Plan for today: TT2

Q1: Directly from the def, show that the set

$$A = \{(t,t) : t \in \mathbb{Q} \cap [0,1]\} \subset \mathbb{R}^2$$

is of content 0 in $\mathbb{R}^2$.

Solution: $A \subseteq \{(t,t) : t \in [0,1]\}$

Cover of n rectangles $R_i = \left[\frac{i}{n}, \frac{i+1}{n}\right]^2$, $i = 0, \dots, n-1$.

$$\mathrm{Vol}(\text{cover}) = \sum_{i=0}^{n-1} \frac{1}{n^2} = \frac{1}{n} . < \varepsilon \quad \forall \varepsilon, \text{ suff. large } n.$$

Q2: Show every $\underbrace{\text{open set}}_{A}$ in $\mathbb{R}^n$ is the union of countably many compact sets.

Solution:

Method 1: (Nested subsets)

Method 2: Take every rational point in A.

$$RP = \{p \in A : p = (p_1, \dots, p_n),\ p_i \in \mathbb{Q}\ \forall i\}$$

dense in A: $\forall a \in A$, $\forall \varepsilon > 0$, $\exists p \in RP$ st. $\|a - p\| < \varepsilon$.

$C_p := \overline{B_\varepsilon(p)}$, ε is largest positive number st. $\overline{B_\varepsilon(p)} \subset A$.

WTS: $C := \bigcup_{p \in RP} C_p = A$.

By construction, $C \subset A$.

By density, $\forall a \in A$, $\exists r$ st. $B_r(a) \subset A$.

$\exists p$ st. $\|a - p\| < \frac{r}{4}$.

$\Rightarrow \overline{B_{\frac{r}{4}}(p)} \subset B_r(a)$.

$\overline{B_{\frac{r}{4}}(p)} \subseteq C_p$. $\Rightarrow a \in C_p$.

Hence $\bigcup_{p \in RP} C_p \supset A$, so $C = A$.

Q3: Compute $I = \int_{\{x^2+y^2 \le R^2\}} \frac{1}{1+x^2+y^2}\, dx\, dy$.

Solution: $x = \rho\cos\theta \qquad \rho \in (0, \infty)$
$y = \rho\sin\theta \qquad \theta \in [0, 2\pi)$

$$\det J = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho$$

$$\Rightarrow I = \int_0^{2\pi} d\theta \int_0^R d\rho\, \frac{\rho}{1+\rho^2} = 2\pi \int_0^R \frac{\rho}{1+\rho^2}\, d\rho$$

$u = 1 + \rho^2 \Rightarrow du = 2\rho\, d\rho$

$$I = \pi \int_1^{1+R^2} \frac{1}{u}\, du = \pi \ln(1+R^2)$$

Q4: A bounded non-negative function f, cont. on $\mathbb{R}^n$ except for a set of msr 0, st. $\exists$ constant M st. $\int_R f \le M$, $\forall R$ rectangle. Show f is NT-integrable in $\mathbb{R}^n$.

Solution: Let $\mathcal{U} = \{R_i\}_{i=1}^{\infty}$ be a cover of $\mathbb{R}^n$ by open rectangles.

Then $\exists$ POI $\Phi = \{\varphi_i\}$ subordinate to the cover:

1) $0 \le \varphi_i \le 1$
2) local finiteness
3) $\sum_i \varphi_i(x) = 1 \quad \forall x.$
4) $\forall \varphi_i$, $\exists R_i$ st. $\overline{\mathrm{supp}(\varphi_i)} \subset R_i$.

Recall: f is NT-integrable if $\sum_{i=1}^{\infty} \int_{\mathbb{R}^n} \varphi_i |f| < \infty$.

$$\sum_{i=1}^{N} \int_{\mathbb{R}^n} \varphi_i |f| = \int_{\mathbb{R}^n} \sum_{i=1}^{N} \varphi_i f$$

$$= \int_{R^N} \sum_{i=1}^{N} \varphi_i f, \quad R^N \supset \bigcup_{i=1}^{N} R_i \text{ rectangle.}$$

$$\le \int_{R^N} f$$

$$\le M$$

Series is positive bounded above, non-decreasing, it converges.

Q5: A cont. diff. map $g : \mathbb{R}^n \to \mathbb{R}^n$ is called "volume-preserving" if $\forall$ Jordan measurable set B, $g^{-1}(B)$ is also Jordan measurable with $\mathrm{vol}(g^{-1}(B)) = \mathrm{vol}(B)$.

Show that $h : \mathbb{R}^2 \to \mathbb{R}^2$, $h(x,y) = (y, x+y)$ is volume-preserving.

Solution: 1) B Jordan $\Rightarrow h^{-1}(B)$ Jordan

2) $\int_B 1_B = \int_{h^{-1}(B)} 1_{h^{-1}(B)}$

Once we have 1), $\int_B 1_B = \int_{B^\circ} 1_B$, WLOG B is open.
$h^{-1}(B)$ is open.

$h(x,y) = (y, x+y)$

$h^{-1}(x,y) = (y-x, x)$

$Dh = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ non-singular, $(\det Dh = -1)$ $\quad Dh^{-1} = \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$ also nonsingular. $(\det Dh^{-1} = -1)$.

By COV,

$$\int_B 1_B = \int_{h^{-1}(B)} (1_B \circ h)(x,y) = \int_{h^{-1}(B)} 1_{h^{-1}(B)} .$$

Pf of 1): MVT: $\boxed{\|h^{-1}(x) - h^{-1}(y)\| \le \|x - y\|.}$ Lipschitz cont.
"derivative is bounded"

B is Jordan $\Rightarrow \forall \varepsilon > 0$, $\partial B \subset \bigcup_i B_{r_i}$ open balls, st. $\sum \mathrm{vol}(B_{r_i}) < \varepsilon$.

$h^{-1}(B_{r_i}(a)) \subset B_{r_i}(h^{-1}(a)) \Rightarrow \mathrm{vol}(h^{-1}(B_{r_i})) \le \mathrm{vol}(B_{r_i})$.

$h^{-1}(\partial B) \subset \bigcup_i h^{-1}(B_{r_i})$, $\sum \mathrm{vol}(h^{-1}(B_{r_i})) < \sum \mathrm{vol}(B_{r_i}) < \varepsilon$

B° open $\Rightarrow h^{-1}(B^\circ)$ open.

$\Rightarrow \partial h^{-1}(B) \subset h^{-1}(\partial B) \Rightarrow \partial h^{-1}(B)$ msr 0.

$\Rightarrow h^{-1}(B)$ is Jordan.