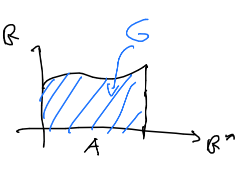
Past TT2:

1. Show "Vol under graph = integral".

ie. If $f: A \to \mathbb{R}$, cont., bounded, non-negative, $A \subset \mathbb{R}^n$ rectangle.

$G \subset \mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_y$, the graph,

$G = \{(x,y) : 0 \le y \le f(x)\}$



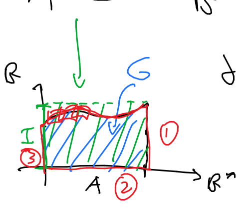
WTS: $\int_A f = \mathrm{Vol}(G)$.

Steps: 1) argue that 1_G integrable

2) $\int 1_G \overset{?}{=} \int_A f$ (Fubini)

1_G integrable $\Leftrightarrow$ ∂G has msr 0.

$1_G : A \times I \to \mathbb{R}^{n+1}$, $I = [0, \sup_x f(x)]$.



∂ graph of a continuous function has msr. 0.

f cont. $\Rightarrow$ f integrable.

$$1_G(x,y) = \begin{cases} 1 & (x,y) \in G \\ 0 & (x,y) \notin G. \end{cases} = 1_A(x)\, 1_{[0,f(x)]}(y)$$

$$\mathrm{Vol}(G) = \int_{\mathbb{R}^{n+1}} 1_G \, dx\,dy = \int_{\mathbb{R}^n} dx \int_{\mathbb{R}} dy \; 1_A(x)\, 1_{[0,f(x)]}(y)$$

$$= \int_{\mathbb{R}^n} 1_A(x) \underbrace{\int_{\mathbb{R}} 1_{[0,f(x)]}(y)\, dy}_{\int_0^{f(x)} dy = f(x)}\, dx$$

$$= \int_A f(x)\, dx$$

$\blacktriangle$

Common Mistakes: 1) no justification for 1_G integrable

2) $\int_0^{f(x)} 1_G \, dy = f(x)$

$\int_0^{f(x)} 1_G \, dy = 1_A(x) f(x)$

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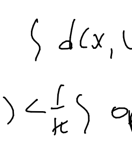
2. Show that every open set $U \subseteq \mathbb{R}^n$ can be presented as the union of a sequence of compacts $C_1, C_2, \dots$ st. $C_k \subset \mathrm{int}\, C_{k+1}$ $\forall k \ge 1$.



$B_1(0) = \bigcup_{n=2}^{\infty} \overline{B_{1-\frac{1}{n}}(0)}$

$C_k := \{x \in U : \underbrace{|x| \le k}_{x \in B_k(0)} \text{ and } \underbrace{d(x, U^c) \ge \frac{1}{k}}\}$

ie C_k bounded

$d(x, U^c) = \inf_{y \in U^c} \|x - y\|$

C_k compact since bounded & closed.

closed: $C_k^c = \{|x| > k\} \cup \{d(x,U^c) < \frac{1}{k}\}$

U^c closed $\Rightarrow$ $\{d(x,U^c) < \frac{1}{k}\}$ open.

$\{|x| > k\}$ also open.

C_k^c open $\Rightarrow$ C_k closed.

Hence, C_k compact.

$B_k(0) \subset \mathrm{int}\, B_{k+1}(0) = \{\|x\| < k+1\}$

$\{d(x,U^c) \ge \frac{1}{k}\} \subset \mathrm{int}\, \{d(x,U^c) \ge \frac{1}{k+1}\}$

$= \{d(x,U^c) > \frac{1}{k+1}\}$

So $C_k \subset \mathrm{int}\, C_{k+1}$.

WTS: $U = \bigcup_{k=1}^{\infty} C_k$. Let $C = \bigcup_{k=1}^{\infty} C_k$.

1) $C \subset U$:

$x \in C \Rightarrow \exists k$ st. $x \in C_k \subset U$.

$\Rightarrow x \in U$.

2) $U \subset C$:

$x \in U$. 1) $\exists R > 0$ st $x \in B_R(0)$.

$\Rightarrow \exists k$ sufficiently large st $x \in B_k(0)$. $(k = \lceil R \rceil)$

2) $\exists B_\varepsilon(x) \subset U$. $\Rightarrow d(x, U^c) > \varepsilon$.

$\Rightarrow k'$ sufficiently large st $k' > R$, $\frac{1}{k'} < \varepsilon$.

$\Rightarrow x \in C_{k'}$. $\Rightarrow x \in C$

$\Rightarrow U = C$.

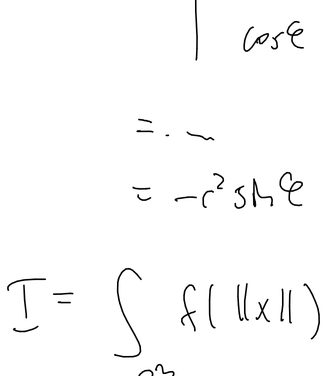
$\blacktriangle$

3. $f: \mathbb{R}_{>0} \to \mathbb{R}$ cont., $\mathrm{Supp} f$ is compact $\subset \mathbb{R}_{>0}$.

Let $g: \mathbb{R}^3 \to \mathbb{R}$, $g(x) = f(\|x\|)$. Show

$$\int_{\mathbb{R}^3} g = \int_0^\infty 4\pi r^2 f(r)\, dr.$$

$$I = \int_{\mathbb{R}^3} g = \int_{\mathbb{R}^3} f(\|x\|)\, dx.$$



$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases} \quad (x^2+y^2+z^2 = r^2.)$$

$r \ge 0$, $\varphi \in [0, \pi]$, $\theta \in [0, 2\pi]$.

$$\det J = \begin{vmatrix} \sin\varphi\cos\theta & -r\sin\varphi\sin\theta & r\cos\varphi\cos\theta \\ \sin\varphi\sin\theta & r\sin\varphi\cos\theta & r\cos\varphi\sin\theta \\ \cos\varphi & 0 & -r\sin\varphi \end{vmatrix}$$

$= \dots$

$= -r^2 \sin\varphi$

$$I = \int_{\mathbb{R}^3} f(\|x\|)\, dx = \int_0^\infty dr \int_0^\pi d\varphi \int_0^{2\pi} d\theta \; f(r)\, r^2 \sin\varphi$$

$$= 2\pi \int_0^\pi \sin\varphi \, d\varphi \int_0^\infty r^2 f(r)\, dr$$

$$= \int_0^\infty 4\pi r^2 f(r)\, dr$$

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4. Prove the set of reals between 0 & 1 whose decimal expansion does not contain the digit 4 is msr 0. Denote this set by C.

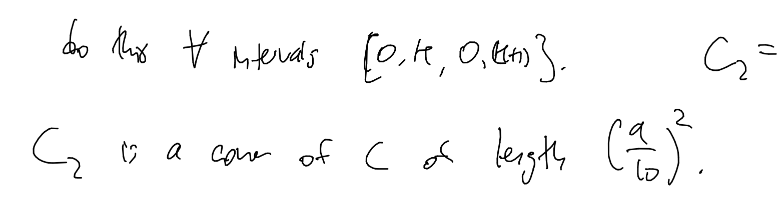
Where can the digit 4 appear?

1) 0.4 __ 1^{st} digit

2) 0._ 4... 2^{nd} digit

3) 0._ _ 4... 3^{rd} digit

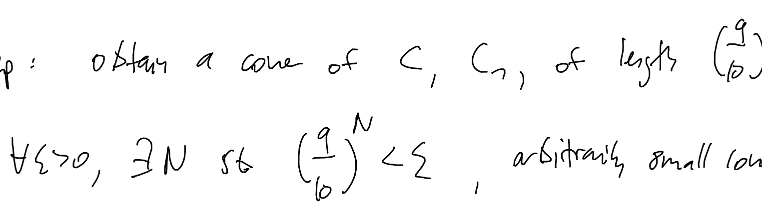
1)



$C_1 = [0,1] \setminus \{0.4\dots\}$

C_1 is a cover of C of length $\frac{9}{10}$.

2)



do this $\forall$ intervals $[0.k, 0.(k+1)]$.

$C_2 = C_1 \setminus \{0.\_4\dots\}$

C_2 is a cover of C of length $(\frac{9}{10})^2$.

n) step: obtain a cover of C, C_n, of length $(\frac{9}{10})^n$.

$\forall \varepsilon > 0$, $\exists N$ st $(\frac{9}{10})^N < \varepsilon$, arbitrarily small cover C_N.

$\Rightarrow$ C msr 0.

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5. $g: \mathbb{R}^2 \to \mathbb{R}^2$ cont., diff.

$\exists a \in \mathbb{R}^2$ st $D_2 g_2(a) \neq 0$.

Prove that near a, g can be written as a composition of 2 cont. diff. functions defined on opens in $\mathbb{R}^2$, taking values in opens in $\mathbb{R}^2$, st. 1 preserves the 1^{st} coord, the other preserves the 2^{nd}.

$g(x,y) = (g_1(x,y), g_2(x,y))$.

WTS $\exists f, h$ st

$g(x,y) = (f \circ h)(x,y)$

$h(x,y) = (x, h_2(x,y)) := \boxed{(x, g_2(x,y))}$ C^1

$f(x,y) = (f_1(x,y), y)$

?

$$\underbrace{\begin{pmatrix} x \\ y \end{pmatrix} \xrightarrow{h} \begin{pmatrix} x \\ h_2(x,y) \end{pmatrix} \xrightarrow{f} \begin{pmatrix} f_1(x, h_2(x,y)) \\ h_2(x,y) \end{pmatrix}}_{g} = \begin{pmatrix} g_1(x,y) \\ g_2(x,y) \end{pmatrix}$$

$f = g \circ h^{-1}$.

$$Dh = \begin{pmatrix} 1 & 0 \\ D_1 g_2 & D_2 g_2 \end{pmatrix} \Rightarrow \det Dh = D_2 g_2.$$

(cont.)

At a, $D_2 g_2(a) \neq 0 \Rightarrow \exists$ nbhd $U_a \ni a$ st $D_2 g_2(x) \neq 0$ $\forall x \in U_a$.

IVT, $\exists$ open nbhd $V_{h(a)} \ni h(a)$ st

$h^{-1}: V_{h(a)} \to U_a$, h^{-1} is C^1.

So $f := g \circ h^{-1}$, f is C^1, $f \circ h = g$.

Since $h_2 = g_2$, f must fix the 2^{nd} coord.