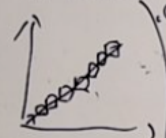
MAT257.

TT2. Q1 $S=\{(t,t)\in\mathbb{R}^n : t\in\mathbb{Q}\cap[0,1]\}$ is content zero.



Or: Countable, cover by $\{B(x_i, \frac{\sqrt{\varepsilon}}{2^{2i}\pi})\}$.

↳ total volume $=\sum\left(\frac{\sqrt{\varepsilon}}{2^{2i}\pi}\right)^2\pi=\sum_{i=1}^{\infty}\frac{\varepsilon}{2^i}=\varepsilon$.

$S\subset\{(t,t)\in\mathbb{R}^n : t\in[0,1]\}$

↳ this is compact.

$\{B(x,\frac{\sqrt{\varepsilon}}{2^{2i}\pi})\}$ covers the diagonal ($\bar{S}$ = diagonal).

$\Rightarrow$ there is a finite subcover of the diagonal and therefore S.

Q5. A Jordan msrable.

$h(x,y)=(y,x+y)$.

Then $h(A)$ is Jordan msrable and $\text{vol}(h(A))=\text{vol}(A)$. $\quad Dh=\begin{pmatrix}0&1\\1&1\end{pmatrix}$

Proof. CoV using h.

$\text{vol}(A)=\int_{\mathbb{R}^n}\chi_A=\int_A 1=\int_{h(A)}(1\circ h^{-1})|\det Dh^{-1}|$, $\quad |\det Dh^{-1}|=\frac{1}{|\det Dh|}$

integrable

$=\int_{h(A)}1$

$=\int_{\mathbb{R}^n}\chi_{h(A)}$

↳ this is integrable $\Leftrightarrow$ $h(A)$ Jordan msrable.

$\Rightarrow \text{vol}(A)=\text{vol}(h(A))$.

Baby Sard. $\Rightarrow$ CoV doesn't need Dg invertible everywhere.

$g:\mathbb{R}^n\to\mathbb{R}^n$.

$\{g(x): x \text{ is a critical pt of } g\}$ is msr 0.

$\int_{g(A)} f$ differs from $\int_A (f\circ g)|\det Dg|$ at a set of msr 0.

$\Rightarrow \int_{g(A)} f=\int_A (f\circ g)|\det Dg|$.

Adult Sard. $g:\mathbb{R}^n\to\mathbb{R}^m$.

g diff'bl sufficiently many times. ($\max(n-m+1,1)$).

then $\{g(x): x \text{ critical pt of } g\}$ is msr 0.

Fixed pts.

Prop. f is a smooth map $D^n\to D^n$ ← unit disk.

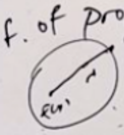
then f has a fixed pt somewhere on D^n.

↳ Brouwer's fixed pt thm. ↑ for continuous f.

Lemma. There is no smooth map $f: D^n\to S^n$ st. $f|_{S^n}=\text{id}$.

Proof. (Appl of Sard for manifolds).

Proof of Prop. Suppose f has no fixed pt. ($f(x)\neq x$ for any x)



Consider the map $g(x)$ = the point of intersection of the ray from $f(x)$ through x w/ the unit circle S^n. $g(x)=x$ when $x\in S^n$. Lemma $\Rightarrow$ g DNE. $\Rightarrow\Leftarrow$

V is an n-dim vs.

$V^*=\{\varphi: V\to\mathbb{R},\ \varphi \text{ linear functional}\}$ dual of V. also a vs of dim n.

What's $(V^*)^*$? Should be a vs of dim n.

Given $\{v_1,\dots,v_n\}$ a basis for V. can get a dual basis $\{\varphi_1,\dots,\varphi_n\}$ for V^*.

Consider the map:

$f: V\to(V^*)^*$.

$f(v): \varphi\mapsto\varphi(v)$.

Claim. $\{f(v_1),\dots,f(v_n)\}$ is the dual basis of $\{\varphi_1,\dots,\varphi_n\}$.

Check: $f(v_i)(\varphi_j)=\varphi_j(v_i)=\delta_{ij}$.

Claim 2: f is an isomorphism $V\to(V^*)^*$.

- f maps basis vectors to basis vectors
- $f(au+bv)(\varphi)$
  $=\varphi(au+bv)=a\varphi(u)+b\varphi(v)$
  $=af(u)(\varphi)+bf(v)(\varphi)$
  $=(af(u)+bf(v))(\varphi)$.

$\Rightarrow$ f is linear

$\Rightarrow$ f is bijective

$\Rightarrow$ f is an isomorphism $V\to(V^*)^*$.