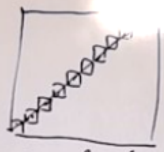
MAT237

T.T2.

Q1. $S=\{(t,t) : t\in\mathbb{Q}\cap[0,1]\}\subset\mathbb{R}^2$
is content zero
(Show from defn).



Def^n of content zero:
For any $\varepsilon>0$, can find a finite cover of S
of rectangles st. total volume $<\varepsilon$.
Let $N \ge 1/\varepsilon$.
Consider the cover $\{[0,\frac{1}{N}]^2, [\frac{1}{N},\frac{2}{N}]^2, \dots, [\frac{N-1}{N},1]^2\}$. this covers S.
total volume $= N\cdot\frac{1}{N^2} = \frac{1}{N} < \varepsilon$.

Q2: Every open set in $\mathbb{R}^n$ is a union of countably many compact sets.

Let U be an open set:

$C_k := \overline{B(0,k)} \cap \{x\in U : d(x,U^c)\ge \frac{1}{k}\}$.

$\bigcup C_k = \{x\in U : d(x,U^c) > 0\} = U$

MAT237

T.T2

Q3. $\int_{x^2+y^2\le R^2} \frac{1}{x^2+y^2+1}\,dx\,dy$.

Change of variables.

$g(r,\theta)=(r\cos\theta, r\sin\theta)$

$|\det Dg| = r$.

$g(A)=\{x^2+y^2\le R^2\}$. $\Rightarrow A=[0,R]\times[0,2\pi]$.

$f(x,y)=\frac{1}{x^2+y^2+1}$. $(f\circ g)(r,\theta)=\frac{1}{r^2+1}$.

$I=\int_{[0,R]\times[0,2\pi]}\frac{1}{r^2+1}\, r\,dr\,d\theta$.

$= 2\pi\int_0^R \frac{r}{r^2+1}\,dr = 2\pi\int_0^{R^2}\frac{1}{u+1}\,du = 2\pi\log(R^2+1)$

Q4: f bdd. nonneg. cts on $\mathbb{R}^n$
except for a set of msr 0.
$\exists$ constant M st. $\int_R f\le M$ for any rectangle R
Then f is integrable over $\mathbb{R}^n$.

Proof. Want a PoI Φ st.
$\sum_{\varphi\in\Phi}\int_{\mathbb{R}^n}\varphi|f|$ cges.
Consider the rectangles $R_m=[-m,m]^n$.
$\{R_m\}$ covers $\mathbb{R}^n$. $\to$ can get a PoI subordinate to $\{R_m\}$. call it Φ.
Take the partial sum
$S_m := \sum_{\varphi\in\Phi,\, \mathrm{supp}\,\varphi\subseteq R_m}\int_{\mathbb{R}^n}\varphi|f|$

MAT237

T.T2

Q4 (Continued).

$S_m = \sum_{\varphi\in\Phi:\, \mathrm{supp}\,\varphi\subseteq R_m}\int_{\mathbb{R}^n}\varphi|f|$
$\le \int_{R_m}|f| = \int_{R_m} f \le M$.

and $S_m\le S_{m+1}$,
so S_m converges.

$\Rightarrow \sum_{\varphi\in\Phi}\int_{\mathbb{R}^n}\varphi|f|$ converges.

$\Rightarrow f$ is integrable over $\mathbb{R}^n$.

Q4: f bdd. nonneg. cts on $\mathbb{R}^n$
except for a set of msr 0.
$\exists$ constant M st. $\int_R f\le M$ for any rectangle R
Then f is integrable over $\mathbb{R}^n$.

Proof. Want a PoI Φ st.
$\sum_{\varphi\in\Phi}\int_{\mathbb{R}^n}\varphi|f|$ cges.
Consider the rectangles $R_m=[-m,m]^n$.
$\{R_m\}$ covers $\mathbb{R}^n$. $\to$ can get a PoI subordinate to $\{R_m\}$. call it Φ.
Take the partial sum
$S_m := \sum_{\varphi\in\Phi,\, \mathrm{supp}\,\varphi\subseteq R_m}\int_{\mathbb{R}^n}\varphi|f|$