5x^2 + 6}$,

(c) $\int_{-\infty}^{\infty} \frac{x^2 - x + 2}{x^4 + 10x^2 + 9}\,dx$, (d) $\int_0^{\infty} \frac{x^2\,dx}{(x^2 + a^2)^3}$, a real,

(e) $\int_0^{\infty} \frac{\cos x}{x^2 + a^2}\,dx$, a real, (f) $\int_0^{\infty} \frac{x \sin x}{x^2 + a^2}\,dx$, a real,

(g) $\int_0^{\infty} \frac{x^{1/3}}{1 + x^2}\,dx$, (h) $\int_0^{\infty} (1 + x^2)^{-1} \log x\,dx$,

(i) $\int_0^{\infty} \log(1 + x^2)\,\frac{dx}{x^{1+\alpha}}$ $(0 < \alpha < 2)$. (Try integration by parts.)

4. Compute

$$\int_{|z|=\rho} \frac{|dz|}{|z - a|^2}, \qquad |a| \neq \rho.$$

Hint: Use $z\bar{z} = \rho^2$ to convert the integral to a line integral of a rational function.

Not so interesting facts about residues and poles[1]

(and some interesting ones, but without proofs)

Math 115, Oct 9 1991

Singular points are the points where a complex function f is not analytic. A singular point is called *isolated* if it is isolated — namely, if there is some neighborhood thereof in which there are no other singular points. Around an isolated singular point z_0 the function f has a Laurent[2] expansion

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}. \qquad (1)$$

The coefficient b_1 is called the *residue* of f at z_0, and is denoted by

$$b_1 = \mathrm{Res}_{z_0} f.$$

2)

Theorem 1 *If C is a positively oriented simple*[3] *closed contour within and on which a function f is analytic except at the points $z_1, \ldots, z_n$, then*

$$\int_C f(z)dz = 2\pi i \sum_{z_i} Res_{z_i} f.$$

1

Theorem 2 *If C is a positively oriented simple closed contour out of which and on which a function f is analytic then*

Firs do example:

$$\int_C f(z)dz = 2\pi i Res_{z=0} \frac{1}{z^2} f\left(\frac{1}{z}\right).$$

$$\int \frac{z^{1000} + 2z - 4}{z^{1001} - 1} dz$$

5

If in (1) only finitely many of the b_n's are non-zero, the point z_0 is called a *pole*[4], and the largest m for which $b_m \neq 0$ is called the *order* of that pole. A pole of order 1 is called a *simple* pole. A pole of order 0 is called a *removable* singular point, and it isn't *really* a singular point — by a simple redefinition of $f(z_0)$ such a singularity can be removed. If a singular point is not a pole, it is called an *essential* singular point. A very hard theorem due to Picard says that in each neighborhood of an essential singular point a function assumes every finite value, with at most one exception, an infinite number of times.

3

Theorem 3 *If a function f can be written in the form*

$$f(z) = \frac{\phi(z)}{(z - z_0)^m}$$

where $\phi(z)$ is non-vanishing an analytic around z_0, then

$$Res_{z_0} f = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}.$$

In particular if $m = 1$ then $Res_{z_0} f = \phi(z_0)$.

example

$$\frac{1}{z^n (1-z)^m}$$

Good idea: remember

$$\frac{1}{(1-x)^m} = \sum_{n=0}^{\infty} \binom{m+n-1}{m-1} x^n$$

5

---

[1] If you want to read the same said in more words, consult sections 53-57 of the textbook.

[2] Spelled correctly

[3] non-self-intersecting

[4] WARNING: in class I called every singular point "a pole". The correct naming is given on this page.

If z_0 is not an essential singularity of f (or maybe not a singularity at all), then one can write $f(z) = (z - z_0)^m \phi(z)$ with ϕ an analytic function (around z_0) *which does not vanish at* z_0. If m is negative, z_0 is a pole. If $m > 0$, then z_0 is called a *zero of order* m, and if $m = 1$, this is a *simple* zero. Clearly, poles and zeros are opposite notions — if $f(z)$ has a pole of order 17, then $1/f(z)$ would have a zero of order 17 and vice versa. 3

Theorem 4 *If p and q are analytic at z_0, $p(z_0) \neq 0$, and q has a simple zero at z_0, then*

$$Res_{z_0} \frac{p}{q} = \frac{p(z_0)}{q'(z_0)}.$$

example:
tgz at z=0 3

Midterm: Oct 28 8-10 PM room 209

HW: Read -62

Do. 55.7; 59.6, 18; 61. 5, 10, 12, 13, 16

Math 115, Oct 9 1991

Go over handout. ~~da Res $\frac{1}{z^m(1-z)^n}$~~

10 $\int_0^{\pi/2} \frac{dx}{a+\sin x}$ $|a|>1$ what if $\sin^2 z$?

10 $\int_0^\infty \frac{\log x}{x^2+a^2}\,dx$

10 $\int_0^\infty \frac{x^{1/3}}{1+x^2}\,dx$ Evaluation: $(1-e^{2\pi i/3})I = 2\pi i\left(\frac{e^{i\pi/6}}{2i} + \frac{e^{\frac{\pi}{2}i}}{-2i}\right)$

HW: Read -62

Do 55.7; 59.6, 18; 61. 5, 10, 12, 13, 16

Midterm Oct 28 8-10 room <u>209</u>

Math 115, oct 11 1991. page 2

$$\int_0^\infty \frac{\log x}{x^2+a^2} = A$$

$$2A + \pi i \int_0^\infty \frac{1}{x^2+a^2}dx = 2\pi i \left(\frac{\log ia}{2ia}\right)$$

$$2A = \pi \frac{\log a + \frac{\pi}{2}i}{a} - \frac{\pi}{2} i \int_{-\infty}^{\infty} \frac{1}{(z-ia)(z+ia)} = \pi \frac{\log a}{a}$$

$$A = \frac{\pi \log a}{2a}$$

Math 115, Oct 11 1991

Review: $\int_{-\infty}^{\infty} \frac{P(x)}{q(x)} dx$ deg P + 2 ≤ deg Q

$\int_{-\infty}^{\infty} \frac{P(x)}{q(x)} \begin{matrix} \sin \\ \cos \end{matrix} x \, dx$ deg P + 1 ≤ deg Q

$\int_{0}^{2\pi} R(\sin\theta, \cos\theta) d\theta$

News:

$\int_{0}^{\infty} \frac{\log x}{x^2+a^2}$ { whenever you have a log, thing about something like that

$\int_{0}^{\infty} \frac{x^{1/3}}{1+x^2}$ {

$\int_{-\infty}^{\infty} \frac{\sin x}{x}$ { whenever a pole at zero, try going around it

---

Logarithmic residues & Rouché's theorem

Def.: log residue $= \operatorname{res} \frac{f'}{f}$

$\frac{1}{2\pi i} \int \frac{f'(z)}{f(z)} dz = \begin{pmatrix} \text{zeros} \\ \text{w. multiplicities} \end{pmatrix} - \begin{pmatrix} \text{Poles w/} \\ \text{multiplicities} \end{pmatrix} = N_f$

Rouché's thm: $|f| > |g|$ along cont. $\Rightarrow N_f = N_{f+g}$

HW: Wednsday's HW.

Math 115 - Second half of complex analysis.

7 classes available

1. Preview of next 7 classes; Linear fns, $1/z$, preservation of circles

2. linear fractional transformations.

3. e^z; $\log z$; $\sin z$; z^α; $\sqrt{\text{Poly}}$.

4. Temperature

5. Electric potential

6. Fluid flow

7. The Poisson integral formula; same for upper-half plane.

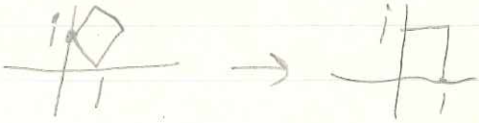
Math 115, Oct 16 1991

Mention project! double t! HW!

Preview: Heat; Harmonic functions; pullbacks, Conformal trans.

linear trans are expansion; rotation; translation.

Example: Find



The Rimann sphere & stereographic projection.

Infinity & north pole

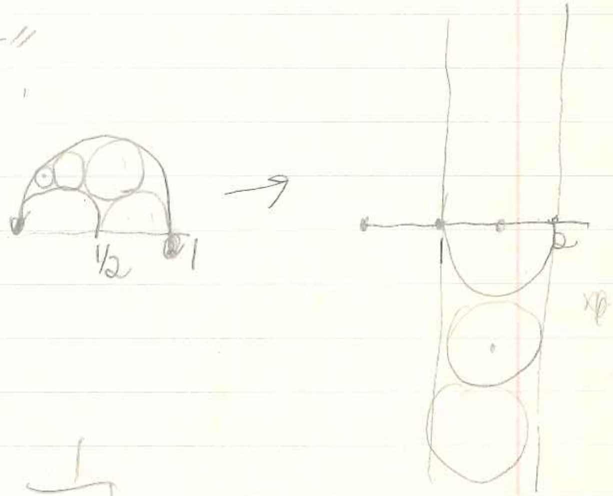
preservation of circles

lines & circles through infinity.

$z \to -\frac{1}{z}$ (HW)

$z \to 1/z$ presurves "circles"

Example:



Lin Frac: $w = \frac{az+b}{cz+d} =$

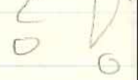
$= \frac{a}{c} + \frac{bc-ad}{c} \frac{1}{cz+d}$ ...

HW: read 64,65 & Ahlfors 18-20; Do #14; Handout.

Math 115, oct 18 1991

Review $1/z$ preserves "circles"

centers?



$w = \frac{az+b}{cz+d}$ "lin frac" "Möbius"

if $\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0$

$\begin{cases} c = 0 \Rightarrow \text{linear} \\ c \neq 0 \quad w = \frac{a}{c} + \frac{bc-ad}{c} \frac{1}{cz+d} \Rightarrow \text{circles are preserved} \end{cases}$

$M_{AB} = M_A \circ M_B$; $M_I = Id$; $M_{A^{-1}} = M_A^{-1}$

$z_0, z_1, z_2 \to 0, 1, \infty$ by $w = \frac{z - z_0}{z - z_2} \cdot \left(\frac{z_1 - z_2}{z_1 - z_0} \right)$

Example. $-1 \xrightarrow{0} -i$ $\quad M_A(z) = \frac{z+1}{z}\left(\frac{1}{2}\right)$ $\quad A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$

$0 \xrightarrow{\infty} 1$

$1 \xrightarrow{1} i$ $\quad M_B(z) = \frac{z+i}{z-1} \cdot \frac{i-1}{2i}$ $\quad B = \begin{pmatrix} \frac{1+i}{2} & \frac{i-1}{2} \\ 1 & -1 \end{pmatrix}$

$B^{-1}A = i \begin{pmatrix} -1 & \frac{1-i}{2} \\ -1 & \frac{1+i}{2} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix} = i \begin{pmatrix} -\frac{i}{2} & -\frac{1}{2} \\ \frac{i}{2} & -\frac{1}{2} \end{pmatrix} \cong \begin{pmatrix} 1 & -i \\ -1 & -i \end{pmatrix}$

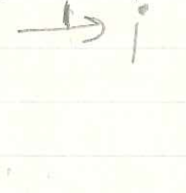
<u>claim</u> $w = \frac{az+b}{cz+d}$ a, b, c, d real, $\det > 0$

maps upper half plane to upper half.

<u>claim</u> $w = e^{i\alpha} \frac{z - z_0}{z - \bar{z}_0}$ upper → disk α real, $\operatorname{Im} z_0 > 0$

<u>claim</u> $w = \frac{e^{i\theta} z + a}{\bar{a} z + e^{-i\theta}} = e^{i\alpha} \frac{z + b}{\bar{b} z + 1}$ $\quad \theta, \alpha$ real $\quad |a|, |b| < 1$

<u>Exercise</u>:



into concentric circles

$z \xrightarrow{1/2} \cdot \xrightarrow{?} w$

$(\circ\!\cdot) \to (\circ)$

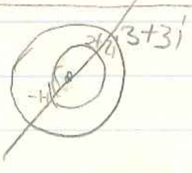
$\frac{z+b}{\bar{b}z+1}$ s.t. $\frac{b}{1} + \frac{\frac{1}{2}+b}{b\frac{1}{2}+1} = 0$

$b^2 + 2b + 1 + 2b = 0$ $\quad b = \frac{-4+\sqrt{20}}{2} = -2+\sqrt{5}$

$b^2 + 4b + 1 = 0$ $\quad$ rearrange!

HW: Read 66, 67

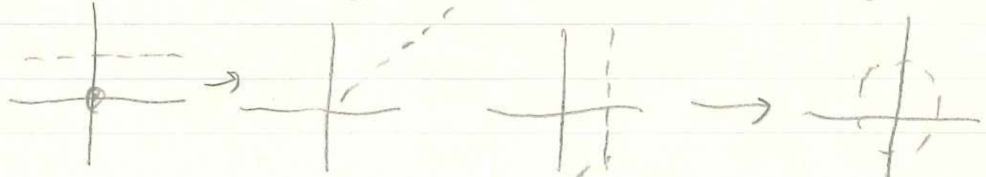
DO: 67. 2, 6, 10, map

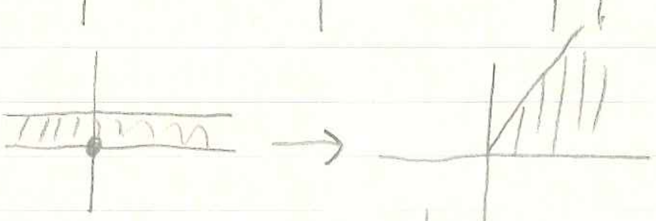


Math 115, Oct 21 1991

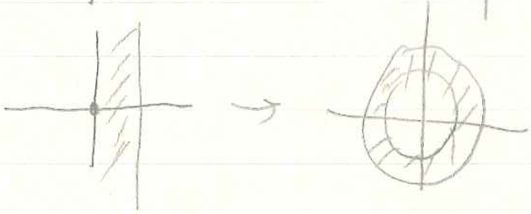
Midterm:
1. upto Electric potential (Friday before)
2. Open everything
3. HW exercises ~ old exams ; harder than final
4. Sunday 8PM - Premid party @ 426G

e^z :

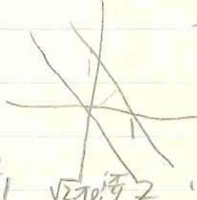




(in part. strip → upper half)



example : map to unit circle

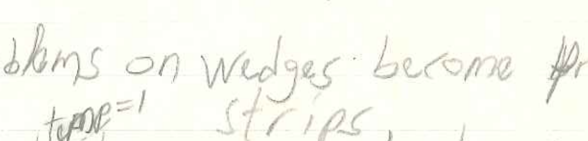


$z_1 = e^{i\frac{\pi}{4}}z$; $z_2 = \sqrt{2}\pi z_1$

$z_3 = e^{z_2}$ $\quad W = \frac{z_3 - i}{z_3 + i} = \frac{e^{\sqrt{2}\pi e^{i\frac{\pi}{4}}z} - i}{\bigcirc + i}$

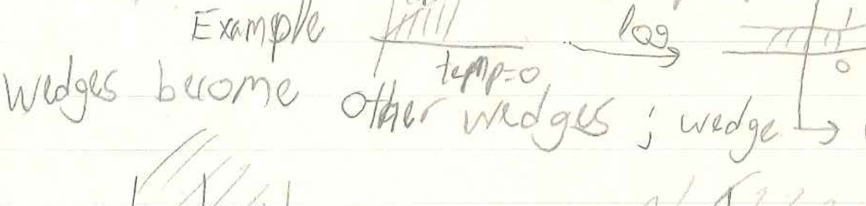
log z opposite. problems on wedges become ~~p~~problems on strips.

Example



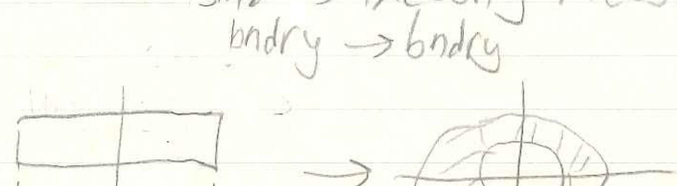
z^α wedges become other wedges ; wedge → upper half.

sin z :



sin z = sin x cosh y + i cos x sinh y (derive using addition thm)

bndry → bndry



(important for problems where heat conductivity is not isotropic !)

Read -71

Do: 69.6, 9, 12, 72.7,

cos z ? sinh z ?

If time, do first problem of temp:

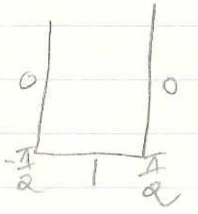
Math 115, oct 23 1991

Thm:

z_0 is harmonic
pullback of harmonic is harmonic.

Example

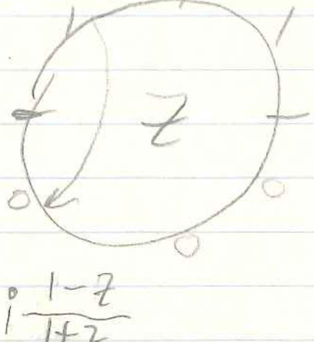
$T = \frac{1}{\pi} \text{arctg}\left(\frac{2y}{x^2+y^2-1}\right)$



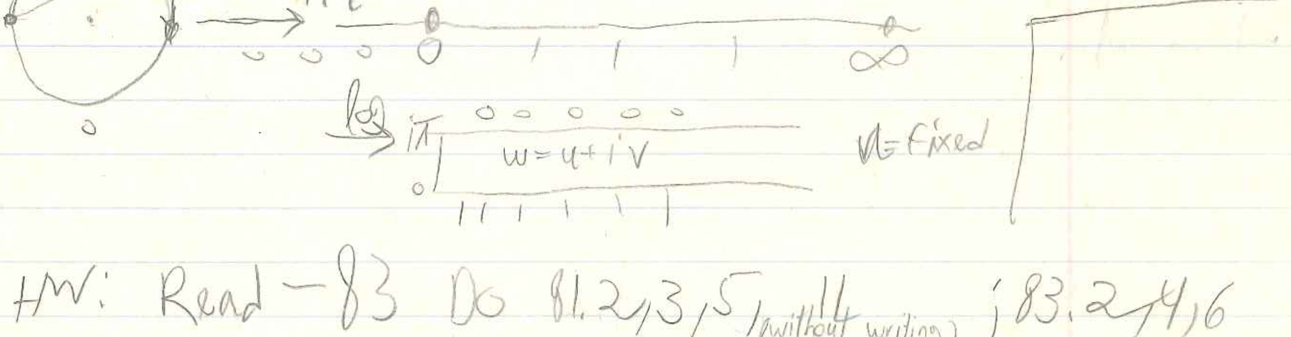
- boring $T = \frac{1}{\pi}\arctan\left(\frac{2\cos x\sinh y}{\sin^2 x\cosh^2 y+\cos^2 x\sinh^2 y - 1}\right)$

Example

Find the trajectory of an electron (in a massless apr.)



$z = \frac{i-w}{i+w}$ $w=i$



HW: Read – 83 Do 81.2,3,5,(without writing the explicit formula) 11 ; 83.2,4,6

Math 115, Oct 25, 1991

Reminder: Exam: Monday 8PM room 209
party: Sunday 8PM by room 4266

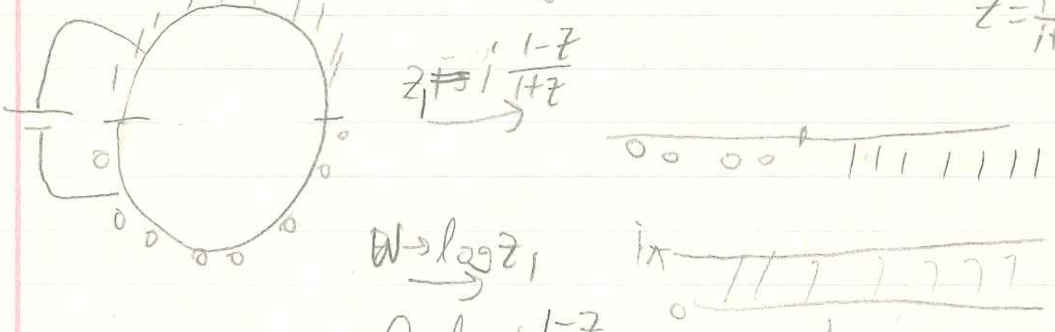
Electric potential:

Facts of life. In the absence magnetic fields and varying electric fields, there is something called "electric potential" s.t.

1. harmonic
2. Movement determined.

Example Trajectory of an elec. in a cylinder

$z = \frac{1-w}{1+w}$ sends circle to uhp



$\mathrm{Re}\log i\frac{1-z}{1+z} = \text{const}$.

apologize about not knowing fluid dynamics.

incomprissible ; No viscosity $V = p + iq$

$p_x + q_y = 0$ $q_x - p_y = \text{const} = 0$

Green's $\Rightarrow$ $\phi(z_1) = \int_{z_0}^{z} p\,dx + q\,dy$ is well defined,

$\phi_x = p$; $\phi_y = q$ $\Rightarrow$ Velocity potential is harmonic!

$F = \phi + i\underset{\text{h.c.}}{\psi}$ analytic

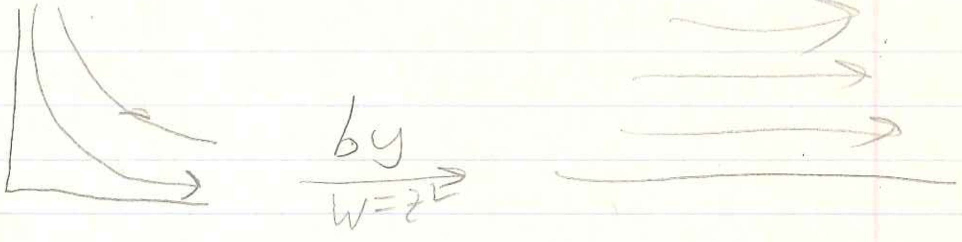
ψ is the stream function satisfying $\psi = c$ is a streamline.

Math 115, oct 25 1991 Page 2

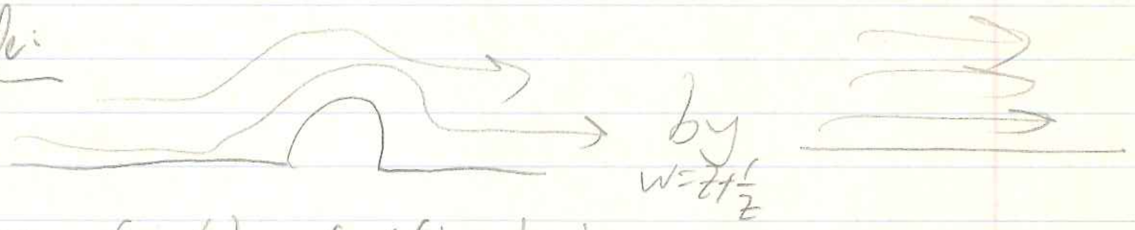
Recover velocity by $V = \overline{F'(z)}$ } Pressure P sat $\frac{P}{\rho} + \frac{1}{2}|V|^2 = \text{cond}$

Example $F(z) = Az \quad A > 0$

Example



Example:



$F = A(z + \frac{1}{z}) \quad V = A(1 - \frac{1}{z^2})$

$\Psi = A(r - \frac{1}{r})\sin\theta = C$

sym. respect to y.

HW: Elec: read -§3 Do §3.2,4,6

Flow: read -§6 Do. §6.2,3,8,9.

If time allows - talk about brownian motion

Name:____________________________

First Midterm — Complex Analysis

Math 115, Oct 28 1991
Dror Bar-Natan

You have 120 minutes to answer the following 7 questions. The weight of each question is marked on it, plan your time wisely! It is a good idea to read the entire exam before answering any question. Notice that the maximum possible total is 150 points, which is about twice of what I expect most people to get. You may use any material you wish to use other than your friends. At the end of the 120 minutes, return this form together with your work and don't forget to sign your name on anything you submit.

1. (25 points)

   (a) Obtain the Taylor series expansion of the function
   $$f(z) = \frac{z}{(1-z)^2}$$
   around $z_0 = 0$. (Life is somewhat simpler if one remembers what the derivative $\left(\frac{1}{1-z}\right)'$ is, but is still bearable even if one doesn't).

   (b) Use your result to derive a formule for
   $$S = \sum_{n=1}^{\infty} nr^n \cos n\theta.$$

   (c) For what values of r and θ is your formula valid?

2. (15 points) Use Green's theorem to prove that
   $$\int_{\partial D} \bar{z}dz = 2i \cdot \text{Area}(D)$$
   whenever ∂D is the positively oriented smooth boundary of some simply connected domain D.

3. (15 points) Can the argument of a non-vanishing analytic function have a local maximum *inside* a domain? Why?

4. Use residues to compute the following integrals:

   (a) (6 points)

$$I_a = \int_{|z|=e^{\pi}} \frac{z^{1990} + 30z - 25}{z^{1991} + 28z^{10}} dz$$

   (b) (12 points) Let m be a positive even integer. Compute

$$I_b = \int_{-\pi}^{\pi} \cos^m \theta \, d\theta$$

   (c) (12 points)

$$I_c = \int_0^{\infty} \frac{\sin x}{x(x^2+1)} dx$$

5. (20 points) The circle A is centered at $+i$ and has a radius 1, the circle B is centered at $-i$ and has the same radius, and the circle C_1, which is tangent to A and B, is also tangent to the line $\mathrm{Re}(z) = 2$. If for each $n > 1$ the circle C_n is tangent to A, B and C_{n-1}, what is the *radius* of C_n? (See figure 1)

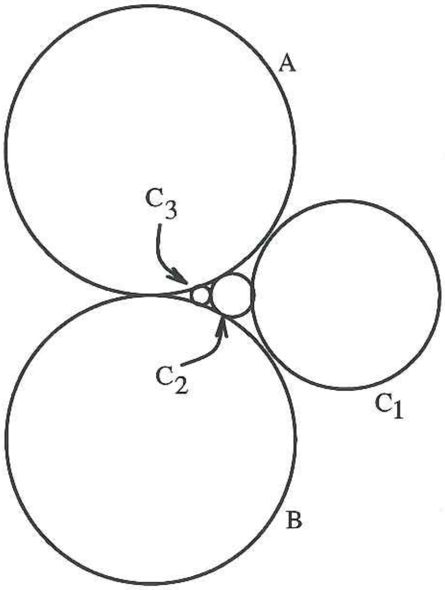


Figure 1. The circles A, B, and C_i.

6. (25 points) Find the temperature distribution on a thin disk of metal whose radius is $1\,in$ if its boundary is held at temperature $T = 0^c$ and a cup of boiling water whose radius is $\frac{2}{5}in$ is placed on the disk so that it is centered $\frac{2}{5}in$ away from the center of the disk. It is enough to write the result in terms of a complex variable z and there is no need to re-express it in terms of x and y. (See figure 2)

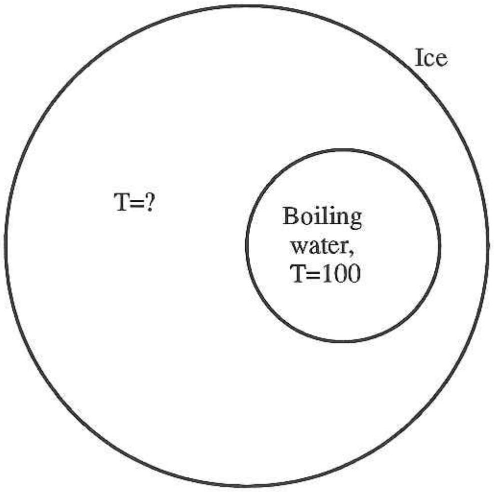


Figure 2. A thin disk of metal.

7. (20 points)

   (a) Explain why if h_2 is the harmonic conjugate of the harmonic function h_1 then their level sets $h_1 = c_1$ and $h_2 = c_2$ intersect at right angles. (Actually, this is only *generically true*, namely, true away from various singularities. But you may ignore this fact for now.)

   (b) The survival algorithm of roaches is quite simple — they just run away from the light into darkness by moving at any given time to the direction precisely opposite to the gradient of the lighting function. If the lighting function of your kitchen floor is $h_1(x,y) = x^2 - y^2$ (in some coordinate system and some units), can you draw the trajectories that roaches will follow?

Extra credit: (20 points) Use the contour made of the line segments connecting 0 to R to $Re^{2\pi i/n}$ and back to 0 to compute

$$\int_0^\infty \frac{dx}{1+x^n}$$

for an arbitrary positive integer n.

— GOOD LUCK —

Math 115 First Midterm Solution; oct 28, 1991.

Problem #1

a) $\frac{1}{1-z} = 1+z+z^2+\ldots\ldots$

differentiating term by term one gets

$$\frac{1}{(1-z)^2} = 1+2z+3z^2+4z^3+\ldots$$

and therefore

$$\frac{z}{(1-z)^2} = z+2z^3+3z^3+\ldots$$

b) Set $z = re^{i\theta}$ and get

$$S = Re\frac{z}{(1-z)^2} = Re\frac{re^{i\theta}}{(1-e^{i\theta})^2} = r\,Re\frac{e^{i\theta}(1-re^{-i\theta})^2}{|1-e^{i\theta}|^4}$$

$$= r\,Re\frac{e^{i\theta}-2r+r^2e^{-i\theta}}{((1-r\cos\theta)^2+r^2\sin^2\theta)^2} = r\cdot\frac{(r^2+1)\cos\theta-2r}{\text{same}}$$

c) Valid for any θ and for $r<1$ – because the first singularity of $z/(1-z)^2$ is at $|z|=1$.

Problem #2

$z = x+iy$; $dz = dx+idy$

$$\int_{\partial D}\bar{z}\,dz = \int_{\partial D}(x-iy)(dx+idy) = \int_{\partial D}\overset{P}{(x-iy)}dx+\overset{Q}{(ix+y)}dy =$$

$$= \iint_D (Q_x-P_y)\,dx\,dy = \iint_D (i+i)\,dx\,dy = 2i\,Area(D)$$

Problem #3

No. otherwise $\operatorname{Im} \log f(z)$ would be a harmonic function, well defined at least in a neighborhood of the offending point, which has a local maximum inside a domain, contradicting the maximum principle.

Problem #4

e^{π} is large enough so that all the zeros of $z^{1991}+28z^{10}$ (today's date) are inside the circle of that radius. Therefore

$$I_a = 2\pi i \operatorname{Res}_{z=0} \frac{1}{z^2}\left(\frac{z^{-1990}+30z^{-1}-25}{z^{-1991}+28z^{-10}}\right) =$$

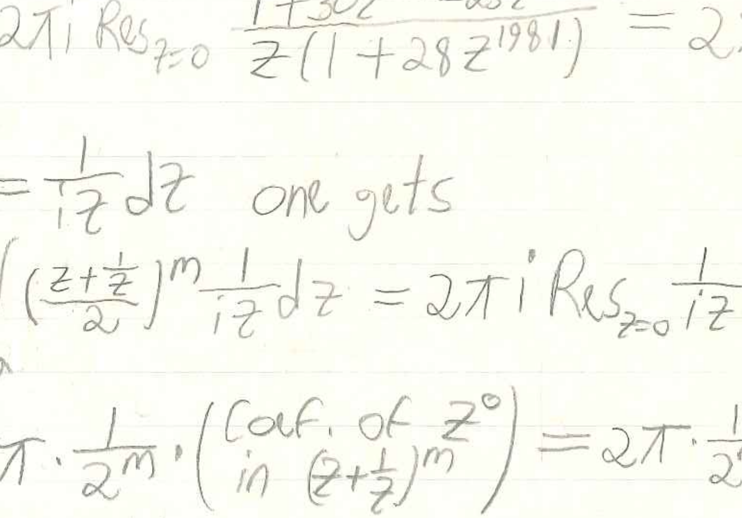
$$= 2\pi i \operatorname{Res}_{z=0} \frac{1+30z^{1989}-25z^{1990}}{z(1+28z^{1981})} = 2\pi i$$

Using $d\theta = \frac{1}{iz}dz$ one gets

$$I_b = \int_{\bigcirc} \left(\frac{z+\frac{1}{z}}{2}\right)^m \frac{1}{iz}dz = 2\pi i \operatorname{Res}_{z=0} \frac{1}{iz}\left(\frac{z+\frac{1}{z}}{2}\right)^m =$$

$$= 2\pi \cdot \frac{1}{2^m} \cdot \left(\begin{matrix}\text{Coef. of } z^0 \\ \text{in } (z+\frac{1}{z})^m\end{matrix}\right) = 2\pi \cdot \frac{1}{2^m}\binom{m}{m/2}$$

Let C_R be the contour



Then

$$\operatorname{Im}\left(\lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{z(z^2+1)}dz\right)=2\cdot I_C+\operatorname{Im}\lim_{R\to\infty}\int_{C\to D}\text{Same}+\operatorname{Im}\lim\int_{A\to B}\text{Same}$$

$$=2\cdot I_C+0+\operatorname{Im}\left(-\tfrac{1}{2}2\pi i\right)=2I_C-\pi$$

on the other hand,

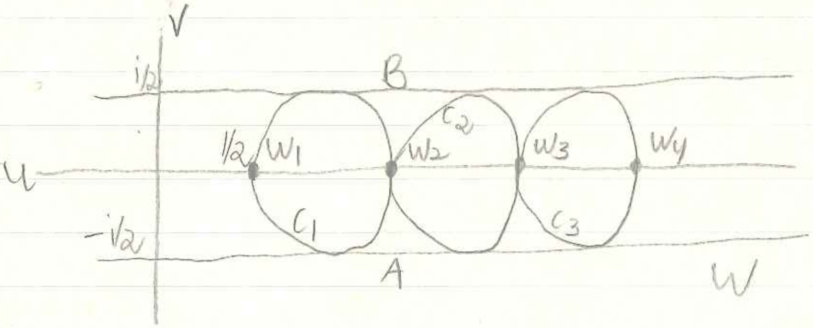
$$\operatorname{Im}\left(\lim_{R\to\infty}\int_{C_R}\right)=\operatorname{Im}2\pi i\left(\operatorname{Res}_{z=i}\frac{e^{iz}}{z(z+i)(z-i)}\right)=$$

$$=\operatorname{Im}2\pi i\frac{e^{i\cdot i}}{i\cdot(i+i)}=-\frac{\pi}{e}$$

Therefore

$$I_C=\frac{1}{2}\left(\pi-\frac{\pi}{e}\right)$$

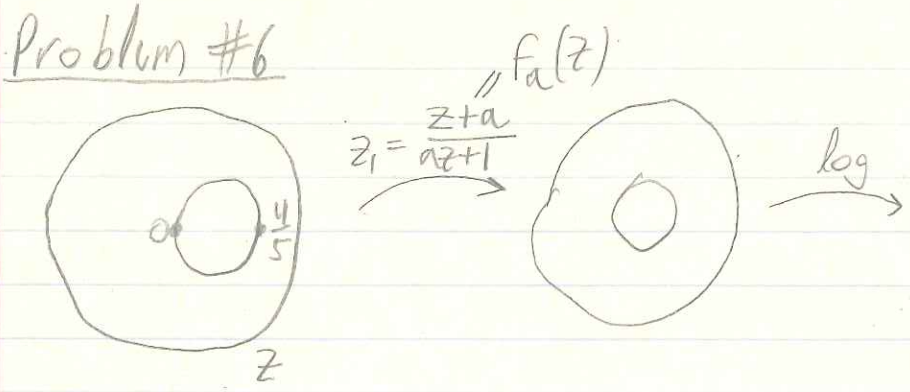
Problem #5 under $w=\frac{1}{z}$ the picture transforms to



Clearly, $w_n=n-\frac{1}{2}$ and

$$r(C_n)=\frac{1}{2}\left(\frac{1}{w_n}-\frac{1}{w_{n+1}}\right)=\frac{1}{2}\left(\frac{1}{n-\frac{1}{2}}-\frac{1}{n+\frac{1}{2}}\right)=\frac{1/2}{n^2-1/4}$$

Problum #6



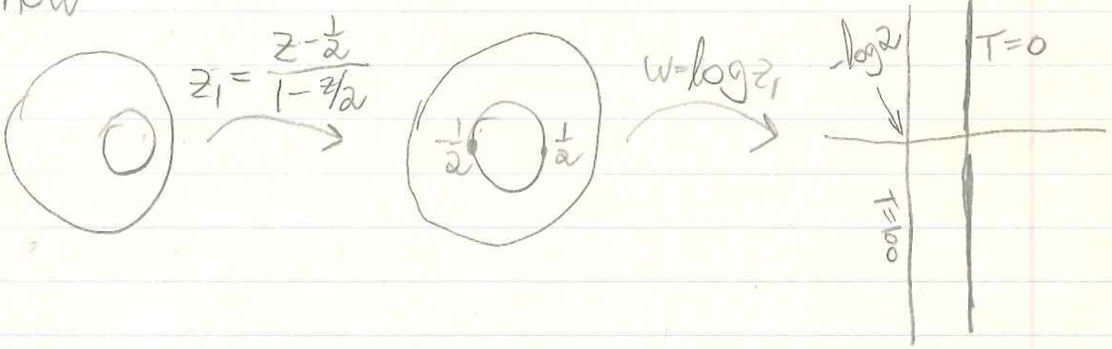
$$0 = f_a(0) + f_a(\tfrac{4}{5}) = a + \frac{4/5 + a}{4/5\, a + 1} \;;\quad \text{multiply by } \tfrac{4}{5}a + 1$$

$$\Rightarrow\quad 0 = \tfrac{4}{5}a^2 + a + a + \tfrac{4}{5} = \tfrac{4}{5}a^2 + 2a + \tfrac{4}{5}$$

$$a_{1,2} = \frac{-2 \pm \sqrt{4 - 64/25}}{2 \cdot 4/5} = \frac{-2 \pm \sqrt{36/25}}{2 \cdot 4/5} = \frac{-2 \pm \frac{6}{5}}{2 \cdot 4/5}$$

$$= -\frac{1}{2},\ -\cancel{2}_{\text{too big}}$$

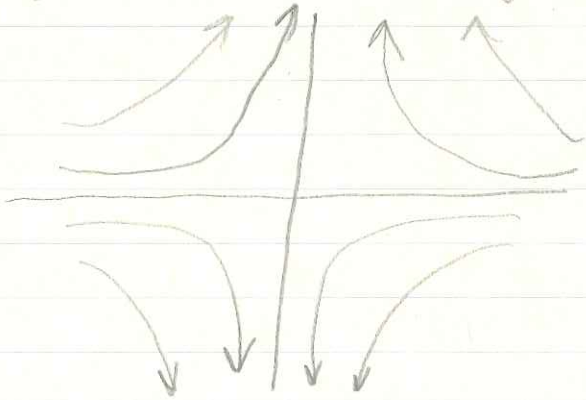
So now



$$T = -\frac{100}{\log 2}\, \text{Re}\, w = -\frac{100}{\log 2}\, \text{Re} \log \frac{z - 1/2}{1 - z/2}$$

Problem #7 Let $h = h_1 + h_2$ be the corresponding analytic function. Then h^{-1} is also analytic (if the inverse function exists, and we are assuming good behavior explicitely) and therfore $h^{-1}($$)$ is made of lines intersecting in right angles. But these are precisely the lines we are considering.

b) $h_1 = x^2 - y^2 \Rightarrow h_2 = 2xy \Rightarrow$



These are the trajectories.

Math 115 - First midterm grading key:

Problem #1

a +10 — +5 did Taylor right to order 4 but failed to generalize.

b +10 — -3 did not simplify.

c +5

Problem #2: -2 confused P & Q

Problem #3:

Problem #4: a: used right residue but didnt finish 3

used right thm.

b: correct setup, wrong residue. 9

c: Ignored trouble at $z=0$ -5

did not $\sin z \to e^{iz}$, no treatment of the problem at 0 3/12

Math 115 - midterm 1 key, page 2.

Problem #5:

12 right image, completely wrong reading back.

(-4) circles going wrong way

Problem #6: (-1) Technical

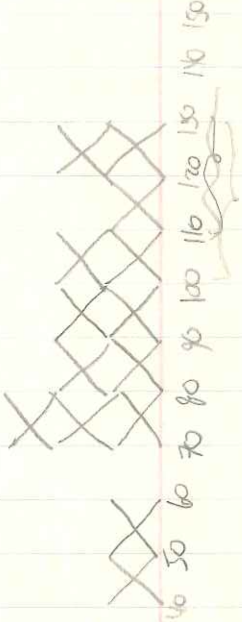
(-5) diameter is preserved

(-7) right circle wrong log

Problem #7:

a 10 pts

b 10 pts



Histogram

Math 115, oct 30 1991

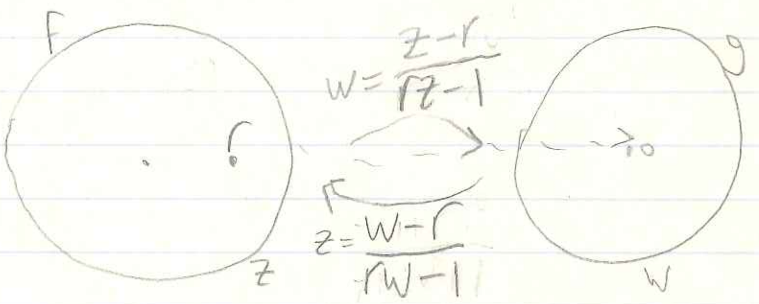
The poisson integral

Theorem:

$$F(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)F(e^{i\phi})}{1-2r\cos(\phi-\theta)+r^2}\,d\phi$$

solves Dirichlet's problem on a disk.

Proof:



$$g(e^{i\alpha}) = F\left(\frac{e^{i\alpha}-r}{re^{i\alpha}-1}\right)$$

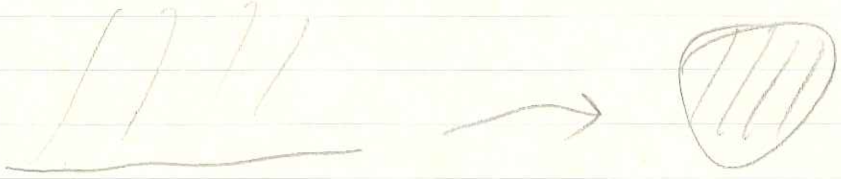
$$F(r) = g(0) = \frac{1}{2\pi}\int g(e^{i\alpha})\,d\alpha = \frac{1}{2\pi}\int F\Bigg(\frac{\overset{\overset{e^{i\phi}}{\|}}{e^{i\alpha}}-r}{re^{i\alpha}-1}\Bigg)\,d\alpha = \#$$

$$\frac{d\alpha}{d\phi} = ? \qquad e^{i\alpha} = \frac{e^{i\phi}-r}{re^{i\phi}-1} \qquad \alpha = -i\log\frac{e^{i\phi}-r}{re^{i\phi}-1}$$

$$\frac{d\alpha}{d\phi} = -i\,\frac{re^{i\phi}-1}{e^{i\phi}-r}\;\frac{ie^{i\phi}(re^{i\phi}-1)-ire^{i\phi}(e^{i\phi}-r)}{(re^{i\phi}-1)^2} = \frac{1-r^2}{1-2r\cos\phi+r^2}$$

$$\# = \frac{1}{2\pi}\int_0^{2\pi}\frac{(1-r^2)F(e^{i\phi})}{1-2r\cos\phi+r^2}\,d\phi \Rightarrow F(re^{i\theta}) = \int_0^{2\pi}\frac{(1-r^2)F(e^{i\phi})}{1-2r\cos(\phi-\theta)+r^2}$$

A very similar computation using



leads to

$$u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\overset{t}{\cancel{(x-y)}}\, u(y,0)}{(x-y)^2+t^2}\, dy$$ (Notice strange naming !)

$$(U_t(u))(x) = \int k(t,x-y)\,u(y)\,dy \quad k(t,\xi) = \frac{1}{\pi}\frac{\xi}{\xi^2+t^2}$$

we are lead to the following expectations:

Probabalistic interp.

1. $\left(\frac{\partial^2}{\partial t^2}+\frac{\partial^2}{\partial \xi^2}\right)k = \Delta k = 0$ (in upper half plane)
2. $\int_{-\infty}^{\infty} k(t,\xi)\,d\xi = 1$
3. $\lim_{t\to 0}\int_{-t}^{t} k(t,\xi)\,d\xi = 1$
4. $U_{t_2}\circ U_{t_1} = U_{t_1+t_2}$

Fundamental Solution

HW: check first thm, check 1-4, read 94.

Math 115 – PDE Part book summary.

chapter 1. Partial differential equations of physics.

1. Two related topics:
   Blah Blah about representation by series & PDE
2. Linear boundary value problems:
   Definition of that notion.
3. The vibrating string –
   – derivation of the wave eqn.
4. Modifications & end conditions
   When force is applied and when one of the ends is a ring.
5. Other examples of wave equations.
   a. Longitudinal vibrations of bars
   b. Transverse vibrations of membranes.
6. Conduction of heat.
   A derivation of the heat equation,
7. Discussion of the heat equation.
   Blah Blah . . . . diffusion.
8. Laplace's equation.
9. Cylindrical & spherical coordinates.
   The necessary changes of variable.
10. Types of equations & conditions.
    The classification of second order equations.

Math 115 – PDE part plan:

9 classes:

1 The equations Nov 1

4

2 Separation of Variables: Heat 4

[illegible]

3 Fourier 6

4 Wave & Laplace 8

5 Fourier integral & Heat 13

6 Other coordinates & the poisson integral 15

7 Sturm-Liouville. 18

8 Bessel 20

9 Legendre 22

Apologize –
$K(\xi,t)=\frac{1}{\pi}\frac{t}{\xi^2+t^2}$

Midterm: 100– A
60–100 B
–60 C
HW average: ~93

Math 115, Nov 1 1991.

name	type	equation	Dirichlet	Neuman	Mixed	Forcing
Wave	Hyper..	$u_{tt}=u_{xx}$ string $u_{tt}=u_{xx}+u_{yy}$ memb.	initial position $u=u_0$ $u_t=0$	initial velocity	both	Forc
Heat (Schrödinger)	parabolic	$u_t=u_{xx}$ $u_t=u_{xx}+u_{yy}$	initial temp	//	//	Heating
Laplace	elliptic	$u_{xx}+u_{yy}=0$ $u_{xx}+u_{yy}+u_{zz}=0$	bndry temp	Isolated body	mixture	Heat

All are linear partial differential equations

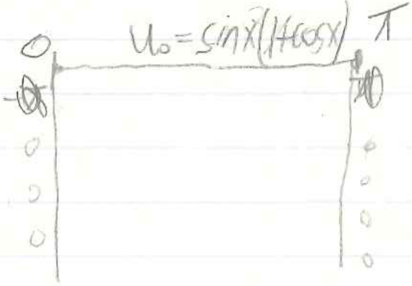
All are boundary value problems

The general bndry value problem

$$Au_{xx}+Bu_{xy}+Cu_{yy}+Du_x+Eu_y+Fu=G \quad \left\} B^2-4AC \begin{cases} >0 & \text{hyperbolic} \\ =0 & \text{parabolic} \\ <0 & \text{elliptic} \end{cases}\right.$$

is one of those.

Example



$u_t=u_{xx}$
What will happen?

HW: Read 1–8, 10 Not too carefully.

Do 5.2, 7.2, 7.8, old book:
understand physics somewhat and do math.

Do §3.6
§6, 2, 3, 8

Math 115, Nov 4 1996.

Separation of variables

Example 1

Mention eigval & eigfuncts!
Sturm-Liouville problems

Example 2/ Same with $u_t=u_{xx}$ &

The Fourier thm

Every "well-behaved" function $f:[-\pi,\pi]\longrightarrow\mathbb{R}$ can be written as

$$\frac{b_0}{2}+\sum a_n\sin nx+\sum b_n\cos nx$$

H.W. Read 11–14 not too seriously
15, 17 well. 17 beginning
16 F.T +17 end.

Do 17.1, 3, 7
F.T: Finish example 2.

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eigenvalues, eigenfunctions
sturm-Liouville

Review :

Piecewise continuous

$u=T(t)\cdot X(t)$

$T'=-\lambda T$
$X''=-\lambda X$ $= X=\sin\frac{n}{2}\cdot x$ $\quad u_0=\sum a_n \sin\frac{nx}{2}$

$T(0)=1$

Fourier Series $F:\mathbb{R}\to\mathbb{C}$ of period 2π

$$F\overset{?}{=}\sum_{n=-\infty}^{\infty}c_n e^{inx} \qquad c_n=\frac{1}{2\pi}\int_{-\pi}^{\pi}F(x)e^{-inx}\,dx$$

"Proof" 1: integrate against.

"proof" 2: Residue Formula : $g(e^{ix})=f(x)$; $g(z)=\sum c_n z^n$; $c_n=\frac{1}{2\pi i}\oint\frac{f(z)}{z^{n+1}}dz$

P1: [illegible] , $a_n=$ [illegible]

$$F=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos nx+\sum_{n=1}^{\infty}b_n\sin nx$$

$$a_0=\frac{1}{\pi}\int_{-\pi}^{\pi}F(x)dx \;;\; a_n=\frac{1}{\pi}\int F(x)\cos nx\,dx \quad b_n= \ldots$$

P4: Period L:

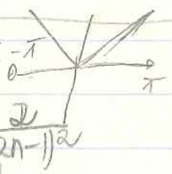
$$F=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\frac{nLx}{2\pi}+\sum_{n=1}^{\infty}b_n\sin\frac{nLx}{2\pi}$$

$$a_0=\frac{2}{L}\int_0^L F(x)dx \;; \ldots\ldots$$

P2: F even $\to b_n=0$

P3: f odd $\to a_n=0$

Example :



$b_n=0 \quad a_0=\pi$

$a_n=\frac{2}{\pi}\int_0^{\pi}x\cos nx\,dx=$

$=\frac{2}{\pi}\left[\frac{x}{n}\sin nx+\frac{1}{n^2}\cos nx\right]_0^{\pi}$

$=\frac{2}{\pi}\frac{1}{n^2}(\overset{(-1)^n}{\cos n\pi}-1)$

$0=\frac{\pi}{2}+\frac{2}{\pi}\sum\frac{2}{(2n-1)^2}$

$\frac{\pi}{8}=\sum\frac{1}{(odd)^2}$

P5: Riemann Lebesgue

P6: Piecewise cont & uniform conv.

Do problem from prev. class.

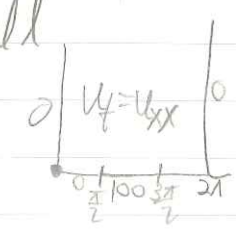
HW: Read 22-31 Lots ! Do: 26.2,4,8,10
31.3.

Post RL: 30.1,6

Math 115, Nov 8 1991

the α riddle?

Recall



$u_n^{(x,t)} = e^{-\frac{n^2}{4}t} \cdot \sin\frac{n}{2}x$

$u_0(x) = \sum b_n \sin\frac{n}{2}x$

$f(x) = \frac{a_0}{2} + \sum a_n \cos\frac{2\pi n x}{L} + \sum b_n \sin\frac{2\pi n x}{L}$

$L = 4\pi$

$$b_n = \frac{2}{L}\int_{-L}^{L} u_0(x) \sin\frac{2\pi n x}{L}\,dx = \frac{1}{\pi}\int_0^{L} u_0(x)\sin\frac{n}{2}x\,dx =$$

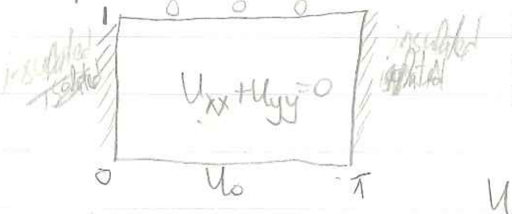
$$= \frac{1}{\pi}\int_{\frac{\pi}{2}}^{3\pi/2} \sin\frac{n}{2}x\,dx = \frac{1}{\pi}\frac{2}{n}\cdot\left[-\cos\frac{n}{2}x\right]_{\pi/2}^{3\pi/2} = \frac{1}{\pi}\frac{2}{n}\left(\cos\frac{\pi}{4}n - \cos\frac{3\pi}{4}n\right)$$

equidistribution of $n\alpha$!

P5: Rieman Lebesgue

P6: Piecewise cont. functions.

Laplace



λ shown to be > 0

$X'' = -\lambda X \quad X'(0) = X'(\pi) = 0 \quad \lambda = n^2 \quad X = \cos nx$

$Y'' = \lambda Y \quad Y = Ae^{ny} + Be^{-ny} \quad A = e^{-n} \quad B = -e^{n}$

$Y(0) = e^{-n} - e^{n} \quad u = \sum a_n u_n$

$u_0 = \underbrace{\sum a_n(e^{-n} - e^{n})}\cos nx$

$\frac{2}{\pi}\int_0^{\pi} u_0(x)\cos nx\,dx \Rightarrow a_n = \frac{2}{\pi(e^{-n} - e^{n})}\int_0^{\pi} u_0(x)\cos nx\,dx$

what if ice instead of insulation?

what if set temps on all sides?

Hit a membrane with a hammer:

The two variable Fourier expansion:

$u_{tt} = u_{xx} + u_{yy}$

HW: Reading isn't going to do you much good,

Do 30.1, 37.4,5 $u_t = ku_{xx}$, 44.2,6 all questions in section 48.

Math 115, Nov 13 1991

the 2^n riddle

Laplace from Nov 8,

2 variable from Nov 8,

Fourier integral & Heat

Example - $e^{-\frac{\lambda}{2}x^2}$

HW: 44. 2,6 ; 46. 1,2 ; Fourier integral : 65. 2,3,4
Laplace 2-var

A little about the Fourier transform

Math 115, No~~w~~v 15 1991

Definition 1 *Let f be an integrable function on* $\mathbf{R}$*. Define its Fourier transform $\tilde{f}$ by:*

$$\tilde{f}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ipx} f(x)dx.$$

Theorem 1 *(The Fourier inversion theorem) One can reconstruct f from $\tilde{f}$ using:*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ipx} \tilde{f}(p)dp.$$

Remark Notice that it follows that $\tilde{\tilde{f}}(x) = f(-x)$ and that $\tilde{\tilde{\tilde{\tilde{f}}}} = f$.
Fact Let

$$f_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma}} e^{-\frac{x^2}{2\sigma}}.$$

Then

$$\tilde{f}_\sigma(p) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\sigma p^2}{2}}.$$

Claim 1 *If $f(x) = g(x - x_0)$ then*

$$\tilde{f}(p) = e^{-ipx_0} \tilde{g}(p).$$

Claim 2 *If $f(x) = e^{ip_0x} g(x)$, then $\tilde{f}(p) = \tilde{g}(p - p_0)$.*

Definition 2 *The* convolution *of two functions f and g is defined by:*

$$(f * g)(x) = \int_{-\infty}^{\infty} dy\, f(x - y)g(y) = \int_{-\infty}^{\infty} dy\, f(y)g(x - y).$$

Claim 3 $\widetilde{f * g} = \sqrt{2\pi}\tilde{f}\tilde{g}$ *and* $\widetilde{fg} = \sqrt{2\pi}\tilde{f} * \tilde{g}$.

Remark Pick $g = f_\sigma$ and you can prove the Fourier inversion theorem!!

Claim 4 $\tilde{f'}(p) = ip\tilde{f}(p)$ *and* $\widetilde{xf}(x) = i\frac{d}{dp}\tilde{f}$.

Problems:

1. Read section 62 of the textbook and do problems 65.2,3,4.
2. Compute the Fourier transform of the function χ defined by

$$\chi(x) = \begin{cases} 1 & |x| < 1 \\ 0 & \text{otherwise} \end{cases}.$$

3. Compute the convolution $\chi * \chi$.
4. Check that indeed $\widetilde{\chi * \chi} \overset{\sqrt{2\pi}}{=} \tilde{\chi} * \tilde{\chi}$.

(hard) ?

5. Prove the *Plancherel identity*: Let f be a complex-valued function on $\mathbf{R}$.
    (a) Let $g(x) = \overline{f(-x)}$. Prove that $\tilde{g}(p) = \overline{\tilde{f}(p)}$.
    (b) Evaluate $(f * g)(0)$ and $(\widetilde{\tilde{f}\tilde{g}})(0)$ and deduce that

$$\int |f(x)|^2 dx = \int |\tilde{f}(p)|^2 dp.$$

Midterm: 2 Divinity Av
Room 18, Nov 25
7PM

Math 115, Nov 15 1991.

Fourier theory as a theory of frequencies.

expectation: if $\tilde{f}(p)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-ipx}f(x)dx$ then $f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{ipx}\tilde{f}(p)dx$

(Every answering machine, quantum particle, speak & spell, violin)

Namely, $\tilde{\tilde{f}}(x)=f(-x)$
$\tilde{\tilde{\tilde{\tilde{f}}}}(x)=f(x)$

Hard to believe: 

Yet if $f_\sigma=\frac{1}{\sqrt{2\pi\sigma}}e^{-\frac{x^2}{2\sigma}}$ then $\tilde{f}_\sigma=\frac{1}{\sqrt{2\pi}}e^{-\frac{\sigma p^2}{2}}$ and indeed $\underbrace{!}_{}$ thm holds

In fact, if $f(x)=g(x-x_0)$ then $\tilde{f}(p)=e^{-ipx_0}\tilde{g}(p)$

$f(x)=e^{ip_0x}g(x)$ then $\tilde{f}(p)=\tilde{g}(p-p_0)$

and so thm holds for all Gaussians!

Convolutions: $(f*g)(x)=\int dy\, f(x-y)g(y)=\int dy\, f(y)g(x-y)$

$\Rightarrow$ $f*f_\sigma$ (σ small) is almost f, yet it is a sum of gaussians!

claim: $\widetilde{(f*g)}=\sqrt{2\pi}\tilde{f}\cdot\tilde{g}$

$\tilde{f}_\sigma=\frac{1}{\sqrt{2\pi}}e^{-\frac{\cancel{\sigma}p^2}{2}}$ proves Fourier's thm!

claimless $\widetilde{fg}=\sqrt{2\pi}\tilde{f}*\tilde{g}$

claim $\widetilde{f'}(p)=ip\tilde{f}(p)$ $\widetilde{xf(x)}=i\frac{\partial}{\partial p}\tilde{f}(p)$

Solve heat eqn if time permits!

HW: TBA

Math 115, Nov 18 1991

Review: $F(x) \longrightarrow \tilde{F}(p) = \frac{1}{\sqrt{2\pi}}\int e^{-ixp} F(x)\,dx$

$g(p) \longrightarrow \hat{g}(x) = \frac{1}{\sqrt{2\pi}}\int e^{ixp} g(p)\,dp$

We are after the Fourier inv. thm: $\hat{\tilde{F}} = F$

$f_\sigma = \frac{1}{\sqrt{2\pi\sigma}} e^{-x^2/2\sigma} \Rightarrow \tilde{f}_\sigma = \frac{1}{\sqrt{2\pi}} e^{-\sigma p^2/2} \Rightarrow \hat{\tilde{f}}_\sigma = f$

<u>claim</u> $\widetilde{F(x-x_0)} = e^{-ipx_0}\tilde{F}$; $\widehat{e^{-ipx_0} g} = \hat{g}(x-x_0)$

$\widehat{\widetilde{F(x-x_0)}} = \widehat{e^{-px_0}\tilde{F}} = \hat{\tilde{F}}(x-x_0) \Rightarrow$ (inv thm hold for $F \Rightarrow$ holds for $F(x-x_0)$).

Continue with convolutions as planned. define, two interps, Proves FIT.

$$\widetilde{(f*g)}(p) = \frac{1}{\sqrt{2\pi}}\int dx\, e^{-ipx}\int dy\, f(x-y)g(y) =$$

$$= \frac{1}{\sqrt{2\pi}}\iint dx\,dy\, e^{-ip(x-y)} e^{-ipy} f(x-y)\, g(y) \overset{x-y=z}{=\!=}$$

$$\frac{1}{\sqrt{2\pi}}\iint dz\,dy\, e^{-ipz} f(z)\, e^{-ipy} g(y) = \sqrt{2\pi}\,\tilde{f}(p)\,\tilde{g}(p)$$

$$\begin{tikzcd} f * f_\sigma \arrow[r, "\sim"] \arrow[d, "\sigma\to 0"] & \tilde{f}\cdot e^{-\frac{\sigma p^2}{2}} \arrow[d, "\sigma\to 0"] \\ f & \tilde{f} \end{tikzcd}$$

~~hat of right = left~~

H/W: See handout.

Nov 20 1991: HW for math 115

1. Show that indeed if $\binom{x}{y}=\binom{r\cos\theta}{r\sin\theta}$ then

$$u_{xx}+u_{yy}=u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}$$

*2. Solve the heat equation on an infinite string if the initial temperature of the string is give by

$$u_0(x)=100\,e^{-x^2+2x}$$

hint: it is a good idea to write $x^2-2x=(x-1)^2-1$, and to think backward in time for a while.

3. Prove that

$$P(r,\theta)=\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\theta+r^2}$$

is a "Fundamental solution" to the Laplace equation on the unit disk by an explicit verification.

4. The temperature on a unit-length interval on an infinite string is 100° at $t=0$. when do you expect the maximal temperature of a point on the string to go under 1°?

5. (unrelated to classwork) Compute the logarithms (to base 10) of as many physical quantities you can put your hands on. Do they still behave funny? Does 2^n behave funny? Does $\log_{10} 2^n$ behave funny?

Dror Bar-Natan

```
Mathematica 2.0 for SPARC
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 -- Terminal graphics initialized --

<< Miscellaneous/ChemicalElements.m
l=N[(({AtomicNumber[#],MeltingPoint[#],BoilingPoint[#],HeatOfFusion[#],
        HeatOfVaporization[#],Density[#],ThermalConductivity[#]}&
                /@ Elements) /. {Kilo -> 1, Joule -> 1, Kilogram -> 1,
                          Meter -> 1, Kelvin -> 1, Watt -> 1, Mole -> 1}),3]

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    {2., 0.95, 4.22, 0.021, 0.082, 125., 0.152},
    {3., 454., 1620., 4.6, 148., 534., 84.7},
    {4., 1550., 3240., 9.8, 309., 1850., 200.},
    {5., 2570., 3930., 22.2, 504., 2340., 27.},
    {6., 3820., 5100., 105., 711., 3510., 1960.},
    {7., 63.3, 77.4, 0.72, 5.58, 1030., 0.026},
    {8., 54.8, 90.2, 0.444, 6.82, 2000., 0.0267},
    {9., 53.5, 85., 1.02, 3.26, 1520., 0.0279},
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    {11., 371., 1160., 2.64, 99.2, 971., 141.},
    {12., 922., 1360., 9.04, 128., 1740., 156.},
    {13., 933., 2740., 10.7, 291., 2700., 237.},
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    {24., 2130., 2940., 15.3, 342., 7190., 93.7},
    {25., 1520., 2240., 14.4, 220., 7440., 7.82},
    {26., 1810., 3020., 14.9, 340., 7870., 80.2},
    {27., 1770., 3140., 15.2, 382., 8900., 100.},
    {28., 1730., 3000., 17.6, 375., 8900., 90.7},
    {29., 1360., 2840., 13., 307., 8960., 401.},
    {30., 693., 1180., 6.67, 114., 7130., 116.},
    {31., 303., 2680., 5.59, 270., 5910., 40.6},
    {32., 1210., 3100., 34.7, 328., 5320., 59.9},
    {33., 83.8, 889., 27.7, 31.9, 5780., 50.},
    {34., 490., 958., 5.1, 90., 4790., 2.04},
    {35., 266., 332., 10.8, 30.5, 4050., 0.122},
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    {54., 161., 166., 3.1, 12.7, 3540., 0.00569},
    {55., 302., 952., 2.09, 66.5, 1870., 35.9},
    {56., 1000., 1910., 7.66, 151., 3590., 18.4},
```

Here is a list of the Atomic number, Melting point, Boiling point, Heat of Fusion, Heat of vaporization, Density & Thermal conductivity of the first 106 elements, in the units

Kilo, Joule, Meter, Kelvin, Watt, Mole.

```
  {57., 1190., 3730., 10., 402., 6150., 13.5},
  {58., 1070., 3700., 8.87, 398., 8240., 11.4},
  {59., 1200., 3790., 11.3, 357., 6770., 12.5},
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  {61., 1440., 3000., 12.6, Unknown, 7220., 17.9},
  {62., 1350., 2060., 10.9, 165., 7520., 13.3},
  {63., 1090., 1870., 10.5, 176., 5240., 13.9},
  {64., 1590., 3540., 15.5, 301., 7900., 10.6},
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  {68., 1800., 3140., 17.2, 280., 9070., 14.3},
  {69., 1820., 2220., 18.4, 247., 9320., 16.8},
  {70., 1100., 1470., 9.2, 159., 6970., 34.9},
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  {73., 3270., 5700., 31.4, 758., 16700., 57.5},
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  {82., 601., 2010., 5.12, 178., 11300., 35.3},
  {83., 545., 1880., 10.5, 179., 9750., 7.87},
  {84., 527., 1230., 10., 101., 9320., 20.},
  {85., 575., 610., 23.8, Unknown, Unknown, 1.7},
  {86., 202., 211., 2.7, 18.1, 4400., 0.00364},
  {87., 300., 950., Unknown, Unknown, Unknown, 15.},
  {88., 973., 1410., 7.15, 137., 5000., 18.6},
  {89., 1320., 3470., 14.2, 293., 10100., 12.},
  {90., 2020., 5060., 19.2, 514., 11700., 54.},
  {91., 2110., 4300., 16.7, 481., 15400., 47.},
  {92., 1410., 4020., 15.5, 417., 19000., 27.6},
  {93., 913., 4170., 9.46, 337., 20200., 6.3},
  {94., 914., 3500., 2.8, 343., 19800., 6.74},
  {95., 1270., 2880., 14.4, 239., 13700., 10.},
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  {97., Unknown, Unknown, Unknown, Unknown, 14800., 10.},
  {98., Unknown, Unknown, Unknown, Unknown, Unknown, 10.},
  {99., Unknown, Unknown, Unknown, Unknown, Unknown, 10.},
  {100., Unknown, Unknown, Unknown, Unknown, Unknown, 10.},
  {101., Unknown, Unknown, Unknown, Unknown, Unknown, 10.},
  {102., Unknown, Unknown, Unknown, Unknown, Unknown, 10.},
  {103., Unknown, Unknown, Unknown, Unknown, Unknown, 10.},
  {104., Unknown, Unknown, Unknown, Unknown, Unknown, Unknown},
  {105., Unknown, Unknown, Unknown, Unknown, Unknown, Unknown},
  {106., Unknown, Unknown, Unknown, Unknown, Unknown, Unknown}}
```

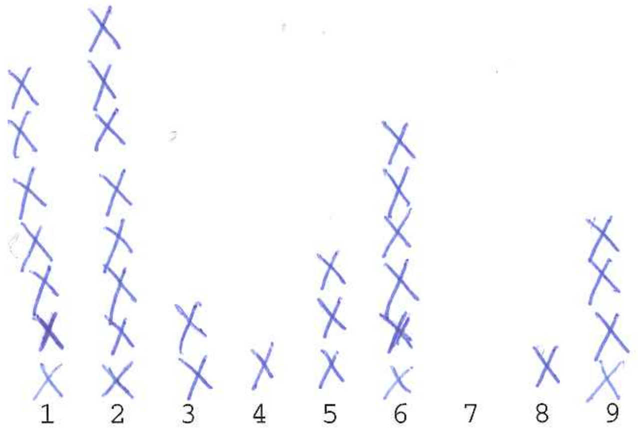
```
l1=Flatten[Drop[#,1]& /@ l] /. {Unknown -> 0.};
d1=(RealDigits[#][[1,1]])& /@ l1;
Count[d1,#]& /@ Range[9]

  {199, 103, 79, 41, 45, 23, 29, 24, 32}
```

1 2 3 4 5 6 7 8 9.

These are the number of times each of the digits 1-9 appears as the most-significant-digit in the numbers of the above list. (Atomic numbers were excluded from the computation)

```
SpeedOfLight = 2.99792458 10^8 Meter/Second
PlanckConstant = 6.6260755 10.^-34 Joule Second
ElectronCharge = 1.60217733 10^-19 Coulomb
ElectronMass = 9.1093897 10^-31 Kilogram
ProtonMass = 1.6726231 10^-27 Kilogram
AvogadroConstant = 6.0221367 10.^23 Mole^-1
FaradayConstant = 9.648456 10^4 Coulomb/Mole
GravitationalConstant = 6.67260 10^-11 Newton Meter^2 Kilogram^-2
FineStructureConstant = 1/137.0359895
PlanckMass = 2.17671 10^-8 Kilogram
BohrRadius = 0.529177249 10^-10 Meter  (* infinite mass nucleus *)
RydbergConstant = 1.09737318 10^7 Meter^-1
ElectronComptonWavelength = 2.426309 10^-12 Meter
ClassicalElectronRadius = 2.817938 10^-15 Meter
ThomsonCrossSection = 6.652245 10^-29 Meter^2
ElectronMagneticMoment = 9.284832 10^-24 Joule/Tesla
ElectronGFactor = 1.0011596567
MagneticFluxQuantum = 2.0678506 10^-15 Weber  (* h / ( 2 e ) *)
BoltzmannConstant = 1.380658 10^-23 Joule/Kelvin
MolarGasConstant = 8.3144 Joule Kelvin^-1 Mole^-1
MolarVolume = 22.41410 10^-3 Meter^3/Mole  (* ideal gas, STP *)
StefanConstant = 5.67051 10^-8 Watt Meter^-2 Kelvin^-4
IcePoint = 273.15 Kelvin
WeakMixingAngle = 0.230    (* Sin[ThetaW]^2 *)
AgeOfUniverse = 4.7 10^17 Second
HubbleConstant = 3.2 10^-18 Second^-1
AccelerationDueToGravity = 9.80665 Meter/Second^2
SolarRadius = 6.9599 10^8 Meter
SolarConstant = 1.37 10^3 Watt/Meter^2
EarthMass = 5.976 10^24 Kilogram
EarthRadius = 6.378164 10^6 Meter   (* equatorial radius *)
SpeedOfSound = 340.29205 Meter/Second  (* standard atmosphere *)
```



Math 115, Nov 20 1991

Review : $\widetilde{f'} = ipf$; $\widetilde{f*g} = \sqrt{2\pi}\,\tilde{f}\cdot\tilde{g}$; $f_a = \frac{1}{\sqrt{2\pi a}}e^{-\frac{x^2}{2a}}$
$\tilde{f}_a = \frac{1}{\sqrt{2\pi}}e^{-\frac{ap^2}{2}}$

Heat equation:

$$u_t = u_{xx} \qquad u(x,0) = u_0$$
$$\tilde{u}_t = -p^2\tilde{u} \qquad \tilde{u}(p,0) = \tilde{u}_0$$
$$\tilde{u}(p,t) = e^{-p^2 t}\tilde{u}_0 = \sqrt{2\pi}\frac{1}{\sqrt{2\pi}}e^{-\frac{(2t)}{2}p^2}\tilde{u}_0$$
$$u(x,t) = f_{2t} * u_0 = \int dy \frac{1}{\sqrt{4\pi t}}e^{-\frac{(x-y)^2}{4t}}u_0(y)$$
$$= \int dy\, U(t, x-y)\, u_0(y)$$

e.g. $u_0 = \delta(x)$

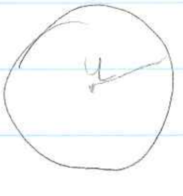
Properties:
1. $\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)U = 0$
2. $\int_{-\infty}^{\infty} U(t,x)\,dx = 1$
3. $\lim_{t\to 0}\int_{-\epsilon}^{\epsilon} U(t,x)\,dx = 1$

} U is called "A Fundamental solution"

4. $U(t_1,\cdot) * U(t_2,\cdot) = U(t_1+t_2,\cdot)$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} \Rightarrow$$

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$



$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \qquad u = R(r)\cdot\Theta(\theta) \qquad \Theta(0) = \Theta(2\pi)$$

$$r^2R''\Theta + rR'\Theta + R\Theta'' = 0 \qquad /: R\Theta$$

$$u_n = A_n r^n \cos n\theta + B_n r^n \sin n\theta \qquad u = \sum u_n \quad A_n = \frac{1}{\pi}\int_0^{2\pi} u_1(\phi)\cos n\phi\, d\phi$$

$$u(r,0) = \sum r^n \cdot \frac{1}{\pi}\int_0^{2\pi} u_1(\phi)\cos n\phi\, d\phi = \frac{1}{\pi}\int_0^{2\pi} u_1(\phi)\sum r^n\cos n\phi\, d\phi = \ldots,$$

HW.

$$r = \sqrt{x^2+y^2} \qquad \theta = \operatorname{atg} y/x$$
$$\frac{\partial}{\partial x}r = \frac{x}{\sqrt{x^2+y^2}} = \cos\theta \qquad \partial/\partial x\,\theta = \frac{-y}{x^2+y^2} = -\frac{\sin\theta}{r}$$
$$\frac{\partial}{\partial x} = (\cos\theta)\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}$$
$$\frac{\partial^2}{\partial x^2} = (\qquad)^2$$

Separation of variables problems from old exams:

Math 115, Now 22 1991

1. Use the method of separation of variables and Fourier analysis to solve the following boundary value problem for $u(x,t)$ on $\{0 < x < \pi,\ t > 0\}$:

$$u_t(x,t) = u_{xx}(x,t) - u(x,t)$$

$$u(0,t) = 0, \qquad u(\pi,t) = 1, \qquad u(x,0) = 0.$$

2. (a) Let $f(x)$ be defined by $f(x) = 0$ for $-\pi < x \leq 0$ and $f(x) = x$ for $0 < x < \pi$. Write down the Fourier series for $f(x)$. Find $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$ by evaluating the Fourier series at a suitable point. Justify your answer by stating and applying Fourier's theorem on the convergence of the Fourier series of piecewise continuous functions.

   (b) Use the above result to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

3. Use the method of Fourier to solve the heat equation $\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$, $(0 < x < \pi,\ t > 0)$ with boundary condition $u(0,t) = 0$ and $u(\pi,t) = 0$ and $u(x,0) = \pi - x$.

4. Consider the heat equation $\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$ for functions $u(x,t)$ which are periodic in x with period 2π. Let $a(x) = u(x,0)$ be the initial value of u.

   (a) Prove that when $t \to \infty$, $u(x,t)$ tends to a constant function u_0.

   (b) Compute the constant u_0 and give an estimate for the difference $u(x,t) - u_0$ for large t in terms of the function $a(x)$.

And some more problems worth considering, from the textbook:

1. (page 98)

$$u_t = ku_{xx}, \qquad u_x(0,t) = 0, \qquad u_x(c,t) = 0, \qquad u(x,0) = f(x).$$

2. (page 101, varied a little)

$$u_t = u_{xx}, \qquad u(0,t) = l, \qquad u(\pi,t) = r, \qquad u(x,0) = f(x).$$

3. (page 102, varied a little)

$$u_t = u_{xx}, \qquad u(0,t) = 0, \qquad u_x(\pi,t) = 0, \qquad u(x,0) = f(x).$$

4. (page 105, varied a little)

$$u_t = u_{xx} + ax, \qquad u(0,t) = 0, \qquad u(\pi,t) = 0, \qquad u(x,0) = f(x).$$

And one last problem, all of my own: (Oh well, it's a well known one)
One the real line **R**, solve:

$$u_t = u_{xx} + E(x,t), \qquad u(x,0) = f(x).$$

Midterm 2: Monday Fairchild 102 7:00PM
open books.

Math 115, Nov 22 1991

Review $\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial^2}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$

$u=R(r)\cdot\Theta(\theta)$ $\Theta(0)=\Theta(2\pi)$ ..... $r^2R''+rR'=\lambda R$ .....

$u_n=A_n r^n\cos n\theta+B_n r^n\sin n\theta$

$u(r,\theta)=\sum r^n\frac{1}{\pi}\int_0^{2\pi}u_1(\phi)\cos n\phi\, d\phi=\frac{1}{\pi}\int_0^{2\pi}u_1(\phi)\sum r^n\cos n\phi\, d\phi=$

$\quad P(r,\theta)=\frac{1}{2\pi}\frac{1-r^2}{1-2r\cos\phi+r^2}$

claim: 1. $\Delta P=(P_{xx}+P_{yy})=0$

2. $\int_{-\pi}^{\pi}P(r,\theta)d\theta=1$

3. $\lim_{t\to 0}\int_{-\epsilon}^{\epsilon}P(r,\theta)\,d\theta=1$

4 $P(1-t_1,\cdot)$ "$*$" $P(1-t_2,\cdot)=P(1-(t_1+t_2),\cdot)$

Jefferson 461

Calculus of Variations

Gelfand ; Fomin

1 Elements of the theory

1. Functionals – some simple variational problems:

Definition of a functional

Examples 1. length of a curve
- 2. Center of mass of a curve
- 3. time of travel w/ a give velocity fcth
- 4. $\int (y')^2 dx$
- 5. $\int F(x,y,y')dx$

Variational problems

Examples: 1. Min length
- 2. The brachistochrone
- 3. The isoperimetric problem

The localization property
- example for a non-local functional

The lattice method

2. Function spaces

Definition of a linear space.

Definition of a norm

Examples: 1. Continuous functions in the L^∞ norm.
- 2. $\mathcal{D}_1$ – Sobolev space
- 3. $\mathcal{D}_n$ – Sobolev spaces

Definition of continuity & semi-continuity

remark: Normally we will not be considering functionals defined on the whole of

$$y(2x-y') - \frac{d}{dx} y^2 (y' - 2x) =$$

$$2xy - yy' - 2yy'(y'-2x) - y^2(y''-2)$$

$$y^2 y'' = 2xy(1-2y') - yy'(1+2y') + 2y^2$$

The variation of a functional. A necessary condition for an extermum

Definition of a continuous linear functional.

Examples: 1. evaluation at a point.
2. The integral.
3. Weighted integral.
4. Weighted distributional integral.

Lemma. $\forall h \int \alpha h\, dx = 0 \rightarrow \alpha \equiv 0$.

Lemma $\forall h \int \alpha h'\, dx = 0 \rightarrow \alpha \equiv C$

Lemma $\forall h \int \alpha h''\, dx = 0 \rightarrow \alpha \equiv C_1 x + C_0$

Lemma $\forall h \int (\alpha h + \beta h') = 0 \rightarrow \beta' \equiv \alpha$

Definition of a differentiable functional & the variation (i.e. differential, principal linear part).

Theorem Uniqueness of the variation.

Definition of weak & strong extremas

Theorem Variation $\equiv 0$ a necessary condition for an extrema

4. The simplest variational problem, Euler's equation.

The simplest variational problem - extremize

$$J(y) = \int_a^b F(x, y, y')dx$$

Theorem: Euler's equation

Bernstein's theorem on the existence of solutions to Euler's equation.

Example for a solution of the Euler equation which is not everywhere twice differentiable.

. The case $\int F(x,y')$

2. The case $\int F(y,y')$

3. The case $\int F(x,y)$

4. The case $\int F(x,y)\sqrt{1+y'^2}$

Examples 1. $\int \frac{\sqrt{1+(y')^2}}{x}dx$

2. Curve whose rotation is of minimal area.

discussion.

BEUTIFUL.

3. $\int (x-y)^2 dx$

5. The case of several variables

Lemma $\forall h \iint \alpha(x,y)h(x,y)dxdy = 0 \to \alpha \equiv 0$

Euler's equation in the multivariable case.

Example: surface of minimal area spanned by a contour.

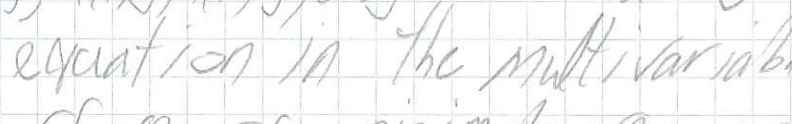
result - zero mean curvature.

NICE BUT HARD.

6. A simple variable end point problem

minimize $\int F(x,y,y')$ without endpoint constrains.

Example



BEUTIFUL.

7

7. The varitional derivative

A lattice derivation of the trivial notion.

Invariance of Euler's equation

The trivial invariace under change of coordinates accompanied by a corresponding change of everything else.

Example $\int \sqrt{r^2+r'^2}\, d\varphi$ is a transformed straight line.

2. Further generalizations

9. The fixed end point problem for n unknown functions

Geodesics, Fermat's principle

Theorem: System of Euler equations.

remark: The freedom to add total derivatives.

remark: The freedom to multiply the Lagrangian by a constant.

Examples: 1. propagation of light in an inhomogeneous medium

2. Geodesics

10: Variational problems in Parametric form

Definition of homogeneity

Theorem: $\int \Phi(t,x,y,\dot{x},\dot{y})dt$ is parametrization independend iff Φ is positive homogeneous of degree 1.

In that case, an identity relates the two Euler equations obtained from Φ

11. Functionals depending on higher order derivatives

Euler's equation in this case.

2 Variational Problems with subsidiary Conditions

12.1 The isoperimetric problem.

Definition of the general isoperimetric problem:

Extermize $J=\int F(x,y,y')$ under $K=\int G(x,y,y')=l$

Theorem: Lagrange multiplier λ.

How to use Lagrange multipliers

The case of n variables.

12.2 Finite subsidiary conditions

Theorem: minimize $\int F(x,y,z,y',z')$ on $g(x,y,z)=0$ by minimizing

$$\int(F+\lambda(x)\cdot g)$$

remark: g could have been a differential equation.

remark: 12.2 is a limiting case of 12.1

Examples 1. The famous isoperimetric problem NICE

2. Geodesics on a sphere.

remark Constrained systems can also be solved by elimination of variables.

3. The general variation of a functional

13. derivation of the basic formula

For $J=\int_{x_0}^{x}F(x,y_1,\ldots y_n,y_1',\ldots y_n')\,dx$

$$\delta J=\int_{x_0}^{x_1}\sum_1(F_{y_i}-\frac{d}{dx}F_{y_i'})h_i(x)dx+\sum F_{y_i'}\delta y_i\Big|_{x_0}^{x_1}+(F-\sum_{i=1}^{n}y_i'F_{y_i'})\delta x\Big|_{x_0}^{x_1}$$

Where $p_i = F_{y_i'}$ & $H = -F + \sum y_i' F_{y_i'}$

14. End points lying on two given curves or surfaces

Explicit formulae for something better remaining unexplicit.

15. Broken extremals. The weierstrass-Erdmann Conditions

Example for a functional with a non-smooth minima: $\int_{-1}^{1} y^2 (1-y')^2 dx$

The Weierstrass-Erdmann Conditions

$$F_{y'}|_{x=c-} = F_y|_{x=c+}$$

$$F - yF_{y'}|_{x=c-} = F - y'F_y|_{x=c+}$$

Namely - The canonical variables have to be continuous

4. The canonical form of the Euler equations and related topics

16. The canonical form of the Euler equations

Hamilton's equations

17. First integrals of the Euler equations

If H is x-independent then it is an integral of motion

Poisson brackets

bracketing with H is the criteria for being integral.

18. The Legendre transformation

Definition of the Legendre transform and

Young's inequality

The variational problem of the Legendre transform of F is equivalent (~) to that of F.

19. Canonical transformations

Canonical transformations & generating functions

20. Noether's theorem

Noether's theorem

Applying Noether's theorem to time translations

21. The principle of least action

A derivation of Newton's equations in the variational problem corresponding to classical mechanics.

22. Conservation laws

Conservation of energy, momentum, and angular momentum

23. The Hamilton-Jacobi equation, Jacobi's theorem

The geodesic distance (the minimal action) (as a function of the "right" boundary conditions)

The Hamilton-Jacobi equation

theorem: "The canonical equations are the characteristic system of the Hamilton-Jacobi equation"

Jacobi's theorem: A way of generating the equations from the full set of solutions of Hamilton-Jacobi's equation

5. The second variation. Sufficient conditions for a weak extremum

24. Quadratic functionals. The second variation of a functional

bilinear & quadratic functionals

positive definiteness

Theorem: positive definiteness of the second
variation is a necessary condition
for a minimum

strong positivity

Theorem: strong positivity is a sufficient
condition for a minimum.

25. The formula for the second variation. Legendre's condition.

$$\delta^2 J(h) = \frac{1}{2}\int_a^b (F_{yy} h^2 + 2F_{yy'} h h' + F_{y'y'} (h')^2) dx$$

$$= \int [P(h')^2 + Q h^2] dx \qquad P = \frac{1}{2} F_{y'y'}$$

$$Q = \frac{1}{2}\left(F_{yy} - \frac{d}{dx} F_{yy'}\right)$$

Theorem. a necessary condition for the positivity
of $\delta^2 J$ is $P \geq 0$, namely $F_{y'y'} \geq 0$
(Legendre's theorem)

26. Analysis of the quadratic functional $\int_a^b (Ph'^2 + Qh^2) dx$

Conjugate points.

Theorem: No conjugate points $\Rightarrow$ Pos. definite.

Theorem: Pos. definite $\Rightarrow$ no conjugate points.

Theorem: pos def. ⟺ no conjugate pts.

27 Jacobi's necessary condition. More on conjugate points.

Definition: Jacobi's equation of a variational problem.

Definition: Conjugate points.

Theorem: (Jacobi's necessary condition) extremal is a minimum ⇒ no interior conjugate pts.

The "variational equation" of a non-linear equation

Two other characterizations of conjugate points.

28. Sufficient conditions for a weak extremum

Theorem: Euler's eqn & $P > 0$ & no conjugate pts ⇒ a weak minimum.

29. Generalization to n unknown functions

The matrix generalization of 24-28

30. Connection between Jacobi's condition and the theory of quadratic forms

The lattice version of 24-28.

Math 115 - Calcus of Variation Siu's way:

1 Euler Lagrange, solve Power lines
2 Variable Endpoints, the brachistochrone.
3 Endpoints on curves
4 Corners, constrains.
5 Lagrange multipliers, many Function, many variables, many derivatives.
6 More Lagrange multipliers.
7 Cannonical trans., Legendre's trans., Hamilton's eqn.
8 Hamilton's eqn's, Hamilton-Jacobi, Generating functions.
9 Hamilton Jacobi, Noether's theorem, second variation.
10 Second variation, Legendre's condition,
11 Conjugate pts ...
12 - - - ,
13 Strong minimas.

Separation of variables problems from old exams:

Math 115, Now 22 1991

$V_t = V_{xx} - V$
$V_t + V = V_{xx}$
$T' = -(\lambda+1)T \quad X'' = -\lambda X$
Cooling is faster and that's all.

1. Use the method of separation of variables and Fourier analysis to solve the following boundary value problem for $u(x,t)$ on $\{0 < x < \pi,\ t > 0\}$: $u = \frac{x}{\pi} + v$

$$u_t(x,t) = u_{xx}(x,t) - u(x,t)$$

$$u(0,t) = 0, \qquad u(\pi,t) = 1, \qquad u(x,0) = 0.$$

std.

2. (a) Let $f(x)$ be defined by $f(x) = 0$ for $-\pi < x \le 0$ and $f(x) = x$ for $0 < x < \pi$. Write down the Fourier series for $f(x)$. Find $\sum_{n=1}^{\infty} \frac{1}{(2n+1)^2}$ by evaluating the Fourier series at a suitable point. Justify your answer by stating and applying Fourier's theorem on the convergence of the Fourier series of piecewise continuous functions.

   (b) Use the above result to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

std.

3. Use the method of Fourier to solve the heat equation $\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$, $(0 < x < \pi,\ t > 0)$ with boundary condition $u(0,t) = 0$ and $u(\pi,t) = 0$ and $u(x,0) = \pi - x$.

do

4. Consider the heat equation $\frac{\partial}{\partial t}u(x,t) = \frac{\partial^2}{\partial x^2}u(x,t)$ for functions $u(x,t)$ which are periodic in x with period 2π. Let $a(x) = u(x,0)$ be the initial value of u.

   (a) Prove that when $t \to \infty$, $u(x,t)$ tends to a constant function u_0.

   (b) Compute the constant u_0 and give an estimate for the difference $u(x,t) - u_0$ for large t in terms of the function $a(x)$.

And some more problems worth considering, from the textbook:

1. (page 98)

triv move k to u_t

$$u_t = ku_{xx}, \qquad u_x(0,t) = 0, \qquad u_x(c,t) = 0, \qquad u(x,0) = f(x).$$

2. (page 101, varied a little)

$u = l + \frac{r-l}{\pi}x + v$
note that it changes bndry cond.

$$u_t = u_{xx}, \qquad u(0,t) = l, \qquad u(\pi,t) = r, \qquad u(x,0) = f(x).$$

3. (page 102, varied a little)

triv

$$u_t = u_{xx}, \qquad u(0,t) = 0, \qquad u_x(\pi,t) = 0, \qquad u(x,0) = f(x).$$

4. (page 105, varied a little)

$v = \phi(x) + u$

$$u_t = u_{xx} + ax, \qquad u(0,t) = 0, \qquad u(\pi,t) = 0, \qquad u(x,0) = f(x).$$

And one last problem, all of my own: (Oh well, it's a well known one)
One the real line **R**, solve:

$$u_t = u_{xx} + E(x,t), \qquad u(x,0) = f(x).$$

$\tilde{u}_t = -p^2\tilde{u} + \tilde{E}(p,t) \quad \tilde{u}(p,0) = \tilde{f}(p)$
$\tilde{u} = e^{-p^2t}\left(\int_0^t e^{p^2\tau}\tilde{E}(p,\tau)\,d\tau + \tilde{F}(p)\right)$
See other side

side thing
$f' = \lambda f + g \qquad F = e^{\lambda t}\cdot\phi$
$\cancel{\lambda e^{\lambda t}\phi} + e^{\lambda t}\phi' = \cancel{\lambda e^{\lambda t}\phi} + g$
$\phi' = e^{-\lambda t}g$
$\phi = A + \int_0^t e^{-\lambda\tau}g\,d\tau$

Namely $\tilde{U}_t(p) = e^{-p^2 t}$ $U_t(x) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$

$$\tilde{u} = e^{-p^2 t}\tilde{F} + \int_0^t d\tau\, e^{-p^2(t-\tau)}\tilde{E}(p,\tau)$$

$$u = U_t * F + \int_0^t U_{t-\tau} * E(\cdot,\tau)\, d\tau$$

This has a neat physical interpretation!

Math 115, Nov 25 1991

Midterm: Fairchild 102, 7PM

mention cal of var, I'm willing to lend a book!

Enough of PDEs!

Here is what we are skipping:

1. multi-variable Fourier theory

   $f:\mathbb{R}^n \to \mathbb{C}$ $\quad \tilde{f}:\mathbb{R}^n \to \mathbb{C}$ $\quad \tilde{f}(p) = \frac{1}{(2\pi)^{n/2}} \iint_{\mathbb{R}^n} e^{-ip\cdot x} f(x)\, d\vec{x}$

   otherwise everything is same -

   $u_t = u_{xx} + u_{yy} + u_{zz}$ . . .

2. Cylinder with temp. depending only on r:

   $$u_t = u_{rr} + \frac{1}{r}u_r \qquad u(R,t)=0$$
   $$u(r,0)=f(r)$$

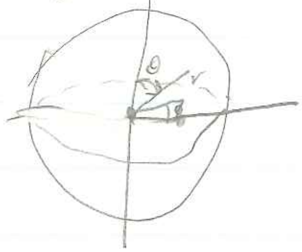
   (Vibrations of a circular membrane)

   $u = R(r)T(t) \Rightarrow rR'' + R' + \lambda r R = 0$

   $\Rightarrow$ Bessel's equation.

   Bessel functions, similar theory to Fourier's.

3. spherical Laplace

   write Laplacian, (bndry fcn of θ)

   $$r(ru)_{rr} + \frac{1}{\sin\theta}(\sin\theta \cdot u_\theta)_\theta = 0$$

   sep of var. $x = \cos\theta$

   $$((1-x^2)\Theta_x)_x + \lambda\Theta = 0$$

   $\Rightarrow$ Legendre's equation, Legendre's polynomials, simil. to F.

4. Sturm Liouville problems:

   $$(rX)_x + (q+\lambda p)X = 0$$ r,p,q functions + bndry

   similar solutions & properties.

   detailed qualitative analysis

5. Hermite, Laguerre

   @ Harmonic osc.

20

Second Midterm — Linear Partial Differential Equations

Math 115, Nov 25 1991

Dror Bar-Natan

You have 120 minutes to answer the following 4 questions. Each question is worth 25 points, plan your time wisely! It is a good idea to read the entire exam before answering any question. You may use any material you wish to use other than your friends. At the end of the 120 minutes don't forget to sign your name on anything you submit.

1. Solve the Laplace equation $u_{xx} + u_{yy} = 0$ on the annulus $1 \leq r \leq 2$ with the following boundary conditions:
$$u(1,\theta) = 3\cos\theta,$$
$$u(2,\theta) = -2\sin\theta.$$

2. In an experiment performed last week in the Harvard University Mathematical Laboratories, one end of a metallic string of length 2π was held in ice and its other end was held in boiling water for a long time, until a stable equilibrium was achieved. Then the string was removed from the ice and the water, was bent to form a circle (and its two ends were thus in contact), and was put on a table made of a heat-insulating material.
   (a) If x measures the distance to the (initially) iced end of the string and t measures time, write the boundary conditions for the heat equation $u_t = u_{xx}$ which describe the above situation.
   (b) Solve the equations you just got.
   (c) At time $t = 20$, the string will have an almost constant temperature. What will this temperature be? Estimate the difference between the actual temperature of the string at time $t = 20$ and the constant you just found.

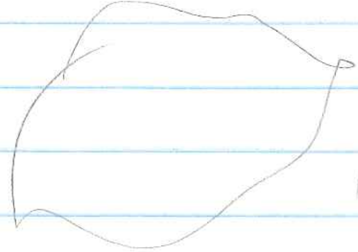
3. (a) Compute the Fourier transform (with respect to the variable x) of the function
$$f_t(x) = \frac{1}{\pi}\frac{t}{t^2 + x^2},$$
where t is some positive constant. (If you choose to use residues for computing integrals, you might want to spend some time picking a *correct* contour. It might be that the choice of contour should be different for different values of p).
   (b) Use this result to compute the convolution $g * h$, where
$$g(x) = \frac{1}{x^2+1} \qquad \text{and} \qquad h(x) = \frac{1}{x^2+4}.$$

4. Write the equations for the flow lines of an incompressible irrotational fluid, which flows inside a wedge of opening $\frac{\pi}{3}$ radians.

— GOOD LUCK —

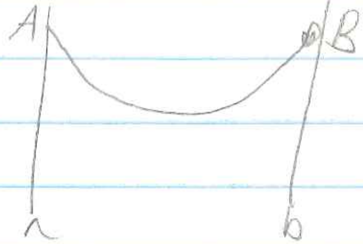
Math 115, Nov 27 1991.

I have an extra book!



$\xrightarrow{\text{minimize a functional}} \mathbb{R}$ (Possibly subject to constraints)

Ex:



$F = \int_a^b m y\sqrt{1+y'^2}\,dx$

constraint: $l = \int dx \sqrt{1+y'^2}$

bndry: $y(a) = A \quad y(b) = B$

derive Euler Lagrange: $F_y - \frac{d}{dx}F_{y'} = 0$
(by adding ϵh, $h(a) = h(b) = 0$)

1. F indep. of y $\quad F_{y'} = \text{const}$

" " y' $\quad F_y = 0$

F indep of x:

☐ $0 = F_y - (F_{y'})' = F_y - F_{y'y}y' - F_{y'y'}y'' = 0 \quad /\cdot y'$

$\Rightarrow \quad y'F_y - F_{y'y}y'^2 - F_{y'y'}y''y' = 0$

$\Rightarrow \frac{d}{dx}(F - y'F_{y'}) = 0 \Rightarrow F - y'F_{y'} = \text{Const.}$

In our case $y\sqrt{1+y'^2} - y'\frac{yy'}{\sqrt{1+y'^2}} = C$

$\Rightarrow y'^2 = \frac{y^2 - C^2}{C^2} \Rightarrow y = C\cosh\frac{x - C'}{C}$

what's wrong here?

HW: read 1-4 Do ☐ explicitly, 1.14, 15 b,d

Math 115 - Second midterm solution 11/25/91

1. By separation of variables, these are all harmonic in the required domain:

$$u_1 = r\cos\theta \quad u_2 = r\sin\theta \quad u_3 = \frac{1}{r}\cos\theta \quad u_4 = \frac{1}{r}\sin\theta$$

(alternatively, these are the real and imaginary parts of z and $1/z$)

Set $u = \sum_{i=1}^{4} a_i u_i$. We need to solve:

$$\begin{tikzcd}
 & a_1+a_3=3 \arrow[dl] & \\
a_1=-1 & a_2+a_4=0 \arrow[dr] & \\
a_3=4 & 2a_1+\frac{1}{2}a_3=0 \arrow[l] & a_2=-\frac{4}{3} \\
 & 2a_2+\frac{1}{2}a_4=-2 \arrow[r] & a_4=\frac{4}{3}
\end{tikzcd}$$

2. a. $u(x,t) = u(2\pi+x,t)$; $u(x,0) = \frac{100}{2\pi}x$

b. $u = \frac{a_0}{2} + \sum_{n=1}^{\infty} e^{-n^2 t}(a_n\cos nx + b_n\sin nx)$

with $a_n = \frac{1}{\pi}\int_0^{2\pi}\frac{100}{2\pi}x\cos nx\,dx = \begin{cases} 100 & n=0 \\ 0 & n>0 \end{cases}$

$b_n = \frac{1}{\pi}\int_0^{2\pi}\frac{100}{2\pi}x\sin nx\,dx = -\frac{100}{\pi n}$

c. $T = 50^c$, diff $\sim \frac{100}{\pi}e^{-20} \sim 6.56\cdot 10^{-8}$

3. a. $\tilde{f}_t(p) = \frac{t}{\pi\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-ipx}}{(x+it)(x-it)} = \frac{t}{\pi\sqrt{2\pi}}2\pi i \,\mathrm{Res}_{\text{in}\, C}\left(\frac{e^{-ipx}}{(x+it)(x-it)}\right)$

where the contour C is

and so if $p>0$

$\tilde{f}_t(p) = -it\sqrt{\frac{2}{\pi}}\left(\frac{e^{-ip\cdot(-it)}}{-it-it}\right) = \frac{e^{-pt}}{\sqrt{2\pi}}$

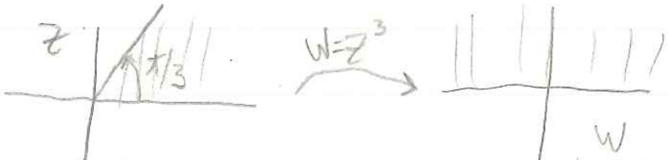
and for $p<0$ it is easy to get $\tilde{f}_t(p) = \frac{e^{pt}}{\sqrt{2\pi}}$.

Therefore $\tilde{f}_t(p) = \frac{e^{-|p|t}}{\sqrt{2\pi}}$

b. by a, $\tilde{g} = \pi\tilde{f}_1 = \pi\frac{e^{-|p|}}{\sqrt{2\pi}}$; $\tilde{h} = \frac{\pi}{2}\tilde{f}_2 = \frac{\pi}{2}e^{-|p|\cdot 2}/\sqrt{2\pi}$

and thefore $\widetilde{g*h} = \sqrt{2\pi}\,\tilde{g}\tilde{h} = \frac{\pi^2}{2}\frac{e^{-3|p|}}{\sqrt{2\pi}} = \frac{\pi^2}{2}\tilde{f}_3$

$\Rightarrow (g*h)(x) = \frac{\pi^2}{2}f_3(x) = \frac{\pi}{2}\frac{3}{9+x^2}$

4. 

$F(z) = Aw = Az^3$

streamlines are $\mathrm{Im}\, w = c \Rightarrow \mathrm{Im}\, z^3 = c \Rightarrow 3x^2y - y^3 = c$
$\Rightarrow y(3x^2-y^2) = c$.

Math 115 - Second Midterm grading key, Nov 1991

1\. 15 - right Σ, no cont.
15 - wrong Σ, cont. right
3 - total mess.

2\. a. 6

b. 11 prefour 6
four 5 2 write
3 do

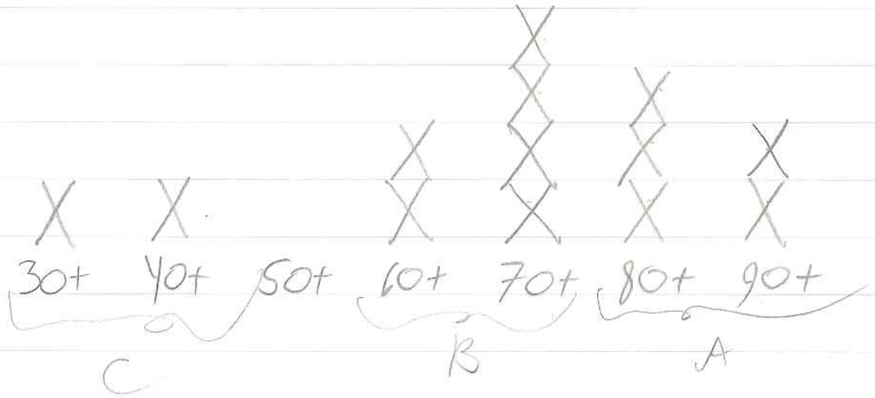
c. 8

3\. a. 15 (-2) negative orient. unnoticed.
(-5) no $p \gtrless 0$ seperation.
(-2) sep incorrect.
(-2) took residues from out of contour,

b. 10 -3 incorrect inversion

4\.

Histogram: Average: 74 ~ median.



Math 115, Dec 2 1991.

Return Exams: 80^+ A 60^+ B 40^+C

Av: 74

__Review:__ $J(y) = \int_a^b F(x,y,y')\,dx \quad y(a)=A\,;\ y(b)=B$

$$F_y - \frac{d}{dx}F_{y'} = 0 \qquad \text{(Euler Lagrange)}$$

HW:
$F=y'$
$F=xyy'$
$F=(y')^2/x^3$

__Power lines:__ $F = y\sqrt{1+y'^2}$ (EL is too hard)

__If F is indep of x:__ $0 = F_y - F_{y'y}y' - F_{y'y'}y'' \quad /\cdot y'$

$$y'F_y - F_{y'y}y'^2 - F_{y'y'}y''y' = 0$$

□ $$(F - y'F_{y'})' = 0 \Rightarrow F - y'F_{y'} = C_1$$

$$\frac{dy}{dx} = g(y) \Rightarrow \frac{dy}{g(y)} = dx \quad \int\frac{dy}{g(y)} = x + C_2$$

Our case: $y\sqrt{1+y'^2} - y'\frac{yy'}{\sqrt{1-y'^2}} = C_1$

$$y\frac{1}{\sqrt{1-y'^2}} = C_1 \qquad \sqrt{1-y^2} = y/C_1$$

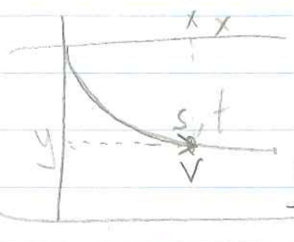
$$y' = \sqrt{1 - y^2/C_1^2}$$

$$C_1\int\frac{d\frac{y}{C_1}}{\sqrt{1-\left(\frac{y}{C_1}\right)^2}} = x + C_2 \Rightarrow C_1\cdot\cosh^{-1}\frac{y}{C_1} = x + C_2$$

$$y = C_1\cosh\frac{x+C_2}{C_1}$$

If time, generalities about gradients & Free ends

The Brachistochrone:

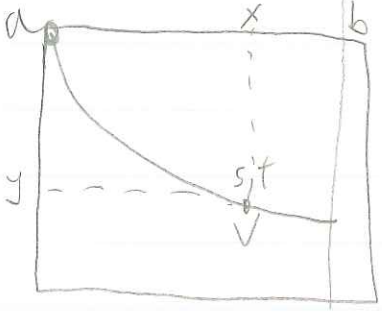


$\frac{1}{2}mv^2 = gy$
$ds = v\,dt$
$ds = \sqrt{dx^2+dy^2}$
$\int dt = \ldots$

H.W.: redo □ For 1. $F = y\sqrt{1+y'^2}$
& 2. $F = \sqrt{\frac{1+y'^2}{y}}$
*Solve EL for 2.
do 15a,c,e or 20 if time permitted

Math 115, Dec 4 1991

The Brachistochrone:



$\frac{1}{2}mv^2=gy$

$ds=v\,dt$

$ds=\sqrt{dx^2+dy^2}=\sqrt{1+y'^2}\,dx$

$\int dt$ ...

$$J(y)=\int\sqrt{\frac{1+y'^2}{y}}\,dx$$

Conditions: $\left.\begin{array}{l} y(0)=0 \\ F_{y'}(b)=0 \\ F-y'F_{y'}=c_1^{-1/2} \end{array}\right\}$ derive by first doing a finite dim analog.

$$F-y'F_{y'}=\sqrt{\frac{1+y'^2}{y}}-y'\frac{y'}{\sqrt{(1+y'^2)y}}=\frac{1}{\sqrt{y(1+y')^2}}=c_1^{-\frac{1}{2}}$$

$y(1+y'^2)=c_1$

$y'=\sqrt{\frac{c_1}{y}-1}$ $\qquad \frac{dy}{\sqrt{\frac{c_1}{y}-1}}=dx$

Snell's law:

$\frac{v_1}{\sin\alpha_1}=\frac{v_2}{\sin\alpha_2}$

$\frac{v}{\sin\alpha}=\text{const}$

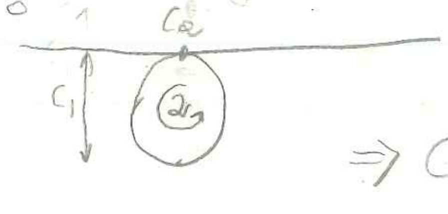
$v=\sqrt{y}$ $\quad \sin\alpha=\frac{1}{\sqrt{1+y'^2}}$

$x-c_2=\int\underbrace{\sqrt{\frac{y}{c_1-y}}}_{\operatorname{tg} t}\,dy$ ... in priciple soluble, in practice hard

trick: $\Rightarrow y=c_1\sin^2 t=\frac{c_1}{2}(1-\cos 2t)$ $\qquad dy=\dots$

$$x=c_2+\frac{c_1}{2}(2t-\sin 2t)$$

This is the cycloid



Center $=\begin{pmatrix}c_2+c_1/2\\ c_1/2\end{pmatrix}$

disp $=\frac{c_1}{2}\begin{pmatrix}-\sin 2t\\ -\cos 2t\end{pmatrix}$

!

$F_{y'}=0\Rightarrow y'=0$

Problem solved !

$\Rightarrow c_2=0$; at b $2t=\pi$

$b=\frac{c_1}{2}\pi$

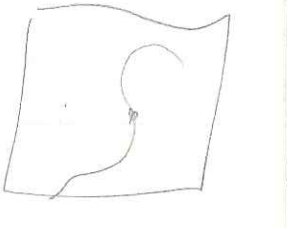
HW: 1. complete details
2. read 6 3. Do 18, 20

Math 115, Dec 6 1991

The isoperimetric ineaquallity:

"Among all domains with boundary length l the disk has the most area".

Lagrange multipliers: maximize $f(x,y)$ under $g(x,y)=0$



stupid way;

smart way $h_\lambda(x,y)=f(x,y)+\lambda g(x,y)$ $\begin{cases}\nabla h_\lambda=0\\ g(x,y)=0\end{cases}$

Example. Find the point nearest to the origin on the curve $x^2+xy+y^2=1$

$h_\lambda=x^2+y^2+\lambda(x^2+xy+y^2-1)$

$\partial_x h_\lambda = 2x+2\lambda x+y=0$

$\partial_y h_\lambda=2y+2\lambda y+x=0$

$x^2+xy+y^2=1$

$y=-2(1+\lambda)x$ $\quad y=x$ $\quad 3x^2=1$

$x=-2(1+\lambda)y$ $\quad y=-x$ $\quad x^2=1$

[illegible]

Example

$y(a)=0$
$y(b)=0$

$J=\int_a^b y\,dx \qquad G=\int_a^b\sqrt{1+y'^2}\,dx=l$

$J+\lambda G=\int_a^b(y+\lambda\sqrt{1+y'^2})\,dx \qquad F_\lambda=y+\lambda\sqrt{1+y'^2}$

Rare case of Euler-Lagrange is simpler than its simplification:

$$0=F_y-\frac{d}{dx}F_{y'}=1-\lambda\frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}} \Rightarrow \frac{\lambda y'}{\sqrt{1+y'^2}}=x-c_1$$

solve for y', get $y'=\frac{x-c_1}{\sqrt{\lambda^2-(x-c_1)^2}} \Rightarrow y=c_2-\sqrt{\lambda^2-(x-c_1)^2}$

$\Rightarrow (x-c_1)^2+(y-c_2)^2=\lambda^2$

! ! !

HW. 2.17-19, 22

Math 115, Dec 9 1991

Five more classes, Five more topics:

1. other types
2. Second derivatives
3. Hamiltonian formulation
4. Symmetries
5. "integral calculus":

The Hamiltonian Formulation.

(Dict:
$L \leftrightarrow J$
$y \leftrightarrow q$
$x \leftrightarrow t$)

motivation: $\underset{J}{L} = \int \underbrace{(\overset{\uparrow}{\tfrac{1}{2}}m\overset{\circ}{v^2} - V(\overset{\nearrow}{q}))}_{y'\ F\ \ y} \overset{\uparrow}{dt}_{x}$ $\quad q = q(t),\ v = \dot q$, describing a particle in a pot. field.

E-L: $F_q \frac{d}{dt}(F_{\dot q}) = 0 \Rightarrow m\ddot q = -V'(q)$ ← Newton's law

maybe we should look at E-L as an initial value problem rather than a boundary value problem.

$$F_{q_i} - \frac{d}{dt}F_{\dot q_i} \quad ; \quad \begin{array}{l} q_i(0) = q_{0i} \\ \dot q_i(0) = v_{0i} \end{array}$$ a 2nd order ODE w/ initial cond.

in higher dims just add subscripts & natural

Theorem: In the variables $q_i,\ p_i \triangleq F_{\dot q_i}$ $(= m\dot q = \text{momentum})$ the E-L eqns are equivalent to the Hamilton equations:

$$\dot q = \frac{\partial H}{\partial p} \quad \dot p = -\frac{\partial H}{\partial q}$$ where the "hamiltonian" H is

$$H(q_i, p_i) \triangleq \sum \dot q_i p_i - F \quad \left(= \tfrac{1}{2}mv^2 + V(q) = \text{total energy} \right)$$

$\frac{p^2}{2m} + V(q)$

Proof:

compute ∂H in two ways. $\quad p_i = F_{\dot q_i} \quad H = \sum p_i \dot q_i - F$

$F_{q_i} - \frac{d}{dt}F_{\dot q_i} = 0$; consider $dq_i, d\dot q_i, dp_i, dH, dF$

$$\sum \frac{\partial H}{\partial p_i}dp_i + \sum \frac{\partial H}{\partial q_i}dq_i = dH = \cancel{\sum p_i d\dot q_i} + \sum \dot q_i dp_i - \sum \frac{\partial F}{\partial q_i}dq_i - \cancel{\sum \frac{\partial F}{\partial \dot q_i}d\dot q_i}$$

$$= \sum \dot q_i dp_i - \sum \dot p_i dq_i$$ Q.E.D.

Definition: P.B: (of fncts of p, q):

$$\{f, g\} = \sum \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right)$$

claim:
1. $\frac{\partial f}{\partial t} = 0 \Rightarrow \frac{df}{dt} = \{f, H\}$
2. $\{H, H\} = 0$ ("H is an integral of motion")
3. $\frac{\partial F}{\partial t} = 0 \Rightarrow$ conservation of energy ; (we actually know that already)

HW: Read 16 | if P.B. is done: Do 4.1, 2 & Read 17.

Do everything explicitly for the harmonic oscillator $F = \frac{1}{2}m\dot q^2 - \frac{1}{2}kq^2$

Math 115, Dec 11 1991.

Conservation laws

Review Hamilton's equations

Definition: P.B:

$$\{f,g\} = \sum\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$$

claim: 1. $\frac{\partial f}{\partial t} = 0 \Rightarrow \frac{df}{dt} = \{F, H\}$

2. $\{H, H\} = 0$

3. $\frac{\partial F}{\partial t} = 0 \Rightarrow$ conservation of energy, energy is an integral of motion, } we already know that

4. $\{q, p\} = 1$ This the beginning of QM!

5. anything that can be said about classical mechanics can be said using P.B. ⇒ symplectic geometry

Noether's theorem:

1. If F is invariant under time trans_ E is conserved
2. If F is invariant under coordinate momentum is conserved — Know but didn't prove yet
3. If F is inv. under rotations, angular momentum is conserved

Let us ~ things which are functions.

Naother's thm If $J(\tilde{q}_i) = \int_{t_0}^{t_1} F(t, \tilde{q}_i, \dot{\tilde{q}}_i)\,dt$ is invariand under

$$t^* = T(t, q_i, \dot{q}_i, \epsilon) \; ; \; q_i^* = Q_i(t, q_i, \dot{q}_i, \epsilon)$$ (Namely, ..

then where $T_{\epsilon=0} = t$; . .

$$\left(\sum P_i \frac{\partial Q_i}{\partial \epsilon} - H\frac{\partial T}{\partial \epsilon}\right)\Big|_{\epsilon=0}$$ (

is conserved (Namely ...)

Example: 1. $T = t+\epsilon$, $Q = q \Rightarrow H$ is conserved if F is t indep.

2. $T = t$, $Q = q+\epsilon \Rightarrow p$ is conserved if F is q indep.

HW: Read 17, Noether: 13, 20 Do 4.1, 2, 3 (Noether: 5)

Noether's theorem, Dec 13 1991

Proof of Noether's theorem:

$$0 = \frac{\partial}{\partial \epsilon} J \Big|_{\epsilon=0} = \frac{\partial}{\partial \epsilon} \int_{t_0^*}^{t_1^*} F(t, \tilde{q}_i^*, \dot{\tilde{q}}_i^*) dt \Big|_{\epsilon=0} =$$

$$= F(t_1, \tilde{q}_i(t_1), \dot{\tilde{q}}_i(t_1)) \frac{\partial T(t_1, \tilde{q}_i(t_1), \dot{\tilde{q}}_i(t_1))}{\partial \epsilon} \Big|_{\epsilon=0} - F(t_0) \frac{\partial T}{\partial \epsilon}(t_0) \Big|_{\epsilon=0}$$

$$+ \int_{t_0}^{t_1} \sum_i \left(F_{q_i} \cdot \frac{\partial \tilde{q}^*}{\partial \epsilon} + F_{\dot{q}_i} \left(\frac{\partial \tilde{q}^*}{\partial \epsilon} \right)^{\cdot} \right) =$$

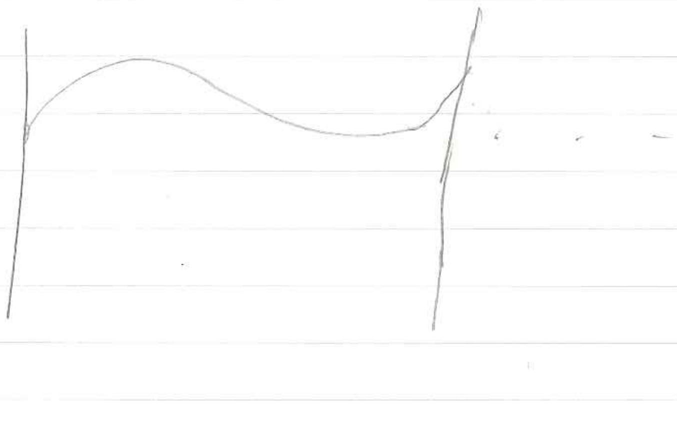
$$= F \frac{\partial T}{\partial \epsilon} \Big|_{t_1} - F \frac{\partial T}{\partial \epsilon} \Big|_{t_0} + \sum p_i \frac{\partial \tilde{q}^*}{\partial \epsilon} \Big|_{t_1} - \sum p_i \frac{\partial \tilde{q}^*}{\partial \epsilon} \Big|_{t_0} + \underbrace{E-L}_{\overset{\parallel}{0}}$$

$$= \left(F - \sum p_i \dot{\tilde{q}}_i \right) \frac{\partial T}{\partial \epsilon} \Big|_{t_0}^{t_1} + \sum p_i \frac{\partial Q}{\partial \epsilon} \Big|_{t_0}^{t_1}$$

Suppose we hold t fixed. What is $\frac{\partial \tilde{q}^*}{\partial \epsilon}$?

$$q^* = \tilde{q}^*(t^*) \implies \frac{\partial Q}{\partial \epsilon} = \frac{\partial \tilde{q}^*}{\partial \epsilon} + \dot{\tilde{q}} \cdot \frac{\partial T}{\partial \epsilon}$$

Geometrically:



Math 115, Dec 16 1991

Review statement of Nöether:

Example: if F is t indep, then L is invariant under

$$T = t+\epsilon, \quad Q = q$$

$$q - q^* = \tilde{q}^*(t^*) = \tilde{q}^*(t+\epsilon)$$

$$L^* = \int_{t_0^*}^{t_1^*} F(\;\tilde{q}^*, \dot{\tilde{q}}^*)\,dt = \int_{t_0+\epsilon}^{t_1+\epsilon} F(\;, \tilde{q}(t+\epsilon), \dot{\tilde{q}}(t+\epsilon)), \ldots$$

Example: if $F = \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) - V(q_1^2 + q_2^2)$

L is inv under $\begin{pmatrix} q_1^* \\ q_2^* \end{pmatrix} = \begin{pmatrix} \cos\epsilon & -\sin\epsilon \\ \sin\epsilon & \cos\epsilon \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$

intuitive proof of Nöether's

Proof of Noether's

If extra time - start integrating.

HW: Read 13,20 Do 5.

4/29/88

what happens to a particle in a quantum harmonic oscillator $\frac{\pi}{2}$ seconds after it was thrown in?

Ofcourse, the above question is not particularily interesting. Luckily, while deriving the answer to that question we will pass by few of the most fundamental ideas in physics, the key one being the idea of integration over infinite dimensional spaces, which is central to quantum field theory. To my understanding, quantum field theory might as well be considered as a part of mathematics exceptional in not being completly rigorous, but yet deep elegant and powerful. So our real purpose here is to see a very simple but yet essential use of the basic ideas of quantum field theory.

Not everything along the way will be acurate and rigorous allthough the discussion below can be made completely so. The reasons for that are lack of time, and as the greater part of QFT is nonrigorous anyway, also lack of motivation. And last comment - few of the exspressions further down are going to look prety horrible, but the end result will be neat, familiar, and maybe a bit unexpected.

The question: Let the complex valued function $\psi = \psi(t,x)$ be a solution of the schrödinger equation

$$\frac{\partial \psi}{\partial t} = -i\left(-\frac{1}{2}\Delta_x + \frac{1}{2}x^2\right)\psi \quad \text{with } \psi|_{t=0} = \psi_0$$

what is $\psi|_{t=T=\pi/2}$?

In fact, big part of our discussion will work just as well for the general schrodinger equation -

$$\frac{\partial\psi}{\partial t}=-iH\psi\ ,\ H=-\tfrac{1}{2}\Delta_x+V(x)\ ,\ \psi|_{t=0}=\psi_0\ ,\ T\text{ arbitrary.}$$

ψ - "the wave function", $|\psi(t,x)|^2$ is the probability of finding our particle at time t in position x.

H - "the Hamiltonian", "the evolution operator".

$\frac{1}{2}\Delta_x$ - "kinetic energy term".

$V(x)$ - "the potential at a point x"

Solution:

$\frac{\partial\psi}{\partial t}=-iH\psi\ ,\ \psi|_{t=0}=\psi_0$ implies formally:

$$\psi(T,x)=(e^{-iTH}\psi_0)(x)=(e^{i\frac{T}{2}\Delta-iTV}\psi_0)(x)=$$

by aside 1, with $n=10^{58}+17$, and for convenience set $x_n=x$

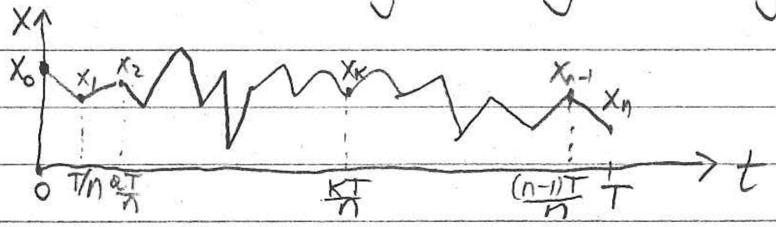
$$=(e^{i\frac{T}{2n}\Delta}\cdot e^{-i\frac{T}{n}V}\cdot e^{i\frac{T}{2n}\Delta}e^{-i\frac{T}{n}V}\cdot\ldots\ldots\cdot e^{i\frac{T}{2n}\Delta}e^{-i\frac{T}{n}V}\cdot\psi_0)(x_n)=$$

$$=c\cdot\int dx_{n-1}e^{i\frac{(x_n-x_{n-1})^2}{2T/n}}e^{-i\frac{T}{n}V(x_{n-1})}\cdot\int dx_{n-2}e^{\frac{i(x_{n-1}-x_{n-2})^2}{2T/n}}e^{-i\frac{T}{n}V(x_{n-2})}\cdot\ldots,$$

$$\ldots\int dx_0e^{i\frac{(x_1-x_0)^2}{2T/n}}e^{-i\frac{T}{n}V(x_0)}\psi_0(x_0)=$$

$$=c\cdot\int dx_0\cdot\ldots\cdot dx_{n-1}\exp\left(i\frac{T}{2n}\sum_{k=1}^{n}\left(\frac{x_n-x_{n-1}}{T/n}\right)^2-i\frac{T}{n}\sum_{k=0}^{n-1}V(x_k)\right)\cdot\psi(x_0)=$$

Now here comes the big novelty - bearing in mind the picture



we can write

$$\hat{=}\,c\int dx_0\underset{\downarrow}{\int}\mathcal{D}x\exp\left(i\int_0^T dt\left(\tfrac{1}{2}\dot{x}^2(t)-V(x(t))\right)\right)\psi(x_0)=$$

$$W_{x_0x_n}=\left\{\begin{matrix}x:[0,T]\to\mathbb{R}\\ x(0)=x_0,\ x(T)=x_n\end{matrix}\right\}$$

$$= c\int dx_0 \Psi(x_0) \int_{W_{x_0 x_n}} \mathcal{D}x \, exp(i\mathcal{L}(x)) =$$

Let x_c be the minimum point of $\mathcal{L}(x)$, write $x = x_c + x_q$, and get

$$= c\int dx_0 \Psi(x_0) \int_{W_{00}} \mathcal{D}x_q \, exp(i\mathcal{L}(x_c + x_q)) =$$

In our particular case, using aside 4, we get

$$= c\int dx_0 \Psi(x_0) \int_{W_{00}} \mathcal{D}x_q \, exp(i\mathcal{L}(x_c) + i\mathcal{L}(x_q)) =$$

The path integral is now independent of x_0, and so it factors out! Therfore

$$= c' \int dx_0 \Psi(x_0) e^{i\mathcal{L}(x_c)} =$$

in our case, with $t = \frac{\pi}{2}$

$$= c' \int dx_0 \Psi(x_0) exp(i\int_0^{\pi/2} \frac{1}{2}[(x_n \sin t + x_0 \cos t)']^2 - \frac{1}{2}(x_n \sin t + x_0 \cos t)^2 dt) =$$

$$= c' \int dx_0 \Psi(x_0) exp(-ix_0 x_n)$$

So how do I know that $|c'| = \frac{1}{\sqrt{2\pi}}$?

Aside 1: If A and B are matrices, then

$$e^{A+B} = \lim_{n\to\infty} (e^{A/n} e^{B/n})^n$$

Proof Just expand both sides as power series, and use some combinatorics to compare the coeficients. For a smoother proof see Glimm-Jaffe page 47. Slightly cheating what they say is that

$$e^{\frac{A+B}{n}} = e^{\frac{A}{n}} e^{\frac{B}{n}} \text{ to } \frac{\text{const}}{n^2} \quad \text{(trivial)}$$

and so

$$(e^{\frac{A+B}{n}})^n = (e^{\frac{A}{n}} e^{\frac{B}{n}})^n \text{ to } \frac{\text{const}}{n}$$

Aside 2: $(e^{itV}\psi_0)(x) = e^{itV(x)}\psi_0(x)$ - Trivial

Aside 3: $(e^{i\frac{t}{2}\Delta}\psi_0)(x) = c\cdot\int dx' e^{i\frac{(x-x')^2}{2t}} \psi_0(x')$

Proof: In fact, the left hand side is just the solution $\psi(t,x)$ of Schrödinger's equation with $V\equiv 0$:

$$\frac{\partial\psi}{\partial t} = i\frac{1}{2}\Delta_x \psi \qquad \psi|_{t=0} = \psi_0 .$$

Taking Fourier transform $\tilde{\psi}(t,p) = \frac{1}{\sqrt{2\pi}}\int e^{-ixp}\psi(t,x)dx$:

$$\frac{\partial\tilde{\psi}}{\partial t} = -i\frac{p^2}{2}\tilde{\psi} \qquad \tilde{\psi}|_{t=0} = \tilde{\psi}_0$$

For a fixed p, this is just a trivial ordinary differential equation with respect to t, and thus:

$$\tilde{\psi}(t,p) = e^{-i\frac{tp^2}{2}}\tilde{\psi}_0(p).$$

Taking inverse Fourier transform, which takes products to convulutions and Gaussians to Gaussians, we get Q.E.D.

Aside 4 Determining the minimum point of $\mathcal{L}(X)$ on $W_{x_0x_n}$:

If x_c is the minimum point in $W_{x_0x_n}$, then for arbitrary $x_q \in W_{00}$ there will be no term in

$$\mathcal{L}(x_c+\epsilon x_q)$$

which is linear in ϵ. Now

$$\mathcal{L}(X)=\int_0^T dt\left(\frac{1}{2}\dot{x}^2(t)-V(x(t))\right)$$

So using $V(x_c+\epsilon x_q)\cong V(x_c)+\epsilon x_q V'(x_c)$, we get that the linear term in ϵ in $\mathcal{L}(x_c+\epsilon x_q)$ is

$$\int_0^T dt(\dot{x}_c\cdot\dot{x}_q-V'(x_c)\cdot x_q)=$$

integrating by parts and using $x_q(0)=x_q(T)=0$:

$$=\int_0^T dt(-\ddot{x}_c-V'(x_c))\cdot x_q \quad .$$

For this integral to vanish independently of x_q, we must have $-\ddot{x}_c-V'(x_c)\equiv 0$, or

$$\ddot{x}_c=-V'(x_c).$$ (The famous F=ma of Newton! We have just rediscovered the Principle of least action!)

In our very particular case $V(x)=\frac{1}{2}x^2$ we get:

$$\ddot{x}_c=-x_c \quad , x_c(0)=x_0, \ x_c\left(\frac{\pi}{2}\right)=x_n$$

and therfore:

$$x_c(t)=x_n\sin t+x_0\cos t$$

Dror Bar-Natan

Math 115, Dec 18 1991.

A bit of integral calculus:

$$\psi(T,q_T) = c\int dq_0\,\psi(q)\,\mathcal{D}q\; e^{iL(q)} = \#$$

$$L(q) = \int_0^T \tfrac{1}{2}\dot{q}^2 - V(q)$$

$\hbar = 1, m = 1, k = 1$

QHO: $V(q) = \frac{1}{2}q^2$

in the particular case where $T = \frac{\pi}{2}$

$$q = q_c + q_q \qquad \begin{matrix} q_c(T) = q_T \\ q_c(0) = q_0 \end{matrix} \Rightarrow q_c =$$

$$\# = c'\int dq_0\,\psi_0(q_0)\,e^{iL(q_c)} \qquad q_c(t) = q_T \sin t + q_0 \cos t$$

$$= c'\int dq_0\,\psi_0(q_0)\,e^{-iq_0 q_T}$$

How do I know that $|c'| = \frac{1}{\sqrt{2\pi}}$?

Math 115, Jan 6 1992

A bit of integral calculus:

ψ - wave function $|\psi|^2$ - probability distribution.

$$\psi_T(q_T) = c\int dq_0\, \psi_0(q_0) \int_{\substack{q(0)=q_0\\ q(T)=q_T}} \mathcal{D}q\, e^{iL(q)} \qquad (\hbar = m = 1)$$

$$L(q) = \int_0^T \left(\tfrac{1}{2}\dot{q}^2 - V(q)\right) dt$$

QHO: $V(q) = \frac{1}{2}q^2 \quad (k=1)$

write $q = q_c + q_a \qquad \begin{matrix} q_c(T) = q_T \\ q_c(0) = q_0 \end{matrix} \Rightarrow q_c = A\sin t + B\cos t$

A, B hard

special case. $T = \frac{\pi}{2} \qquad q_c(t) = q_T \sin t + q_0 \cos t$

$$\# = c\int dq_0\, \psi_0(q_0) \int_{\substack{q_a(0)=q_0\\ q_a(T)=q_T}} \mathcal{D}q_a\, e^{iL(q_c+q_c)}$$

$$= c'\int dq_0\, \psi_0(q_0)\, e^{iL(q_c)} = c'\int dq_0\, \psi_0(q_0)\, e^{-iq_0 q_T}$$

So how do I know that $|c'| = \frac{1}{\sqrt{2\pi}}$

And what is the obvious question that you should ask?

Math 115 Extra Calculus of Variations Problems

Jan 10 1992

Dror Bar-Natan

1. Doodle with the following functionals - for each one find the Euler-Lagrange equation, solve it, write the Hamiltonian, write Hamilton's equations, solve them, and compute the Poisson bracket $\{y, y'\}$ - if you can.

   (a)

   $$\int \sqrt{y(1+y'^2)}dx$$

   (b)

   $$\int y'(1+x^2y')dx$$

   (c)

   $$\int (y^2+y'^2-2y\sin x)dx$$

   (d)

   $$\int_0^1 (xy+y^2-2y^2y')dx, \qquad y(0)=1, \qquad y(1)=2.$$

2. Find the extremals of the functional

   $$J(y)=\int_0^1 \sqrt{1+y'^2}\,dx$$

   under the conditions $\int_0^1 y dx = \frac{\pi}{4}$, $y(0)=0$, $y(1)=1$.

```
MATH 115 SUMMARY:

Complex Analysis:
  Complex arithmetic:
    Addition, Subtraction.
    Multiplication, Division.
    Conjugates and Moduli.
    Polar form.
    Powers and roots.
  The geometrical interpretation of complex numbers.
  The Cauchy-Riemann equations:
    Definition of complex functions.
    Complex Differentiation - all the rules apply.
    Differentiability in a domain = Analiticity.
    The Cauchy-Riemann equations.
    Theorem: C-R is more or less equivalent to analyticity.
  Laplace's equation - harmonic functions:
    What are harmonic functions? What do they mean?
    C-R is equivalent to Laplace:
      C-R --> real and imaginary parts are harmonic.
      Definition and existence of the harmonic conjugate.
  The exponential function:
    exp(z) is entire.
    Trigonometrical functions.
    Hyperbolic functions.
    The logarithm:
      Definition, the principal branch.
      Derivative.
      Non-uni-valuedness.
    Complex powers.
    Inverse trigonometric and hyperbolic functions.
  The Riemann mapping theorem:
    The conformal property of analytic functions.
    * Circle packings.
    * The Riemann mapping theorem.
    * Analytic functions are more or less as many as domains in the
          plane.
  Integration along a contour:
    Integration as a sum.
    Integration using a parametrization.
    Green's theorem.
  Cauchy's theorem:
    The weak form.
    * The strong form.
    Trivial consequences:
      Anti-derivatives.
      Independence of the path.
      Cauchy for 'funny' domains.
    More complicated consequences:
      Cauchy's integral formula.
      Boundary values determine the values inside!
      Higher derivatives.
      Gauss's mean value theorem.
      The maximum principle.
      Liouville's theorem (Bounded entire -> constant).
      The fundamental theorem of algebra.
```

- Series:
  - Existence of the (Taylor) series representation.
  - Convergence of the Taylor series, radius of convergence.
  - Addition, multiplication, differentiation integration and composition of series.
  - Laurent series.
  - Complex functions are functions of z alone!
- Residues:
  - Definition.
  - The residue theorem.
    - The residue theorem at infinity.
  - Zeros and poles.
  - Formula relating residues and derivatives.
  - Examples:
    - Rational functions with numerator smaller than denominator. (Upper or lower half plane).
    - Cos or sin times a rational function. (Replace by exp and use upper or lower half plane).
    - Trigonometric over [0,2pi]. (Do on the unit circle).
    - Involving logs or fractional powers. (Use the multi-valuedness of log).
    - Rational with a pole on the real line. (Try integrating around that pole).
- Conformal mappings:
  - Linear fractional transformations:
    - Linear transformations: expansions, rotations, translations.
    - The Riemann sphere and stereographic projection:
      - Preservation of circles.
      - Infinity.
    - z -> 1/z.
      - Preservation of circles.
      - Solving geometrical problems using inversion.
    - Linear fractional transformations.
    - Preservation of circles.
    - Mapping any 3 points to any other 3 points.
    - Transformations preserving the upper half plane.
    - Transformations preserving the unit disk.
      - Mapping a non-concentric circles to concentric ones.
    - Transformations sending the upper half plane to the unit disk.
  - The exponential map.
  - The logarithm as a map.
  - Powers as maps.
  - Sin as a map.
- Using conformal mappings to solve the Laplace equation:
  - Temperatures.
  - Electric potential and trajectories.
  - Fluid dynamics and flow lines.
- The Poisson integral.

Partial Differential Equations:

- The equations and their boundary conditions:
  - The wave equation.
  - The heat equation.
  - Laplace's equation.
  - Dirichlet conditions.

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    Neumann conditions.
  Separation of variables:
    The basic principle.
    The wave equation.
    The heat equation.
    Laplace's equation.
    Strange boundary conditions.
  Fourier series:
    Using exponentials.
    Using trigonometric functions.
    Non-standard periods.
    Even/odd functions.
    The Riemman Lebesgue lemma.
    Convergence.
    * The equidistribution of multiples of an irrational number.
    * The first digits of physical constants.
  Fourier transforms:
    Definition of the transform and its alleged inverse.
    Gausians.
    Translating and multiplying by exponentials.
    Convolutions.
      Transforming a convolution is multiplying the transforms.
      Transforming a product is convolving the transforms.
    The Fourier inversion theorem.
    Derivatives and multiplication by p.
    * The Plancherel identity.
  Heat on an infinite string:
    Solution using the Fourier transform.
    Fundamental solutions.
    Heating.
  The Poisson integral.

Calculus of Variations:
  The basic Euler-Lagrange.
    F independent of y.
    F independent of y'.
    F independent of x.
    Total derivatives.
    Minimal area of rotational surfaces.
  Free end points:
    The brachistochrone.
  Constraints:
    Lagrange multipliers.
    The isoparametric inequality.
  The Hammiltonian formulation:
    Canonical variables.
    Classical mechanics.
    Hammilton's equation.
    Poisson brackets:
      * Abstract properties. (Anti-symmetry, bilinearity, Leibnitz's
            rule, Jacobi's identity)
      Poisson bracket with the Hammiltonian.
        Conservation of energy.
  Noether's theorem:
    Formulation.
    * Proof.
```

Time translations and energy.
Space translations and momentum.
Rotations and angular momentum.
* Integration.

A '*' indicates topics that were discussed in class 'for fun', and are not required for the final.

Semi-Final

Math 115, Jan 8 1992

Dror Bar-Natan

You have 180 minutes to answer 6 of the following 8 questions, as indicated below. Each question is worth 16 points, except for question 8 which is worth 20 points. Plan your time wisely! It is a good idea to read the entire exam before answering any question. You may use any material you wish to use other than your friends. At the end of the 180 minutes don't forget to sign your name on anything you submit.
Solve 3 out of the four questions (1-4) on complex analysis.
Solve 2 out of the three questions (5-7) on partial differential equations.
Solve question number 8 on the calculus of variations.

1. (a) When is the function

$$h(x+iy) = x^3 + 3xy^2 - 3x + i(y^3 + 3x^2y - 3y)$$

differentiable in the complex sense?

(b) Find a function $\phi(x,y)$ which is harmonic in the upper half plane and satisfies

$$\phi(x,0) = x^2 + 5x + 1$$

for all x

2. What condition is missing in the statement of the following "minimum modulus principle"? Add the necessary condition and use it to prove the statement. Then show an example for a case in which the minimum modulus principle *does not hold* when this extra condition is not added:

> Let f be analytic in a bounded domain D and continuous up to and including the boundary of D. Then the modulus $|f(z)|$ attains its minimum value on the boundary of D.

3. Compute the following integrals:

(a)

$$\int_0^\infty \frac{x^2+1}{x^4+1} dx$$

(b)

$$\int_{-\infty}^\infty \frac{\sin x}{x+i} dx$$

(c)

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx \qquad ; \qquad \text{hint: } 2\sin^2 x = \mathrm{Re}(1 - e^{2ix})$$

4. A metal cup with a large amount of boiling water in it was placed near the edge of a table of high thermal conductivity and low thermal capacity, while the edge itself was kept at the freezing temperature of water. The radius of the cup was 1.5 units, its center was placed 1.25 units away from the edge of the table, and very far from the corners (so you may represent the table as the upper half plane without too much loss of accuracy). Some time has passed, and the temperature of the surface of the table has reached equilibrium. What would this equilibrium temperature be?

5. Solve the following differential equation by the method of separation of variables:

$$u_{xx} = u_t + tu \quad , \quad u_x(0,t) = u_x(\pi,t) = 0 \quad , \quad u(x,0) = 2.$$

6. (a) Express the function $f(x) = x \cos x$ as a Fourier series on the interval $-\pi \leq x \leq \pi$.

   (b) Let $g(x)$ be defined by

$$g(x) = \begin{cases} 1 & -\pi \leq x < -\frac{\pi}{2} \\ 2 & -\frac{\pi}{2} \leq x \leq 0 \\ 3 & 0 < x \leq \pi \end{cases}$$

   Let $S(x)$ denote the function to which the Fourier series for g converges on $-\pi \leq x \leq \pi$. What are $S(-\pi)$, $S(-\frac{\pi}{2})$, $S(0)$, $S(\frac{\pi}{2})$, and $S(\pi)$?

7. Compute the Fourier transform of $f(x) = e^{-x^2/2} \cos x$.

8. (a) Find the extremal of the functional

$$J(y) = \int_0^1 y'^2(1 + y'^2)dx$$

   with fixed end points $y(0) = 1$, $y(1) = 5$.

   (b) Compute the Poisson bracket $\{y, y'^2\}$.

If the above wasn't enough, here are some more relevant questions:

1. Prove that if an harmonic function in the entire plane is bounded from above, then it is constant.

2. Let $f(z) = \sum_{k=0}^{\infty} \frac{k^3 z^k}{3^k}$. Compute each of the following:

   (a)

$$f^{(6)}(0)$$

   (b)

$$\oint_{|z|=1} \frac{f(z)dz}{z^4}$$

   (c)

$$\oint_{|z|=1} \frac{f(z) \sin z}{z^2} dz$$

(d)

$$\oint_{|z|=1} f(z)e^z dz$$

3. Prove that the Laurent series expansion of the function

$$f(z) = e^{\frac{\lambda}{2}(z-\frac{1}{z})}$$

for $|z| > 0$ is given by

$$\sum_{k=-\infty}^{\infty} J_k(\lambda)z^k$$

where

$$J_k(\lambda) = \frac{1}{2\pi}\int_0^{2\pi} \cos(k\theta - \lambda\sin\theta)d\theta.$$

The functions $J_k(\lambda)$ are known as the *Bessel functions* of the first kind.

4. Compute the Taylor series expansion of the main branch of $(1+z)^\alpha$ around $z = 0$ where α is an arbitrary complex number.

5. A harmonic function u is defined on the crescent-shaped region which is the part of the disk $|z - i| < 1$ outside of the unit disk. u is equal to 1 on the large arc bounding this crescent, and to 0 on the small arc. Find an explicit formula for u.

6. Suppose we have two wires, of identical length L. One of these wires is heated to $T = 100°$ and the other is brought to $T = 0°$. The two wires are then joined at their ends to make a circle. If no heat flows out (or in) of the circle of wire, the temperature $u(x,t)$ satisfies $u_t = u_{xx}$ where t is time and x is the distance along the wire from some arbitrary base point. Find $u(x,t)$.

— GOOD LUCK —

Math 115 Extra Calculus of Variations Problems

Jan 10 1992

Dror Bar-Natan

1. Doodle with the following functionals - for each one find the Euler-Lagrange equation, solve it, write the Hamiltonian, write Hamilton's equations, solve them, and compute the Poisson bracket $\{y, y'\}$ - if you can.

   (a)
   $$\int \sqrt{y(1+y'^2)}dx$$

   (b)
   $$\int y'(1+x^2y')dx$$

   (c)
   $$\int (y^2+y'^2-2y\sin x)dx$$

   (d)
   $$\int_0^1 (xy+y^2-2y^2y')dx, \qquad y(0)=1, \qquad y(1)=2.$$

2. Find the extremals of the functional
   $$J(y)=\int_0^1 \sqrt{1+y'^2}\,dx$$
   under the conditions $\int_0^1 y dx = \frac{\pi}{4}$, $y(0)=0$, $y(1)=1$.

Math 115 Final

Jan 15 1992
Dror Bar-Natan

You have 180 minutes to answer 6 of the following 8 questions, as indicated below. Each question is worth 16 points, except for question 8 which is worth 20 points. Plan your time wisely! It is a good idea to read the entire exam before answering any question. You may use any material you wish to use other than your friends. At the end of the 180 minutes don't forget to sign your name on anything you submit, and to indicate clearly which are the questions that are to be graded. If there will be no such indication on your notebook, your answers will be graded in the order in which they appear.
Solve 3 out of the four questions (1-4) on complex analysis.
Solve 2 out of the three questions (5-7) on partial differential equations.
Solve question number 8 on the calculus of variations.

1. (a) Show that if the function v is the harmonic conjugate of the function u in some domain, then uv is harmonic in that domain.

   (b) Find all the solutions of the equation
$$\sin z = 2.$$

2. (a) Prove that an analytic function with a constant modulus in a certain domain is constant.

   (b) Use the above, the maximum principle and the minimum principle of the semi-final to prove that a non-constant analytic function in a domain D that has a constant modulus on the boundary of that domain has at least one zero inside D.

3. Compute two of the following integrals:

   (a)
$$\int_0^{2\pi} \frac{8d\theta}{5+4\cos\theta}$$

   (b)
$$\int_0^\infty \frac{\cos x}{(x^2+1)^2}dx$$

   (c)
$$\int_0^\infty \frac{x^{\lambda-1}}{x+4}dx \qquad ; \qquad 0<\lambda<1$$

4. A harmonic function u defined on the unit disk, but outside of the circle of radius $1/2$ about the point $1/2$, is equal to 1 on the outer boundary of its domain of definition and equal to 0 on the inner boundary of that domain. Find u explicitly. (The u that you find doesn't have to be defined at $z=1$, and isn't even required to have a limit as $z \to 1$. But it better be bounded near $z=1$).

5. Find the solution $U(r,\theta)$ of Laplace's equation

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0 \quad ; \quad 0 \le \theta \le \alpha \quad , \quad R \le r < \infty$$

under the boundary conditions:

$$U(R,\theta) = f(\theta) \quad , \quad U(r,0) = 0 = U(r,\alpha),$$

$$\lim_{r\to\infty} U(r,\theta) = 0.$$

   (a) For a general $f(\theta)$.

   (b) For $f(\theta) = \sin(2\pi\theta/\alpha)$.

6. (a) Find real constants c_n for which

$$x = \sum_{n=1}^{\infty} c_n \sin(nx)$$

   for $-\pi < x < \pi$.

   (b) Show that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} = \frac{\pi}{4}.$$

7. Compute the Fourier transform of $f(x) = x^2 e^{-x^2/2}$.

8. (a) Find the extremals of the functional

$$L(q) = \int_0^1 \frac{1+q^2}{\dot{q}^2}\, dt$$

   under the conditions $q(0) = 0$, $q(1) = \frac{e^2-1}{2e}$.

   (b) Find the momentum p.

   (c) Write the Hamiltonian H in terms of p and q.

   (d) Check that the extremal that you found above indeed satisfies Hamilton's equations.

— GOOD LUCK —

Your final grade will be available at my office on Monday January the 20th.

Math 115 Final

Makeup Examination for

Dror Bar-Natan

You have 180 minutes to answer 6 of the following 8 questions, as indicated below. Each question is worth 16 points, except for question 8 which is worth 20 points. Plan your time wisely! It is a good idea to read the entire exam before answering any question. You may use any material you wish to use other than your friends. At the end of the 180 minutes don't forget to sign your name on anything you submit, and to indicate clearly which are the questions that are to be graded. If there will be no such indication on your notebook, your answers will be graded in the order in which they appear.
Solve 3 out of the four questions (1-4) on complex analysis.
Solve 2 out of the three questions (5-7) on partial differential equations.
Solve question number 8 on the calculus of variations.

1. (a) Show that if the function v is the harmonic conjugate of the function u in some domain in which neither of them vanishes, then $\frac{u}{u^2+v^2}$ is harmonic in that domain.

   (b) Find all the solutions of the equation

$$\cos z = 2i.$$

2. Prove that an entire function f which satisfies

$$\lim_{z\to\infty} \frac{f(z)}{z^2} = 0$$

is a linear function of the form $f(z) = az + b$.

3. Compute two of the following integrals:

   (a)

$$\int_0^\pi \frac{d\theta}{2-\cos\theta}$$

   (b)

$$\int_0^\infty \frac{x^2+1}{x^4+1}dx$$

   (c)

$$\int_{-\infty}^\infty \frac{xe^{2ix}}{x^2-1}dx$$

4. A harmonic function u defined on the unit disk, but outside of the circle of radius 1/3 about the point 2/3, is equal to 1 on the outer boundary of its domain of definition and equal to 0 on the inner boundary of that domain. Find u explicitly. (The u that you find doesn't have to be defined at $z = 1$, and isn't even required to have a limit as $z \to 1$. But it better be bounded near $z = 1$).

5. Find the solution $U(r,\theta)$ of Laplace's equation

$$U_{rr} + \frac{1}{r}U_r + \frac{1}{r^2}U_{\theta\theta} = 0 \quad ; \quad 0 \le \theta \le \alpha \quad , \quad 0 \le r \le R$$

under the boundary conditions:

$$U(R,\theta) = f(\theta) \quad , \quad U_\theta(r,0) = 0 = U_\theta(r,\alpha),$$

(a) For a general $f(\theta)$.

(b) For $f(\theta) = \sin^2(2\pi\theta/\alpha)$.

6. (a) Find real constants c_n for which

$$|x| = \sum_{n=0}^{\infty} c_n \cos(nx)$$

for $-\pi < x < \pi$.

(b) Use the above to prove

$$\sum_{k=0}^{\infty} \frac{1}{(2k+1)^2} = \frac{\pi^2}{8}.$$

7. Compute the Fourier transform of $f(x) = e^{-|x|}\sin x$.

8. (a) Find the extremals of the functional

$$L(q) = \frac{1}{2}\int_0^1 (\dot{q}^2 - q^2)\, dt$$

under the conditions $q(0) = q(\pi) = 0$.

(b) Find the momentum p.

(c) Write the Hamiltonian H in terms of p and q.

(d) Check that the extremal that you found above indeed satisfies Hamilton's equations.

— GOOD LUCK —

2.4. The Spherical Representation. For many purposes it is useful to extend the system **C** of complex numbers by introduction of a symbol ∞ to represent infinity. Its connection with the finite numbers is established by setting $a + \infty = \infty + a = \infty$ for all finite a, and

$$b \cdot \infty = \infty \cdot b = \infty$$

for all $b \neq 0$, including $b = \infty$. It is impossible, however, to define $\infty + \infty$ and $0 \cdot \infty$ without violating the laws of arithmetic. By special convention we shall nevertheless write $a/0 = \infty$ for $a \neq 0$ and $b/\infty = 0$ for $b \neq \infty$.

In the plane there is no room for a point corresponding to ∞, but we can of course introduce an "ideal" point which we call the *point at infinity*. The points in the plane together with the point at infinity form the *extended complex plane*. We agree that every straight line shall pass through the point at infinity. By contrast, no half plane shall contain the ideal point.

It is desirable to introduce a geometric model in which all points of the extended plane have a concrete representative. To this end we consider the unit sphere S whose equation in three-dimensional space is $x_1^2 + x_2^2 + x_3^2 = 1$. With every point on S, except $(0,0,1)$, we can associate a complex number

$$z = \frac{x_1 + ix_2}{1 - x_3}, \tag{24}$$

and this correspondence is one to one. Indeed, from (24) we obtain

$$|z|^2 = \frac{x_1^2 + x_2^2}{(1 - x_3)^2} = \frac{1 + x_3}{1 - x_3},$$

and hence

$$x_3 = \frac{|z|^2 - 1}{|z|^2 + 1}. \tag{25}$$

Further computation yields

$$\begin{aligned} x_1 &= \frac{z + \bar{z}}{1 + |z|^2} \\ x_2 &= \frac{z - \bar{z}}{i(1 + |z|^2)}. \end{aligned} \tag{26}$$

The correspondence can be completed by letting the point at infinity correspond to $(0,0,1)$, and we can thus regard the sphere as a representation of the extended plane or of the extended number system. We note that the hemisphere $x_3 < 0$ corresponds to the disk $|z| < 1$ and the hemisphere $x_3 > 0$ to its outside $|z| > 1$. In function theory the sphere S is referred to as the *Riemann sphere.*

If the complex plane is identified with the (x_1,x_2)-plane with the x_1- and x_2-axis corresponding to the real and imaginary axis, respectively, the transformation (24) takes on a simple geometric meaning. Writing $z = x + iy$ we can verify that

$$x : y : -1 = x_1 : x_2 : x_3 - 1, \tag{27}$$

and this means that the points $(x,y,0)$ (x_1,x_2,x_3), and $(0,0,1)$ are in a straight line. Hence the correspondence is a central projection from the center $(0,0,1)$ as shown in Fig. 1-3. It is called a *stereographic projection.* The context will make it clear whether the stereographic projection is regarded as a mapping from S to the extended complex plane, or *vice versa.*

In the spherical representation there is no simple interpretation of addition and multiplication. Its advantage lies in the fact that the point at infinity is no longer distinguished.

It is geometrically evident that the stereographic projection transforms every straight line in the z-plane into a circle on S which passes through the pole $(0,0,1)$, and the converse is also true. More generally, any circle on the sphere corresponds to a circle or straight line in the z-plane. To prove this we observe that a circle on the sphere lies in a plane $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = \alpha_0$, where we can assume that $\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$ and $0 \leqq \alpha_0 < 1$. In terms of z and $\bar{z}$ this equation takes the form

$$\alpha_1(z + \bar{z}) - \alpha_2 i(z - \bar{z}) + \alpha_3(|z|^2 - 1) = \alpha_0(|z|^2 + 1)$$

or

$$(\alpha_0 - \alpha_3)(x^2 + y^2) - 2\alpha_1 x - 2\alpha_2 y + \alpha_0 + \alpha_3 = 0.$$

For $\alpha_0 \neq \alpha_3$ this is the equation of a circle, and for $\alpha_0 = \alpha_3$ it represents a straight line. Conversely, the equation of any circle or straight line

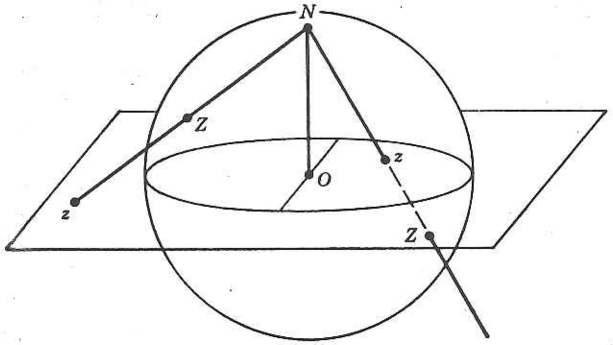


FIG. 1-3. Stereographic projection.

can be written in this form. The correspondence is consequently one to one.

It is easy to calculate the distance $d(z,z')$ between the stereographic projections of z and z'. If the points on the sphere are denoted by (x_1,x_2,x_3), (x_1',x_2',x_3'), we have first

$$(x_1 - x_1')^2 + (x_2 - x_2')^2 + (x_3 - x_3')^2 = 2 - 2(x_1x_1' + x_2x_2' + x_3x_3').$$

From (35) and (36) we obtain after a short computation

$$x_1x_1' + x_2x_2' + x_3x_3' = \frac{(z + \bar{z})(z' + \bar{z}') - (z - \bar{z})(z' - \bar{z}') + (|z|^2 - 1)(|z'|^2 - 1)}{(1 + |z|^2)(1 + |z'|^2)} = \frac{(1 + |z|^2)(1 + |z'|^2) - 2|z - z'|^2}{(1 + |z|^2)(1 + |z'|^2)}.$$

As a result we find that

$$d(z,z') = \frac{2|z - z'|}{\sqrt{(1 + |z|^2)(1 + |z'|^2)}}. \tag{28}$$

For $z' = \infty$ the corresponding formula is

$$d(z, \infty) = \frac{2}{\sqrt{1 + |z|^2}}.$$

EXERCISES

1. Show that z and z' correspond to diametrically opposite points on the Riemann sphere if and only if $z\bar{z}' = -1$.

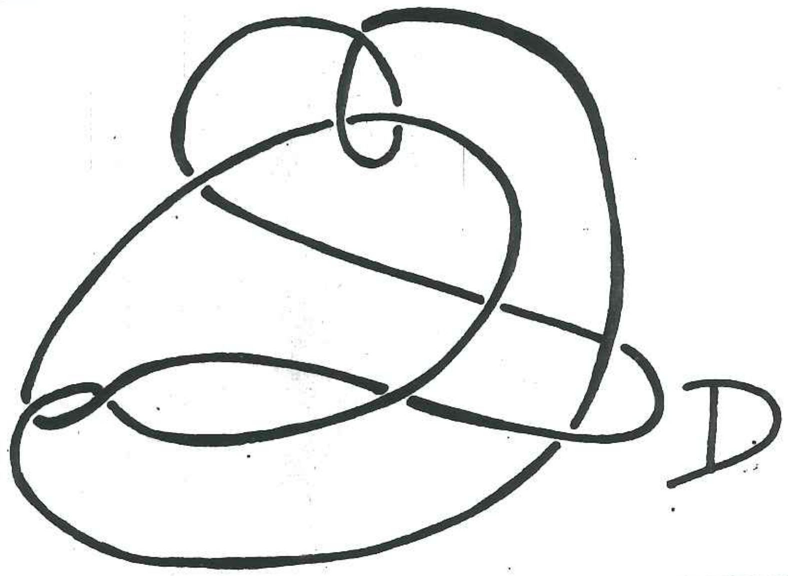
2. A cube has its vertices on the sphere S and its edges parallel to the coordinate axes. Find the stereographic projections of the vertices.

3. Same problem for a regular tetrahedron in general position.

4. Let Z, Z' denote the stereographic projections of z, z', and let N be the north pole. Show that the triangles NZZ' and Nzz' are similar, and use this to derive (28).

5. Find the radius of the spherical image of the circle in the plane whose center is a and radius R.

Remember your project!



Is this a knot or not a knot?

(discovered by Ken Millett)

Math 115 - Final Solution

①

ⓐ method 1: since v is harmonic conjugate of u, we know:

$$u_x = v_y \quad u_y = -v_x \quad \text{(cauchy Riemann)}$$

$$\text{Also } u_{xx}+u_{yy}=0 \quad v_{xx}+v_{yy}=0$$

$$\text{Thus } (uv)_{xx}+(uv)_{yy} = (uv_x+u_xv)_x + (uv_y+u_yv)_y$$

$$= u_xv_x + uv_{xx} + u_{xx}v + u_xv_x + u_yv_y + uv_{yy} + u_{yy}v + u_yv_y$$

$$= \underbrace{2u_xv_x+2u_yv_y}_{0 \text{ by C-Riemann}} + u\underset{\substack{\| \\ 0}}{(v_{xx}+v_{yy})} + v\underset{\substack{\| \\ 0}}{(u_{xx}+u_{yy})}$$

So uv is harmonic

method 2: $u+iv$ is analytic $\Rightarrow (u+iv)^2$ is analytic

$2uv$ is $\operatorname{Im}((u+iv)^2)$

$\Rightarrow 2uv$ is harmonic $\Rightarrow uv$ is harmonic.

ⓑ $$\sin z = \frac{e^{iz}-e^{-iz}}{2i} = 2 \Rightarrow e^{2iz}-4ie^{iz}-1=0$$

$$y = e^{iz}$$

$$y^2-4iy-1=0$$

$$\Rightarrow y = \frac{4i \pm \sqrt{-16+4}}{2}$$

$$= 2i \pm \sqrt{3}i = (2\pm\sqrt{3})i$$

$$= 2\pm\sqrt{3}\, e^{i\pi/2}$$

$$= e^{\ln 2\pm\sqrt{3}}\, e^{i(\pi/2+2\pi n)}$$

$$\Rightarrow z = (\pi/2+2\pi n) - i(\ln(2\pm\sqrt{3}))$$

n is integer

②

ⓐ The result clearly holds if the constant modulus = 0. Otherwise consider log (f(z)). For any z_0, pick a neighborhood N small enough so that f(N) is contained in $\mathbb{C}$ - ½ line through 0 for some half-line. log (f(z)) will then be analytic, and since it has constant real part CRieman $\Rightarrow$ $\log(f(z))$ = constant $\Rightarrow$ $f(z)$ is constant in some neighborhood of z_0 for any z_0 in the domain $\Rightarrow$ f is constant.

ⓑ Assume f has no 0 in D. Maximum principle and semi-final minimum principle $\Rightarrow$ f has constant modulus. ⓐ $\Rightarrow$ f is constant, a contradiction. So f must have a 0 in D.

Question #3:

a) $\int_0^{2\pi} \frac{8d\theta}{5+4\cos\theta} =$ $\qquad z=e^{i\theta} \quad dz=ie^{i\theta}d\theta \quad d\theta=\frac{1}{iz}d\theta$

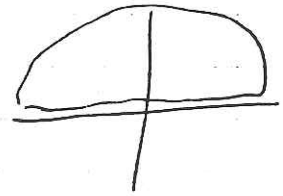
$\cos\theta = \frac{z+z^{-1}}{2}$

$$= \int_0^{2\pi} \frac{dz}{iz}\frac{8}{5+4\left(\frac{z+z^{-1}}{2}\right)} = -i\int_0^{2\pi}\frac{8dz}{2z^2+5z+2}$$

$$= -i\int_0^{2\pi}\frac{8dz}{(2z+1)(z+2)} = -i\cdot 2\pi i\,\text{Res}_{z=-\frac{1}{2}}\frac{4}{(z+\frac{1}{2})(z+2)}$$

$$= 2\pi\cdot\frac{4}{3/2} = \frac{16\pi}{3}$$

b) $\int_0^{\infty}\frac{\cos x}{(x^2+1)^2}dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{e^{iz}}{(z^2+1)^2}dz$ $\qquad C =$



$= \frac{1}{2}\int_C\frac{e^{iz}dz}{(z^2+1)^2} =$ $\qquad z^2+1=0 \Rightarrow z=\pm i$

only $+i$ is inside C.

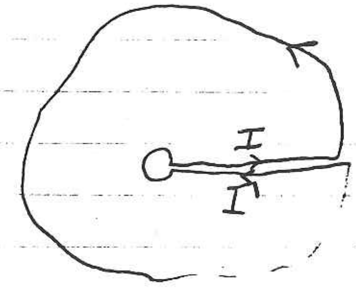
$$= \frac{1}{2}\cdot 2\pi i\,\text{Res}_{z=i}\frac{e^{iz}}{(z-i)^2(z+i)^2} = \pi i\cdot\left(\frac{e^{iz}}{(z+i)^2}\right)'\Bigg|_{z=i}$$

$$= \pi i\left(ie^{iz}(z+i)^{-2} - 2e^{iz}(z+i)^{-3}\right)_{z=i} =$$

$$= \pi i e^{-1}\left(-\frac{i}{4} - 2\cdot\frac{i}{8}\right) = \frac{\pi}{2e}$$

c) Call $I = \int_0^{\infty}\frac{x^{\lambda-1}}{x+4}dx$ and

$C =$



and now

$$X^{\lambda-1} = e^{(\lambda-1)\log x} = e^{(\lambda-1)(\operatorname{Log} x + 2\pi n i)}$$

and so

$$I' = e^{(\lambda-1)(2\pi i)} I = e^{2\pi i\lambda} I$$

or

$$(1-e^{2\pi i\lambda}) I = 2\pi i \operatorname{Res}_{z=-y} \frac{z^{\lambda-1}}{z+y} = 2\pi i(-y)^{\lambda-1} =$$

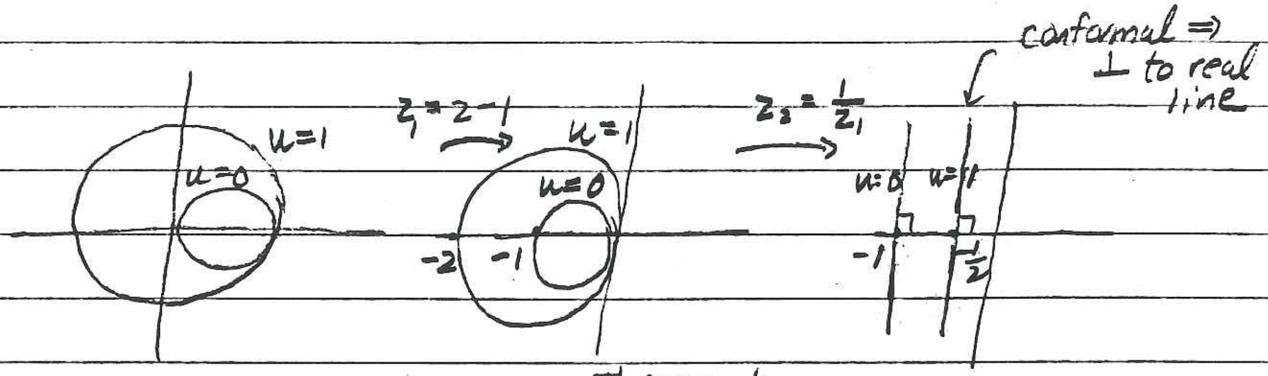
$$= 2\pi i\, e^{(\log -y)(\lambda-1)} = 2\pi i\, e^{(\pi i+\log y)(\lambda-1)} =$$

$$= 2\pi i\, y^{\lambda-1} \cdot e^{\pi i(\lambda-1)} = -2\pi i\, e^{\pi i\lambda}$$

or

$$I = \frac{-2\pi i\, e^{\pi i\lambda}}{1-e^{2\pi i\lambda}} = \pi \frac{2i}{e^{\pi i\lambda}-e^{-\pi i\lambda}} \cdot y^{\lambda-1} = \frac{\pi \cdot y^{\lambda-1}}{\sin(\pi\lambda)}$$

④



These go to parallel lines, since there is only 1 intersection point and it will be at ∞.

$$\Rightarrow T(z_2) = c_1 \operatorname{Re} z_2 + c_2 \qquad 1 = -\frac{c_1}{2} + c_2 \qquad c_2 = 2$$

$$= 2\operatorname{Re} z_2 + 2 \qquad 0 = -c_1 + c_2 \qquad c_1 = 2$$

$$z_2 = \frac{1}{z-1} = \frac{1}{x+yi-1} = \frac{1}{(x-1)+yi} = \frac{(x-1)-yi}{(x-1)^2+y^2}$$

$$\Rightarrow T(x,y) = 2\frac{(x-1)}{(x-1)^2+y^2} + 2$$

check: on unit disk $x^2+y^2=1$

$$\frac{2(x-1)}{1-2x+1} + 2 = 1 \ ✓$$

At origin = -2+2 = 0

Question #5:

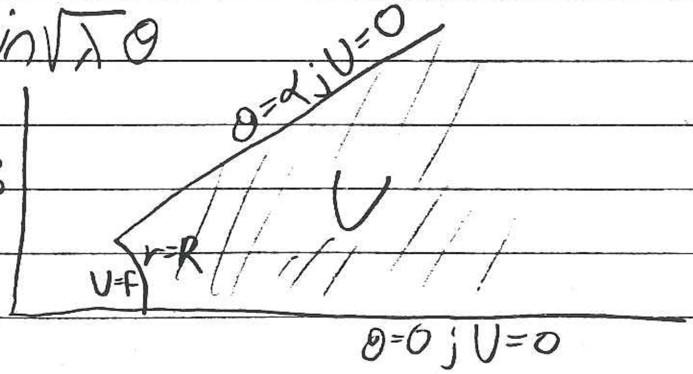
Seperation of variables gives $U = R(r)\Theta(\theta)$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

(It is easy to eliminate the possibility $\lambda \leq 0$ using the boundary conditions)

$$\Rightarrow \Theta(\theta) = A\cos\sqrt{\lambda}\,\theta + B\sin\sqrt{\lambda}\,\theta$$

The boundary conditions are:



$U(r,0)=0 \quad \Rightarrow \quad A=0$

$U(r,\alpha)=0 \quad \Rightarrow \quad \sin\sqrt{\lambda}\alpha=0 \Rightarrow \sqrt{\lambda}\alpha=\pi n \Rightarrow$

$$\Rightarrow \lambda_n=\frac{\pi^2 n^2}{\alpha^2}$$

The R equation

$$r^2R''+rR'=\frac{\pi^2 n^2}{\alpha^2}R$$

try $R=r^{\beta}$ and get

$$\beta(\beta-1)r^{\beta}+\beta r^{\beta}=\frac{\pi^2 n^2}{\alpha^2}r^{\beta}$$

$$\Rightarrow \quad \beta=\pm\frac{\pi n}{\alpha}$$

but $\lim_{r\to\infty} R(r)=0$, and therefore $\beta=-\frac{\pi n}{\alpha}$

and $R_n=r^{-\frac{\pi n}{\alpha}}$

$$U=\sum_{n=1}^{\infty} b_n U_n=\sum_{n=1}^{\infty} b_n r^{-\frac{\pi n}{\alpha}}\sin\left(\frac{\pi n\theta}{\alpha}\right)$$

$$F(\theta)=U(R,\theta)=\sum_{n=1}^{\infty} b_n R^{-\frac{\pi n}{\alpha}}\sin\left(\frac{\pi n\theta}{\alpha}\right)$$

on the other hand, F can be extented to become an odd function on $-\alpha\leq\theta\leq\alpha$, and by Fourier theory on that interval,

$$F(\theta)=\sum_{n=1}^{\infty}\tilde{b}_n\sin\frac{\pi n\theta}{\alpha} \qquad \tilde{b}_n=\frac{1}{\alpha}\int_{-\alpha}^{\alpha}F(\theta)\sin\frac{\pi n\theta}{\alpha}d\theta$$

$$\Rightarrow b_n=R^{\frac{\pi n}{\alpha}}\tilde{b}_n=\frac{2}{\alpha}R^{\frac{\pi n}{\alpha}}\int_0^{\alpha}F(\theta)\sin\frac{\pi n\theta}{\alpha}d\theta$$

Finally, if $F(\theta)=\sin\frac{2\pi\theta}{\alpha}$ then $\tilde{b}_2=1$ and $\tilde{b}_{\neq 2}=0$ and thus

$$U(r,\theta)=R^{\frac{2\pi}{\alpha}}r^{-\frac{2\pi}{\alpha}}\sin\frac{2\pi\theta}{\alpha}$$

Question 6:

a) $$c_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx=\frac{1}{\pi}\left(x\cdot\left(-\frac{\cos nx}{n}\right)\right)\Big|_{-\pi}^{\pi}-\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{-\cos nx}{n}dx}_{0}$$

integral of cos over a bunch of periods = 0

$$=\frac{1}{n\pi}\left(-\pi\cos n\pi-\pi\cos n\pi\right)=$$

$$=-\frac{2}{n}\cos(n\pi)=-(-1)^n\frac{2}{n}$$

and so for $-\pi<x<\pi$

$$x=\sum_{n=1}^{\infty}-(-1)^n\frac{2}{n}\sin nx$$

b) Want an x for which all even terms in the above equation will drop ⇒ take $x=\frac{\pi}{2}$:

$$\frac{\pi}{2}=\sum_{n=1}^{\infty}-(-1)^n\frac{2}{n}\sin\frac{n\pi}{2}=\sum_{\text{odd } n}-(-1)^n\frac{2}{n}\sin\frac{n\pi}{2}$$

$n=2k+1$: $$\frac{\pi}{2}=\sum_{k=0}^{\infty}+\frac{2}{2k+1}(-1)^k$$

and this is what we wanted to prove.

Question 7:

Remember that $\widetilde{f'} = ip\tilde{f}$

$$\Rightarrow \widetilde{xf} = i\frac{d}{dp}\tilde{f}$$

$$\Rightarrow \widetilde{x\cdot xf} = i\frac{d}{dp}\widetilde{xf} = i\frac{d}{dp}i\frac{d}{dp}\tilde{f} = -\frac{d^2}{dp^2}\tilde{f}$$

in our case, $f = e^{-x^2/2}$ and $\tilde{f} = e^{-p^2/2}$
and so

$$\left(x^2e^{-\frac{x^2}{2}}\right)^\sim = -\left(e^{-p^2/2}\right)'' = \left(pe^{-p^2/2}\right)' = e^{-p^2/2} - p^2e^{-p^2/2}$$

The same result can be obtained by an explicit computation involving a few integrations by parts.

Question 8:

a) F indep. of t $\Rightarrow$

$$E.L \Leftrightarrow F - \dot{q}F_{\dot{q}} = \tilde{C}_1$$

$$\frac{1+q^2}{q^2} + 2\dot{q}\,\frac{1+q^2}{q^3} = \tilde{C}_1$$

$$\Rightarrow \quad 3\frac{1+q^2}{q^2} = \tilde{C}_1 \quad \Rightarrow \dot{q} = \sqrt{3\frac{1+q^2}{\tilde{C}_1}} = C_1\sqrt{1+q^2}$$

$$\Rightarrow \frac{dq}{dt} = C_1\sqrt{1+q^2} \Rightarrow \frac{dq}{\sqrt{1+q^2}} = C_1\,dt \Rightarrow \sinh^{-1}q = C_1t + C_2$$

$$\Rightarrow q = \sinh(C_1t + C_2)$$

$q(0) = 0 \Rightarrow C_2 = 0$

$q(1) = \frac{1}{2}(e - e^{-1}) \Rightarrow C_1 = 1$ $\quad \Rightarrow q = \sinh t$

b) $p = F_{\dot q} = -2\frac{1+q^2}{\dot q^3}$ $\left(\Rightarrow \dot q = \sqrt[3]{-2\frac{1+q^2}{p}}\right)$

c) $H = p\dot q - F = -3\frac{1+q^2}{\dot q^2} = -3\frac{1+q^2}{\left(-2\frac{1+q^2}{p}\right)^{2/3}} =$

$= -3\cdot 2^{-\frac{2}{3}}(1+q^2)^{1/3}p^{2/3}$

d) $\frac{\partial H}{\partial p} = -3 2^{-2/3}(1+q^2)^{1/3}\cdot\frac{2}{3}\cdot p^{-1/3} = \sqrt[3]{-2\frac{1+q^2}{p}} = \dot q$ as required.

The other equation is more complicated:

$p = -2\frac{1+q^2}{\dot q^3} = -2\frac{1+\sinh^2 t}{(\cosh t)^3} = -2\frac{\cosh^2 t}{\cosh^3 t} = -\frac{2}{\cosh t}$

$\dot p = 2\frac{\sinh t}{\cosh^2 t}$

$\frac{\partial H}{\partial q} = -3\cdot 2^{-2/3}\cdot 2q\cdot\frac{1}{3}(1+q^2)^{-\frac{2}{3}}p^{2/3} = -2^{1/3}\cdot q\cdot(1+q^2)^{-2/3}\cdot p^{2/3} =$

$= -2^{1/3}\sinh t\,(1+\sinh^2 t)^{-2/3}\left(-\frac{2}{\cosh t}\right)^{2/3} =$

$= -2\sinh t\cdot(\cosh^2 t)^{-2/3}\left(\frac{1}{\cosh t}\right)^{2/3} = -2\frac{\sinh t}{\cosh^2 t} = -\dot p$

as required.

Math 115 - Final Solution

①

ⓐ method 1: since v is harmonic conjugate of u, we know:

$$u_x = v_y \quad u_y = -v_x \quad \text{(cauchy Riemann)}$$

Also $u_{xx}+u_{yy}=0 \quad v_{xx}+v_{yy}=0$

Thus $(uv)_{xx}+(uv)_{yy} = (uv_x+u_xv)_x + (uv_y+u_yv)_y$

$$= u_xv_x + uv_{xx} + u_{xx}v + u_xv_x + u_yv_y + uv_{yy} + u_{yy}v + u_yv_y$$

$$= \underbrace{2u_xv_x + 2u_yv_y}_{0 \text{ by C-Riemann}} + u\underset{\overset{\shortparallel}{0}}{(v_{xx}+v_{yy})} + v\underset{\overset{\shortparallel}{0}}{(u_{xx}+u_{yy})}$$

So uv is harmonic

method 2: $u+iv$ is analytic $\Rightarrow (u+iv)^2$ is analytic

$2uv$ is $\mathrm{Im}((u+iv)^2)$

$\Rightarrow$ $2uv$ is harmonic $\Rightarrow$ uv is harmonic.

ⓑ $\sin z = \frac{e^{iz}-e^{-iz}}{2i} = 2 \Rightarrow e^{2iz} - 4ie^{iz} - 1 = 0$

$y = e^{iz}$

$y^2 - 4iy - 1 = 0$

$\Rightarrow y = \frac{4i \pm \sqrt{-16+4}}{2}$

$= 2i \pm \sqrt{3}i = (2 \pm \sqrt{3})i$

$= 2 \pm \sqrt{3}\, e^{i\pi/2}$

$= e^{\ln 2\pm\sqrt{3}} e^{i(\pi/2 + 2\pi n)}$

$\Rightarrow z = (\pi/2 + 2\pi n) - i(\ln(2 \pm \sqrt{3}))$

n is integer

②

ⓐ The result clearly holds if the constant modulus = 0. otherwise consider log (f(z)). For any z_0, pick a neighborhood N small enough so that f(N) is contained in $\mathbb{C}$ - ½ line through 0 for some half-line. log (f(z)) will then be analytic, and since it has constant real part CRieman ⇒ log (f(z)) = constant ⇒ f(z) is constant in some neighborhood of z_0 for any z_0 in the domain ⇒ f is constant.

ⓑ Assume f has no 0 in D. Maximum principle and semi-final minimum principle ⇒ f has constant modulus. ⓐ ⇒ f is constant, a contradiction. So f must have a 0 in D.

Question #3:

a) $\int_0^{2\pi} \frac{8d\theta}{5+4\cos\theta} =$ $\quad z=e^{i\theta} \quad dz=ie^{i\theta}d\theta \quad d\theta=\frac{1}{iz}d\theta$

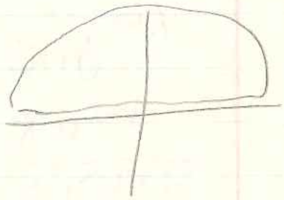
$\cos\theta = \frac{z+z^{-1}}{2}$

$$= \int_0^{2\pi} \frac{dz}{iz}\,\frac{8}{5+4\left(\frac{z+z^{-1}}{2}\right)} = -i\int_0^{2\pi} \frac{8dz}{2z^2+5z+2}$$

$$= -i\int_0^{2\pi} \frac{8dz}{(2z+1)(z+2)} = -i\cdot 2\pi i \,\mathrm{Res}_{z=-\frac{1}{2}} \frac{4}{(z+\frac{1}{2})(z+2)}$$

$$= 2\pi\cdot\frac{4}{3/2} = \frac{16\pi}{3}$$

b) $\int_0^{\infty} \frac{\cos x}{(x^2+1)^2}dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{e^{iz}}{(z^2+1)^2}dz$ $\quad C =$



$= \frac{1}{2}\int_C \frac{e^{iz}dz}{(z^2+1)^2} =$ $\quad z^2+1=0 \Rightarrow z=\pm i$

only $+i$ is inside C.

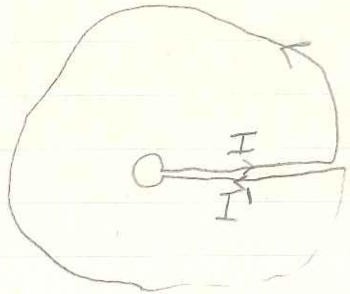
$$= \frac{1}{2}\,2\pi i\,\mathrm{Res}_{z=i} \frac{e^{iz}}{(z-i)^2(z+i)^2} = \pi i\cdot\left(\frac{e^{iz}}{(z+i)^2}\right)'\Big|_{z=i}$$

$$= \pi i\left(ie^{iz}(z+i)^{-2} - 2e^{iz}(z+i)^{-3}\right)_{z=i} =$$

$$= \pi i e^{-1}\left(-\frac{i}{4} - 2\cdot\frac{i}{8}\right) = \frac{\pi}{2e}$$

c) Call $I = \int_0^{\infty} \frac{x^{\lambda-1}}{x+4}dx$ and

$C =$



and now

$$x^{\lambda-1} = e^{(\lambda-1)\log x} = e^{(\lambda-1)(\text{Log}\, x + 2\pi n i)}$$

and so

$$I' = e^{(\lambda-1)(2\pi i)} I = e^{2\pi i \lambda} I$$

or

$$(1 - e^{2\pi i \lambda}) I = 2\pi i \,\text{Res}_{z=-y} \frac{z^{\lambda-1}}{z+y} = 2\pi i (-y)^{\lambda-1} =$$

$$= 2\pi i\, e^{(\log -y)(\lambda-1)} = 2\pi i\, e^{(\pi i + \log y)(\lambda-1)} =$$

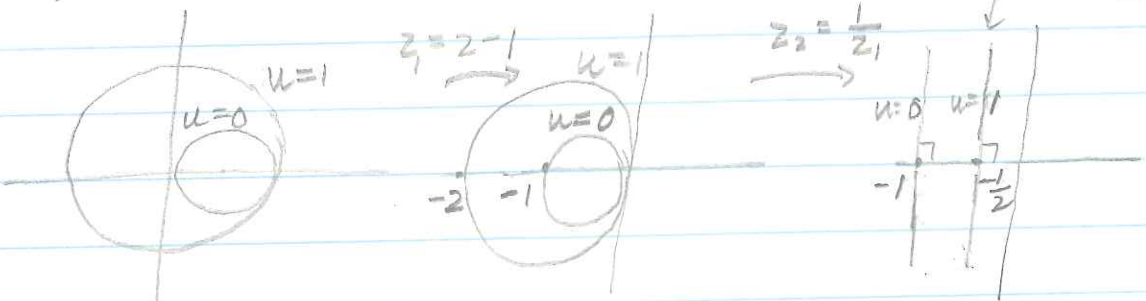
$$= 2\pi i\, y^{\lambda-1} \cdot e^{\pi i(\lambda-1)} = -2\pi i\, e^{\pi i \lambda}$$

or

$$I = \frac{-2\pi i e^{\pi i \lambda}}{1 - e^{2\pi i \lambda}} = \pi \frac{2i}{e^{\pi i \lambda} - e^{-\pi i \lambda}} \cdot y^{\lambda-1} = \frac{\pi \cdot y^{\lambda-1}}{\sin(\pi\lambda)}$$

④

conformal ⇒ ⊥ to real line



These go to parallel lines, since there is only 1 intersection point and it will be at ∞.

$$\Rightarrow T(z_2) = c_1 \operatorname{Re} z_2 + c_2 \qquad 1 = -\frac{c_1}{2} + c_2 \qquad c_2 = 2$$

$$= 2 \operatorname{Re} z_2 + 2 \qquad 0 = -c_1 + c_2 \qquad c_1 = 2$$

$$z_2 = \frac{1}{z-1} = \frac{1}{x+yi-1} = \frac{1}{(x-1)+yi} = \frac{(x-1)-yi}{(x-1)^2+y^2}$$

$$\Rightarrow T(x,y) = \frac{2(x-1)}{(x-1)^2+y^2} + 2$$

check: on unit disk $x^2+y^2=1$

$$\frac{2(x-1)}{1-2x+1} + 2 = 1 \checkmark$$

At origin $= -2+2 = 0$

Question #5:

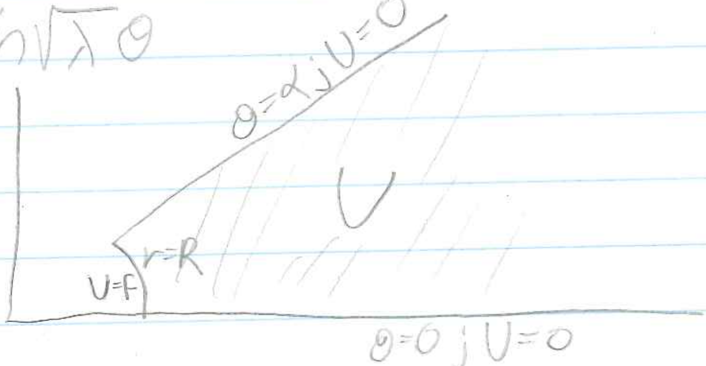
Seperation of variables gives $U = R(r)\Theta(\theta)$

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda$$

(It is easy to eliminate the possibility $\lambda \leq 0$ using the boundary conditions)

$$\Rightarrow \Theta(\theta) = A\cos\sqrt{\lambda}\,\theta + B\sin\sqrt{\lambda}\,\theta$$

The boundary conditions are:



$U(r,0)=0 \Rightarrow A=0$

$U(r,\alpha)=0 \Rightarrow \sin\sqrt{\lambda}\alpha=0 \Rightarrow \sqrt{\lambda}\alpha=\pi n \Rightarrow$

$\Rightarrow \lambda_n=\frac{\pi^2n^2}{\alpha^2}$

The R equation

$$r^2R''+rR'=\frac{\pi^2n^2}{\alpha^2}R$$

try $R=r^\beta$ and get

$$\beta(\beta-1)r^\beta+\beta r^\beta=\frac{\pi^2n^2}{\alpha^2}r^\beta$$

$\Rightarrow \quad \beta=\pm\frac{\pi n}{\alpha}$

but $\lim_{r\to\infty}R(r)=0$, and therefore $\beta=-\frac{\pi n}{\alpha}$

and $R_n=r^{-\frac{\pi n}{\alpha}}$

$$U=\sum_{n=1}^{\infty}b_nU_n=\sum_{n=1}^{\infty}b_nr^{-\frac{\pi n}{\alpha}}\sin\left(\frac{\pi n\theta}{\alpha}\right)$$

$$F(\theta)=U(R,\theta)=\sum_{n=1}^{\infty}b_nR^{-\frac{\pi n}{\alpha}}\sin\left(\frac{\pi n\theta}{\alpha}\right)$$

on the other hand, F can be extented to become an odd function on $-\alpha\leq\theta\leq\alpha$, and by Fourier theory on that interval,

$$F(\theta)=\sum_{n=1}^{\infty}\tilde{b}_n\sin\frac{\pi n\theta}{\alpha} \qquad \tilde{b}_n=\frac{1}{\alpha}\int_{-\alpha}^{\alpha}F(\theta)\sin\frac{\pi n\theta}{\alpha}d\theta$$

$$\Rightarrow b_n=R^{\frac{\pi n}{\alpha}}\tilde{b}_n=\frac{2}{\alpha}R^{\frac{\pi n}{\alpha}}\int_0^{\alpha}F(\theta)\sin\frac{\pi n\theta}{\alpha}d\theta$$

Finally, if $f(\theta)=\sin\frac{2\pi\theta}{\alpha}$ then $\tilde{b}_2=1$ and $\tilde{b}_{\neq 2}=0$ and thus

$$U(r,\theta)=R^{\frac{2\pi}{\alpha}}r^{-\frac{2\pi}{\alpha}}\sin\frac{2\pi\theta}{\alpha}$$

Question 6:

a) $$c_n=\frac{1}{\pi}\int_{-\pi}^{\pi}x\sin nx=\frac{1}{\pi}\left(x\cdot\left(-\frac{\cos nx}{n}\right)\right)\Big|_{-\pi}^{\pi}-\underbrace{\frac{1}{\pi}\int_{-\pi}^{\pi}\frac{-\cos nx}{n}dx}_{0}$$

integral of cos ovr a bunch of periods $=0$

$$=\frac{1}{n\pi}\left(-\pi\cos n\pi-\pi\cos n(-\pi)\right)=$$

$$=-\frac{2}{n}\cos(n\pi)=-(-1)^n\frac{2}{n}$$

and so for $-\pi<x<\pi$

$$x=\sum_{n=1}^{\infty}-(-1)^n\frac{2}{n}\sin nx$$

b) Want an x for which all <u>even</u> terms in the above equation will drop $\Rightarrow$ take $x=\frac{\pi}{2}$:

$$\frac{\pi}{2}=\sum_{n=1}^{\infty}-(-1)^n\frac{2}{n}\sin\frac{n\pi}{2}=\sum_{\text{odd } n}-(-1)^n\frac{2}{n}\sin\frac{n\pi}{2}$$

$n=2k+1$: $$\frac{\pi}{2}=\sum_{k=0}^{\infty}+\frac{2}{2k+1}(-1)^k$$

and this is what we wanted to prove.

Question 7:

Remember that $\widetilde{f'} = ip\tilde{f}$

$$\Rightarrow \widetilde{xf} = i\frac{d}{dp}\tilde{f}$$

$$\Rightarrow \widetilde{x\cdot xf} = i\frac{d}{dp}\widetilde{xf} = i\frac{d}{dp}i\frac{d}{dp}\tilde{f} = -\frac{d^2}{dp^2}\tilde{f}$$

in our case, $f = e^{-x^2/2}$ and $\tilde{f} = e^{-p^2/2}$

and so

$$\left(x^2e^{-\frac{x^2}{2}}\right)^{\sim} = -\left(e^{-p^2/2}\right)'' = \left(pe^{-p^2/2}\right)' = e^{-p^2/2} - p^2e^{-p^2/2}$$

The same result can be obtained by an explicit computation involving a few integrations by parts.

Question 8:

a) F indep. of t $\Rightarrow$

$$E.L \Leftrightarrow F - \dot{q}F_{\dot{q}} = \tilde{C}_1$$

$$\frac{1+q^2}{\dot{q}^2} + 2\dot{q}\,\frac{1+q^2}{\dot{q}^3} = \tilde{C}_1$$

$$\Rightarrow \quad 3\frac{1+q^2}{\dot{q}^2} = \tilde{C}_1 \quad \Rightarrow \dot{q} = \sqrt{3\frac{1+q^2}{\tilde{C}_1}} = C_1\sqrt{1+q^2}$$

$$\Rightarrow \frac{dq}{dt} = C_1\sqrt{1+q^2} \Rightarrow \frac{dq}{\sqrt{1+q^2}} = C_1\,dt \Rightarrow \sinh^{-1}q = C_1t + C_2$$

$$\Rightarrow q = \sinh(C_1t + C_2)$$

$q(0) = 0 \Rightarrow C_2 = 0$

$q(1) = \frac{1}{2}(e - e^{-1}) \Rightarrow C_1 = 1$

$\Rightarrow q = \sinh t$

b) $p = F_{\dot q} = -2\frac{1+q^2}{\dot q^3}$ $\left(\Rightarrow \dot q = \sqrt[3]{-2\frac{1+q^2}{p}}\right)$

c) $H = p\dot q - F = -3\frac{1+q^2}{\dot q^2} = -3\frac{1+q^2}{\left(-2\frac{1+q^2}{p}\right)^{2/3}} =$

$= -3\cdot 2^{-\frac{2}{3}}(1+q^2)^{1/3}p^{2/3}$

d) $\frac{\partial H}{\partial p} = -3 2^{-2/3}(1+q^2)^{1/3}\cdot\frac{2}{3}\cdot p^{-1/3} = \sqrt[3]{-2\frac{1+q^2}{p}} = \dot q$ as required.

The other equation is more complicated:

$p = -2\frac{1+q^2}{\dot q^3} = -2\frac{1+\sinh^2 t}{(\cosh t)^3} = -2\frac{\cosh^2 t}{\cosh^3 t} = -\frac{2}{\cosh t}$

$\dot p = 2\frac{\sinh t}{\cosh^2 t}$

$\frac{\partial H}{\partial q} = -3\cdot 2^{-2/3}\cdot 2q\cdot\frac{1}{3}(1+q^2)^{-\frac{2}{3}}p^{2/3} = -2^{1/3}\cdot q\cdot(1+q^2)^{-2/3}\cdot p^{2/3} =$

$= -2^{1/3}\sinh t\,(1+\sinh^2 t)^{-2/3}\left(-\frac{2}{\cosh t}\right)^{2/3} =$

$= -2\sinh t\cdot(\cosh^2 t)^{-2/3}\left(\frac{1}{\cosh t}\right)^{2/3} = -2\frac{\sinh t}{\cosh^2 t} = -\dot p$

as required.