	1	2	3	4	5	6	7
Solved	3	2	2				1
Want it solved	X			2	2	2	1
				3	2	1	4

(1) (Morning 6) Show that

$$f(x) = x + \frac{2}{3}x^3 + \frac{2}{3}\frac{4}{5}x^5 + \frac{2}{3}\frac{4}{5}\frac{6}{7}x^7 + \cdots = \frac{\arcsin x}{\sqrt{1-x^2}}$$

Note (not on Putnam exam): $\arcsin x$ is the same as $\sin^{-1} x$.

$$f' = 1 + 2x^2 + \frac{2}{3}4x^4 + \frac{2}{3}\frac{4}{5}6x^6 + \cdots$$

$$= 1 + x(xf)' = 1 + xf + x^2 f'$$

(2) (Afternoon 10) Given the power series

$$a_0 + a_1 x + a_2 x^2 + \dots$$

in which

$$a_n = (n^2+1)3^n ,$$

$$(n^{54} + 3n^{42} + \dots)101^n$$

show that there is a relationship of the form

$$a_n + p a_{n+1} + q a_{n+2} + r a_{n+3} = 0 ,$$

in which p, q, r are constants independent of n. Find these constants and the sum of the power series.

$$(x-101)^{55}$$

$$n^k x^n \qquad (Sf)(n) = x f(n-1)$$

$$(I-S)(n^k x^n) = n^k x^n - x(n-1)^k x^{n-1}$$
$$= (n^k - (n-1)^k) x^n$$

$$(I-S)^{k+1}(n^k x^n) = 0$$

$$\sum_{j=0}^{k+1} (-1)^j \binom{k+1}{j} (n-j)^k x^n = 0$$

$$0 = (1-S)^3 a_n = a_n - 3 \cdot 3 a_{n-1} + 3 \cdot 9 a_{n-2} + 27 a_{n-3}$$

$$Ef = x \frac{\partial}{\partial x} f$$

$$\sum n^k x^n$$

$$= E^k \sum x^n = E^k \frac{1}{1-x}$$

(3) (A4) Let k be a positive integer and let $m = 6k - 1$. Let

$$S(m) = \sum_{j=1}^{2k-1} (-1)^{j+1} \binom{m}{3j-1}.$$

For example with $k = 3$,

$$S(17) = \binom{17}{2} - \binom{17}{5} + \binom{17}{8} - \binom{17}{11} + \binom{17}{14}.$$

Prove that $S(m)$ is never zero.

(4) (B4) Let $a_{m,n}$ denote denote the coefficient of x^n in the expansion of $(1+x+x^2)^m$. Prove that for all $k \geq 0$,

$$0 \leq \underbrace{\sum_{i=0}^{[2k/3]} (-1)^i a_{k-i,i}}_{b_k} \leq 1 .$$

$$\sum y^m (1+x+x^2)^m = \sum a_{mn} y^m x^n$$

$$= \frac{1}{1-y(1+x+x^2)} \qquad = \sum a_{k-i,i}\, y^{k-i} x^i = \sum a_{k-i,i}\, y^k \left(\frac{x}{y}\right)^i$$

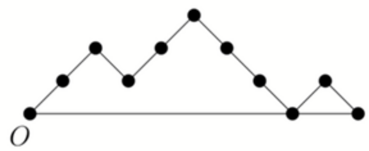
$$\Big\downarrow x=-y$$

$$\frac{1}{1-y+y^2-y^3} = \sum_k y^k \left(\sum_i (-1)^i a_{k-i,i}\right)$$

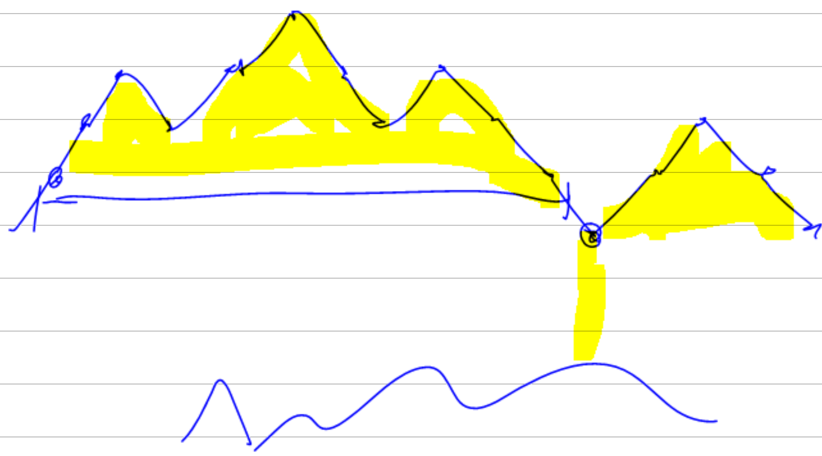
$$= \frac{1}{(1-y)(1+y^2)} = \frac{A}{1-y} + \frac{B+Cy}{1+y^2}$$

$$= A(1+y+y^2+y^3+\ldots) + B(1-y^2+y^4-y^6) + C(y-y^3+y^5 \ldots)$$

(5) (A5) A Dyck n-path is a lattice path of n upsteps $(1,1)$ and n downsteps $(1,-1)$ that starts at the origin O and never dips below the x-axis. A return is a maximal sequence of contiguous downsteps that terminates on the x-axis. For example, the Dyck 5-path illustrated has two returns, of length 3 and 1 respectively.



Show that there is a one-to-one correspondence between the Dyck n-paths with no return of even length and the Dyck $(n-1)$-paths.



(6) (A6) For a set S of nonnegative integers, let $r_S(n)$ denote the number of ordered pairs (s_1, s_2) such that $s_1 \in S$, $s_2 \in S$, $s_1 \neq s_2$, and $s_1 + s_2 = n$. Is it possible to partition the nonnegative integers into two sets A and B in such a way that $r_A(n) = r_B(n)$ for all n ?

0 1 2 3 4 5 6 7 8 9 10

A 0 3 5 6 9 } numbers whose binary has even 1's

B 1 2 4 7 8 } . . . odd 1's.

$$A: 9 = 9+0 = 6+3$$

101 000 110 011

100 001 111 010

$$B: 9 = 8+1 = 7+2$$

$$f(x) = \sum_{k\in A} x^k \qquad g(x) = \sum_{k \in B} x^k$$

$$f(x) + g(x) = \frac{1}{1-x}$$

$$\sum r_A(n) x^n = \frac{1}{2} f^2(x) - \frac{1}{2} f(x^2)$$

$$f^2(x) - f(x^2) = g^2(x) - g(x^2)$$

$$f^2(x) - g^2(x) = f(x^2) - g(x^2)$$

$$\frac{f(x) - g(x)}{1-x} = f(x^2) - g(x^2)$$

$$f(x) - g(x) = (1-x)\left(f(x^2) - g(x^2)\right) =$$

$$= (1-x)(1-x^2)\left(f(x^4)-g(x^4)\right)$$

$$= (1-x)(1-x^2)(1-x^4)(1-x^8)(1-x^{16})\cdots\left(f(x^{2^n})-g(\;)\right) \to 1$$

$$= \prod_{k=0}^{\infty}\left(1-x^{2^k}\right) = \sum c_n x^n$$

$$c_n = \begin{cases} +1 & n \text{ even 1's} \\ -1 & n \text{ odd 1's.} \end{cases}$$

$$f(x)-g(x)$$

(7) (B6) For a set with n elements, how many subsets are there whose cardinality (the number of elements in the subset) is respectively $\equiv 0 \ (mod\ 3)$, $\equiv 1 \ (mod\ 3)$, $\equiv 2 \ (mod\ 3)$? In other words, calculate

$$s_{i,n} = \sum_{k \equiv i \ (mod\ 3)} \binom{n}{k}$$

for $i = 0, 1, 2$.

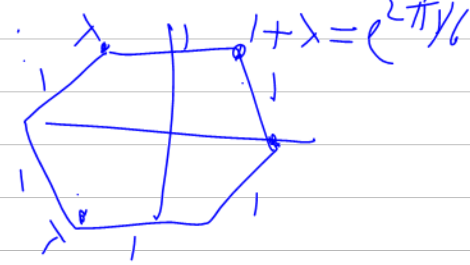
Your result should be strong enough to permit direct evaluation of the numbers $s_{i,n}$ and to show clearly the relationship of $s_{0,n}$ and $s_{1,n}$ and $s_{2,n}$ to each other for all positive integers n. In particular, show the relationships among these 3 sums for $n = 1000$. [An illustration of the definition of $s_{i,n}$ is $s_{0,6} = \binom{6}{0} + \binom{6}{3} + \binom{6}{6} = 22$.]

$\lambda^3 = 1 \quad \lambda = e^{2\pi i/3}$

$S_{0,n} = \frac{1}{3}\left(2^n + (1+\lambda)^n + (1+\bar{\lambda})^n\right)$

$S_{1,n} = \frac{1}{3}\left(2^n + \lambda(1+\lambda)^n + \bar{\lambda}(1+\bar{\lambda})^n\right)$

$1+\lambda = e^{2\pi i/6}$



$$\binom{n}{0} + \binom{n}{5}x^5 + \binom{n}{10}x^{10} + \binom{n}{15}x^{15} \ldots$$

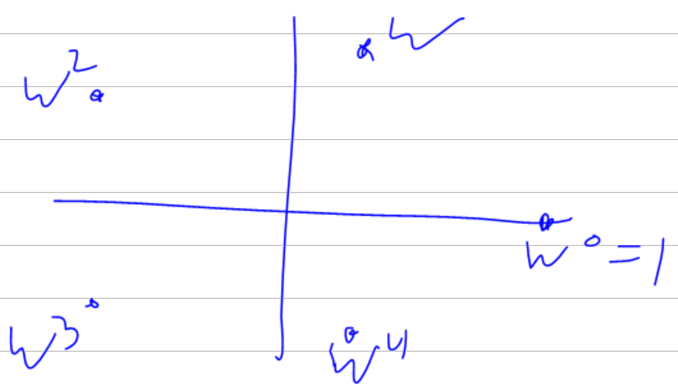
$$= \frac{1}{5}\left((1+x)^n + (1+wx)^n + (1+w^2x)^n + (1+w^3x)^n + (1+w^4x)^n\right)$$

$$w = e^{\frac{2\pi i}{5}}$$

Coeff of x^k:

$$\frac{1}{5}\binom{n}{k}\left(1 + w^k + w^{2k} + w^{3k} + w^{4k}\right)$$

$$= \frac{1}{5}\binom{n}{k}\frac{1-w^{5k}}{1-w^k}$$



$$= \begin{cases} \binom{n}{k} & 5 \mid k \\ 0 & \text{otherwise} \end{cases}$$

$$S_{0,n} = S_{0,n-1} + S_{2,n-1} \qquad \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

$$S_{k,n} = S_{k,n-1} + S_{k-1,n-1} \qquad k \in \mathbb{Z}/3\mathbb{Z}$$

n	0	1
S_{0n}	1	
S_{1n}	0	
S_{2n}	0	

$$S_{k,n} = \frac{1}{3}2^n + P_{k,n}$$

$$P_{k,n} = P_{k,n-1} + P_{k-1,n-1}$$

n	0	1	2	3	4	5	6
P_{0n}	$2/3$	$1/3$	$-1/3$	$-2/3$	$-1/3$	$1/3$	$2/3$
$P_{1,n}$	$-1/3$	$1/3$	$2/3$	$1/3$	$-1/3$	$-2/3$	$-1/3$
$P_{2,n}$	$-1/3$	$-2/3$	$-1/3$	$1/3$	$2/3$	$1/3$	$-1/3$

$$\begin{pmatrix} S_{0n} \\ S_{1n} \\ S_{2n} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} S_{0,n-1} \\ S_{1,n-1} \\ S_{2,n-2} \end{pmatrix}$$

S_{0n}	1	1	1	2	5	11	22	$\frac{1}{3}2^n +$
S_{1n}	0	1	2	3	5	10	21	
S_{2n}	0	0	1	3	6	11	21	
	1	2	4	8	16	32	64	