Jan 12 Tutorial

January 12, 2022 3:53 PM

Today: test prep: going over last year's tt2.

Q1 (20 points)

Show that "the volume under the graph of a function is equal to the integral of that function". Precisely, show that if $f\colon A \to \mathbb{R}$ is continuous, bounded, and non-negative, where $A \subset \mathbb{R}^n$ is a rectangle, and if $G \subset \mathbb{R}^{n+1} = \mathbb{R}^n_x \times \mathbb{R}_y$ is defined by $G := \{(x,y)\colon 0 \le y \le f(x)\}$, then $\int_A f = v(G)$. (Recall that the volume of a set is the integral of its characteristic function).

WTS

$$\int_A f = v(G) = \int_G 1$$

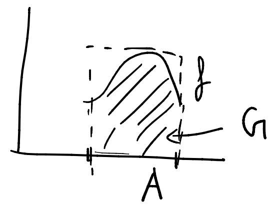
$$= \int_{A \times [0, \sup f]} \chi_G = \int_A \int_0^{\overbrace{\sup f}^{=f(x)}} \chi_G$$

$$= \sup_{\substack{P \\ \text{is partition of } A}} \; \sup_{\substack{Q \\ \text{partition of } [0, \sup f]}} \Big\{ \sum_{\substack{S \in P \\ [q_i, q_{i+1}] \in Q}} \sup(\chi_G)\, \mathrm{vol}(S) \Big\}$$

$\le$

$$\ge \sup_{P,Q} \Big\{ \sum_{S \in P} (\sup_S f)\, \mathrm{vol}(S) \Big\}$$

$$= \int_A f$$

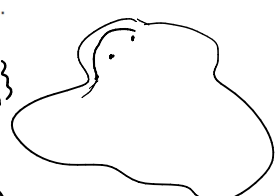


Q2 (20 points)

Show that every open set U in $\mathbb{R}^n$ can be presented as the union of a sequence of compact sets $C_1, C_2, C_3, \ldots$, satisfying $C_k \subset \operatorname{int} C_{k+1}$ for all $k \ge 1$.

Pf from Spivak:

$$C_n = \{x \in U \text{ st. } \underbrace{|x| \le n}_{\text{closed}} \;\&\; \underbrace{d(x, \mathrm{bd}\, U) \ge \tfrac{1}{n}}_{\substack{\text{closed} \\ \text{(should justify)}}}\}$$



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my proof: Let $x_1, x_2, \ldots$ be pts w/ rational coords

Let K_i be compact ball around x_i st. $K_i \subseteq U$ & $d(K_i, \mathrm{bd}\, U) \ge r_i$

$C_1 = K_1$

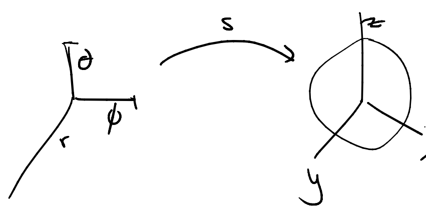
$C_2 = K_2 \cup \{K_1 + \frac{r_1}{2}\}$

$$C_3 = K_3 \cup \{K_2 + \tfrac{r_2}{2}\} \cup \{K_1 + \tfrac{r_1}{2} + \tfrac{r_1}{4}\}$$

$\vdots$

Q3 (20 points)

Let $f\colon \mathbb{R}_{>0} \to \mathbb{R}$ be a continuous function whose support $\operatorname{supp} f$ is a compact subset of $\mathbb{R}_{>0}$, where $\mathbb{R}_{>0}$ denotes the positive real numbers. Define a function $g\colon \mathbb{R}^3 \to \mathbb{R}$ by $g(x) = f(|x|)$. Show that $\int_{\mathbb{R}^3} g = 4\pi \int_0^\infty r^2 f(r)\,dr$.



$x = r\sin\phi\cos\theta$

$y = r\sin\phi\sin\theta$

$z = r\cos\phi$

$r \in [0,\infty)$

$\phi \in [0,\pi]$

$\theta \in [0,2\pi]$

$$\int_{\mathbb{R}^3} g = \int_0^\infty\int_0^{2\pi}\int_0^{\pi} g(s(r,\phi,\theta))\, d\phi\, d\theta\, dr \qquad g$$

$$|\det Ds| = r^2\sin\phi$$

$$= \iiint f(r)\, r^2 \sin\phi \;\; d\phi\, d\theta\, dr$$

$$= \iint -f(r)\, r^2\cos\phi\Big|_0^{\pi}\; d\theta\, dr$$

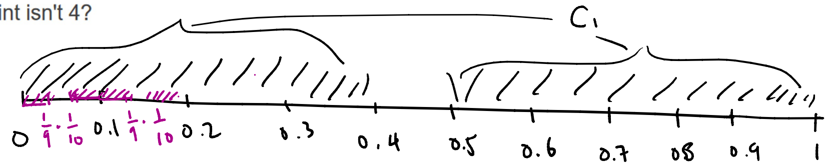
$$= 2\iint_0^{2\pi} f(r)\, r^2\, d\theta\, dr$$

$$= 4\pi \int f(r)\, r^2\, dr$$

Q4 (20 points)

For reasons unknown to me, my apartment building has no floors whose number contains the digit 4. Prove that $\underbrace{\text{the set of real numbers between 0 and 1 whose decimal expansion does not contain the digit 4}}_{=:C}$ is of measure 0.

Hint. What's the length of the set C_1 of real numbers between 0 and 1 whose first digit after the decimal point isn't 4?



length of $C_1 = \frac{1}{9}$

length of $C_2 = \frac{1}{9}$ length of $C_1 = \frac{1}{9^2}$

let C_2 = xs whose 1st AND 2nd digits are not 4

let C_n = xs ... 1st ... n^{th} digits are not 4.

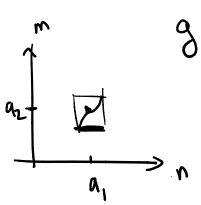
Then length of $C_n = \frac{1}{9^n}$.

Then $C \subseteq C_n \quad \forall n.$

For any ε, take n large enough st. $\frac{1}{9^n} < \varepsilon$.

Q5 (20 points)

Let $g\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a continuously differentiable function, and assume that at some point $a \in \mathbb{R}^2$ we have that $\boxed{D_2 g_2(a) \neq 0,}$ where g_2 is the second component of g. Prove that near a the function g can be written as a composition of two continuously differentiable functions defined on some open sets in $\mathbb{R}^2$ and taking values in some open sets in $\mathbb{R}^2$, and such that one of those functions preserves the first coordinate and the other one preserves the second coordinate.



$$g: \mathbb{R}^2 \to \mathbb{R}^2$$

$$(x, y) \mapsto (g_1(x,y), g_2(x,y))$$

$$\begin{pmatrix} D_1 g_1 & D_2 g_1 \\ D_1 g_2 & D_2 g_2 \end{pmatrix}$$

$D_2 g_2 \neq 0$

$f :=$ ✓

k

$(x, g_2(x,y))$

✗

so we just need to find this

$(g_1(x, \pi_2 \circ k(x,z)), z)$

a_1 $\qquad$ n $\qquad$ $f :=$

$(x, g_2(x,y))$ ← so we just need to find this function

$(g_1(x, \pi_2 \circ k(x,z)), z)$

(x, z)

We'd like to get y back from x & $g_2(x,y)$

Use inv. func. thm on f:

$$\det \begin{pmatrix} 1 & 0 \\ D_1 g_2 & D_2 g_2 \end{pmatrix} = D_2 g_2 \neq 0 \text{ at } a.$$

So $\exists$ open V around $f(a)$
U around a
$k : V \to U$ st. k is inv. of f

So we can write:

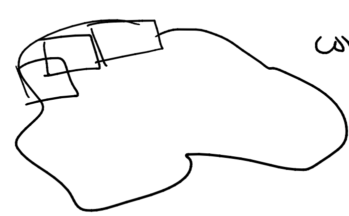
$$y = \pi_2 \circ k(x, g_2(x,y))$$

1. Let A be a rectangle in $\mathbb{R}^n$ and let $f, g : A \to \mathbb{R}$, where f is integrable on A and g is equal to f except on finitely many points. Show from basic definitions that g is also integrable on A and that $\int_A f = \int_A g$.

   Tip. "From basic definitions" means "not using any of the theorems that came after the definitions that are necessary to make the question meaningful". In our case those definitions are those of lower and upper sums, integrability, and the integral. Yet words like "measure-0", whether or not they are relevant, are forbidden.

2. (a) Show that the boundary of a set of content-0 is also of content-0.
   (b) Give an example of a set of measure-0 whose boundary is not of measure-0.

eg. $\mathbb{Q} \cap [0,1]$



content 0

Rect.

finite union of closed set is closed, so covering by closed rect. covers the bd.