MAT257.

Online til Mar 3rd

HW13. Q1. Identify $\Lambda^1(\mathbb{R}^3)$, $\Lambda^2(\mathbb{R}^3)$ w/ $\mathbb{R}^3$.

s.t. $\wedge : \Lambda^1(\mathbb{R}^3) \times \Lambda^1(\mathbb{R}^3) \to \Lambda^2(\mathbb{R}^3)$ is $P : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ and $P$ is the cross product.

$$\begin{array}{ccc} \Lambda^1(\mathbb{R}^3) \times \Lambda^1(\mathbb{R}^3) & \xrightarrow{\wedge} & \Lambda^2(\mathbb{R}^3) \\ \downarrow \quad\quad \downarrow & & \downarrow \\ \mathbb{R}^3 \times \mathbb{R}^3 & \xrightarrow{P} & \mathbb{R}^3 \end{array}$$

$\Lambda^1(\mathbb{R}^3) \simeq T^1(\mathbb{R}^3)$.

basis $\{\varphi_1, \varphi_2, \varphi_3\}$.

$\sum a_i \varphi_i \mapsto (a_1, a_2, a_3)$.

$(a_1\varphi_1 + a_2\varphi_2 + a_3\varphi_3) \wedge (b_1\varphi_1 + b_2\varphi_2 + b_3\varphi_3)$

$= \underline{a_1\varphi_1 \wedge b_1\varphi_1} + a_1\varphi_1 \wedge b_2\varphi_2 + a_1\varphi_1 \wedge b_3\varphi_3 + \ldots$

$= a_1 b_2 \varphi_1 \wedge \varphi_2 + \underline{a_2 b_1 \varphi_2 \wedge \varphi_1} + \ldots$

$= a_1 b_2 \varphi_1 \wedge \varphi_2 - a_2 b_1 \varphi_1 \wedge \varphi_2 + \ldots$

$= \underline{(a_1 b_2 - a_2 b_1)} \varphi_1 \wedge \varphi_2 + \ldots$

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Q2. $L : V \to W$ invertible L.T. between oriented v.s.

$L$ is orientation preserving if $L$ pushes orientation of $V \to$ orientation of $W$.

orientation reversing o/w

orientation of $W$ / orientation of $V$.

$L_*(v_1, \ldots, v_n) = (Lv_1, \ldots, Lv_n)$

vs.

$(w_1, \ldots, w_n)$. orientation of $W$

$(Lv_1, \ldots, Lv_n) \sim (w_1, \ldots, w_n) \iff L$ is orientation preserving".

iff $\det(Lv_1 | \ldots | Lv_n) > 0$.

$\det L > 0$.

$L : (u, v) \mapsto (v, u)$ $u \in \mathbb{R}^m$, $v \in \mathbb{R}^n$.

$L_*(e_1, \ldots, e_m, e_{m+1}, \ldots, e_{m+n})$

$= (e_{m+1}, \ldots, e_{m+n}, e_1, \ldots, e_m)$

matrix repr of $L = \begin{pmatrix} e_{m+1} \\ \vdots \\ e_{m+n} \\ e_1 \\ \vdots \\ e_n \end{pmatrix}$

$\det L = (-1)^{\#\text{ row swaps to get to the matrix}}$

$= (-1)^{\#\text{ swaps to get from } (1, \ldots, m+n) \text{ to } (m+1, \ldots, m+n, 1, \ldots, m)}$

$= (-1)^?$

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Q3. $V$ n-dim v.s. isom: $\chi : \Lambda^n(V) \to \mathbb{R}$.

$0 \le k \le n$. Construct $\Psi_k : \Lambda^{n-k}(V) \to (\Lambda^k(V))^*$.

$\binom{n}{n-k} = \binom{n}{k}$

Want: a map $\Psi_k : \omega \mapsto \varphi_\omega$

s.t. $\varphi_\omega : \Lambda^k(V) \to \mathbb{R}$.

Consider the map $\wedge_\omega(\alpha) = \omega \wedge \alpha$

$\omega \in \Lambda^{n-k}(V)$. $\alpha \in \Lambda^k(V)$. then

$\wedge_\omega : \Lambda^k(V) \to \Lambda^n(V) \xrightarrow{\chi} \mathbb{R}$.

Q4. $V$ n-dim vs. w/ basis $\{v_i\}$ & dual basis $\{\varphi_j\}$.

$0 \le k \le n$. define inner product on $\Lambda^k(V)$.

$\langle \omega_I, \omega_J \rangle = \delta_{IJ}$.

write $\omega_n = \varphi_1 \wedge \ldots \wedge \varphi_n$.

(a). unique isomorphism $* : \Lambda^k(V) \to \Lambda^{n-k}(V)$.

s.t. $\lambda \wedge *\eta = \langle \lambda, \eta \rangle \omega_n$

for any $\lambda, \eta \in \Lambda^k(V)$.

$\omega_I \wedge *\omega_J = \langle \omega_I, \omega_J \rangle \omega_n$

$= \delta_{IJ}\, \omega_n$

in particular, $\omega_I \wedge *\omega_I = \omega_n$

Let $*\omega_I = \omega_J$. $J \in \underline{n}^{n-k}$.

$I^c = J$.

then

$\omega_I \wedge \omega_J = \omega_n$, so $\operatorname{sgn} \sigma_{IJ} = 1$

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$\lambda = \omega_I$. $\eta = \sum_{J \in \underline{n}^k} a_J \omega_J$. $*\eta = \sum_{K \in \underline{n}^{n-k}} b_K \omega_K$

$\lambda \wedge *\eta = \omega_I \wedge \sum b_K \omega_K = \sum b_K (\omega_I \wedge \omega_K)$

$= \langle \lambda, \eta \rangle = \langle \omega_I, \sum a_J \omega_J \rangle \omega_n$

$= \sum a_J \delta_{IJ} \omega_n$.

$\sum \underline{a_J \delta_{IJ}} \omega_n = \sum b_K \omega_I \wedge \omega_K$

$= \sum \underline{b_K (-1)^{\sigma_{IK}}} \omega_n$.

$* : \omega_I \to \omega_J$ w/ $I = J^c$.

s.t. this holds

$* \circ * : \Lambda^k \to \Lambda^{n-k} \to \Lambda^k$ $\omega \mapsto (-1)^{k(n-k)} \omega$

$\omega_I \mapsto (-1)^{\sigma_I} \omega_{I^c} \mapsto (-1)^{\sigma_I \sigma_J} \omega_I$.

$\mapsto \omega_J \mapsto$ some $(-1)^k \omega_I$

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write $\omega_n = \varphi_1 \wedge \ldots \wedge \varphi_n$.

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