MAT257.

Admin: Tutorials are online for 10^{th}, 17^{th} (24^{th} – Reading week. Returning to SS2110 on Mar 3^{rd}).

Alternating tensors.

$\omega \in \Lambda^k(V)$: $\omega \circ \sigma^*(v_1, \dots, v_k) = \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)})$
$= (-1)^\sigma \omega$

$T \in T^k(V)$, $Alt(T) = \frac{1}{k!} \sum_{\sigma \in S_k} (-1)^\sigma T \circ \sigma$.

Basis for $\Lambda^k(V)$: $\{\varphi_{i_1} \wedge \cdots \wedge \varphi_{i_k}, 1 \le i_1 < \cdots < i_k \le n\}$

$\dim \Lambda^k(V) = \binom{n}{k}$.

$\dim V = n$.

Q2. $\det \in \Lambda^n(V)$.

$\dim \Lambda^n(V) = \binom{n}{n} = 1$.

$\Lambda^n(V) = \{c\varphi_1 \wedge \cdots \wedge \varphi_n\}$.

$\det(e_1, \dots, e_n) = c(\varphi_1 \otimes \cdots \otimes \varphi_n)(e_1, \dots, e_n)$
$= 1$.

$\Rightarrow c = 1$.

$v_i = \sum a_{ij} e_j$.

$\det(v_1, \dots, v_n) = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$
$= \sum_{\sigma \in S_n} \varphi_1(v_{\sigma(1)}) \cdots \varphi_n(v_{\sigma(n)})$
$= \sum_{\sigma \in S_n} (\varphi_1 \cdots \varphi_n) \circ \sigma(v_1, \dots, v_n)$.
$= \sum_{\sigma \in S_n} (\varphi_1 \cdots \varphi_n)$
$= \dots$

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4-2. $f: V \to V$ linear transformation.
$\dim V = n$.

$f^*: \Lambda^n(V) \to \Lambda^n(V)$
is multiplication by $\det f$.

$f^*(\varphi_1 \wedge \cdots \wedge \varphi_n)(e_1, \dots, e_n)$
$= (\varphi_1 \wedge \cdots \wedge \varphi_n)(f(e_1) \cdots f(e_n))$.
$= \det(f)(\varphi_1 \wedge \cdots \wedge \varphi_n)(e_1, \dots, e_n)$
Thm 4.X.

Q3. $L: \mathbb{R}^m \to \mathbb{R}^n$ linear transformation.
w/ matrix repn A relative to std basis

ω_I: elementary alt. k-tensor on $\mathbb{R}^n$

Then $L^*\omega_I = \sum_{J \in \underline{m}} c_J \omega_J$.

Consider $L^*\omega_I(e_{j_1}, \dots, e_{j_k})$.

$A = (a_{ij})$. $= \omega_I(Le_{j_1}, \dots, Le_{j_k})$.
$= \omega_I(Ae_{j_1}, \dots, Ae_{j_k})$.
$= \omega_I(A_{*j_1}, \dots, A_{*j_k})$.
$= \dots$

$J = (j_1, \dots, j_k)$.
$L^*\omega_I(e_{j_1}, \dots, e_{j_k}) = c_J \omega_J$.

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Q4. Space of symmetric tensors. $S^k(V)$
$\omega \in S^k(V)$ iff $\omega \circ \sigma^*(v_1, \dots, v_k) = \omega(v_1, \dots, v_k)$

$T \in T^k(V)$. $Sym(T) = \sum_{\sigma \in S_k} T \circ \sigma^*$

What is $S^k(T)$?

Basis of $S^k(T) = \{Sym(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$.
$\varphi_{i_1}, \dots, \varphi_{i_k}$ basis elts in $T^1(V)\}$.

$Alt(\varphi_1 \otimes \varphi_1) = \sum_{\sigma \in S_k} (\varphi_1 \otimes \varphi_1) \circ \sigma^* \, sgn(\sigma)$.
$= \sum_{\sigma \in S_k} \varphi_{\sigma(1)} \otimes \varphi_{\sigma(1)} \, sgn(\sigma)$
$= 0$.

Basis $= \{Sym(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$
$1 \le i_1 \le \cdots \le i_k \le n\}$.

What is the size of this?

$\#$ Basis $= \#\{(i_1, \dots, i_k) : 1 \le i_1 \le \cdots \le i_k \le n\}$.

Consider arranging $\{1, \dots, n\}$ and $\{i_1, \dots, i_k\}$ in a nondecreasing sequence st. i_j is before any number $\le i_j$. and i_j's are in order of j.

(eg. $i_1 = 1$, then we put $i_1 \le 1 \le \dots$)

↳ this gives a unique ordering of $\{1, \dots, n\}$, $\{i_1, \dots, i_k\}$

o.t.o.h. any way to intersperse $i_1, \dots, i_k$ between $1 \le \cdots \le n$ (nothing after n)

Gives a unique $(i_1, \dots, i_k)$.

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Consider a sequence of $n+k$ things.
Any such sequence w/ $1, \dots, n$ among these "things" and n as the last correspond to a seq →

$\{\underset{i_1}{x}, \underset{i_2}{x}, 1, \underset{i_3}{x}, 2, \underset{i_4}{x}, \underset{i_5}{x}, 3, \dots, \underset{i_k}{x}, n\}$.

– The position of n is fixed.

– Among the remaining $n+k-1$ elts we're choosing $n-1$ of them to be numbers.

$\#$ ways $= \binom{n+k-1}{n-1}$.

Basis $= \{Sym(\varphi_{i_1} \otimes \cdots \otimes \varphi_{i_k})$
$1 \le i_1 \le \cdots \le i_k \le n\}$.

What is the size of this?

$\#$ Basis $= \#\{(i_1, \dots, i_k) : 1 \le i_1 \le \cdots \le i_k \le n\}$.

Consider arranging $\{1, \dots, n\}$ and $\{i_1, \dots, i_k\}$ in a nondecreasing sequence st. i_j is before any number $\le i_j$. and i_j's are in order of j.

(eg. $i_1 = 1$, then we put $i_1 \le 1 \le \dots$)

↳ this gives a unique ordering of $\{1, \dots, n\}$, $\{i_1, \dots, i_k\}$

o.t.o.h. any way to intersperse $i_1, \dots, i_k$ between $1 \le \cdots \le n$ (nothing after n)

Gives a unique $(i_1, \dots, i_k)$.