Term Test 1

(Author's name here)

November 2, 2021

Summaries and Notes on Intuition

This solution set consists of handwritten solutions that I wrote during the test. Unfortunately, my solutions are quite messy because they were written under time pressure. Hence, before I present the handwritten solutions, I will summarize the key ideas for each solution, and I will also discuss some methods to approach each problem and motivate each solution. (Note: This section was not submitted for grading.)

1. For this question, we had to use all scale fidelity to prove that f is surjective, using the same method that was done in lecture (while proving the Inverse Function Theorem). The key idea is to use a recursive sequence $\{x_n\}_{n\in\mathbb{N}}$ to find better approximations of x over time. Since all scale fidelity says that $f(x_n) - f(x_{n-1})$ is close to $x_n - x_{n-1}$, and since we want $f(x_n)$ to be close to y, it is reasonable to pick x_n such that $y - f(x_{n-1}) = x_n - x_{n-1}$. This gives us our recursive definition $x_n = x_{n-1} + (y - f(x_{n-1}))$.
   Now, we want our sequence $\{x_n\}_{n\in\mathbb{N}}$ to better approximate x over time, so these terms should also get closer to each other over time. This motivates us to show that $|x_n - x_{n-1}|$ decreases quickly over time. The naive approach of rewriting a single recursion $x_n = x_{n-1} + (y - f(x_{n-1}))$ as $x_n - x_{n-1} = y - f(x_{n-1})$ does not work because we do not know, a priori, how well $f(x_{n-1})$ actually approximates y. Instead, we can subtract consecutive recursive relations. The left-hand side is precisely $x_n - x_{n-1}$, and the right-hand side is $(x_{n-1} - x_{n-2}) - (f(x_{n-1}) - f(x_{n-2}))$. Directly applying all scale fidelity, the right-hand side can be bounded to have a magnitude of at most $\frac{1}{7}|x_{n-1} - x_{n-2}|$. Thus, $|x_n - x_{n-1}| \leq \frac{1}{7}|x_{n-1} - x_{n-2}|$, so the distances between consecutive terms experiences exponential decay. This allows us to prove that $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy (with some technicalities involving geometric series), so it converges to some limit x.
   Next, recall the "naive step" that we tried above: $x_n - x_{n-1} = y - f(x_{n-1})$. Now that we know that $|x_n - x_{n-1}|$ experiences exponential decay, this step becomes helpful because we discover that $\left|y - f(x_{n-1})\right|$ also experiences exponential decay. As a result, $f(x_n)$ must approach y as $x \to \infty$. Finally, since f is continuous, it follows that:

$$f(x) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = y,$$

   as required.

2. This question asks us to prove Spivak's Theorem 2-8. We can prove it using the "axis crawl" solution presented during lecture. In other words, for all $a + h$ near a, we want to travel from a to $a + h$ in n steps, where we travel along the x_k-direction during the k^{th} step. We formalize this idea by defining the points $b_k := (a_1 + h_1, \dots, a_k + h_k, a_{k+1}, \dots, a_n)$ for all indices $0 \leq k \leq n$; the k^{th} step travels along the x_k-direction from b_{k-1} to b_k.
   Next, we are interested in how much f changes at each step. At the k^{th} step, since this step of the axis crawl only travels along x_k-direction, we can apply the MAT157 Mean Value Theorem for the k^{th} partial derivative. This gives us:

$$f(b_k) - f(b_{k-1}) = h_k \frac{\partial f(c_k)}{\partial x_k}, \qquad (*)$$

   where c_k is between b_k and b_{k-1}. Since f has continuous partial derivatives, the term $\frac{\partial f(c_k)}{\partial x_k}$ above should approach $\frac{\partial f(a)}{\partial x_k}$ as $h \to 0$.

Next, we shall guess that the differential of f at a is the linear map:

$$L(h) := \sum_{i=1}^{n} \frac{\partial f(a)}{\partial x_i} h_i,$$

as suggested by Spivak's Theorem 2-7. We can verify this by verifying the definition of the differential:

$$\lim_{h \to 0} \frac{\left| f(a+h) - f(a) - L(h) \right|}{|h|} = 0.$$

Our axis crawl allows us to split the numerator into:

$$\left| (f(b_n) - f(b_{n-1})) + (f(b_{n-1}) - f(b_{n-2})) + \cdots + (f(b_1) - f(b_0)) - L(h) \right|.$$

Since L is linear, we can split $L(h)$ into its n directions to split the numerator further:

$$\left| (f(b_n) - f(b_{n-1}) - L(0, \ldots, h_n)) + \cdots + (f(b_1) - f(b_0) - L(h_1, \ldots, 0)) \right|.$$

The rest of the solution is computational. We apply $(*)$, then the observation $\lim_{h \to 0} \frac{\partial f(c_k)}{\partial c_k} = \frac{\partial f(a)}{\partial x_k}$ gives us the final push to prove that L is the differential of f at a.
As a final note, my handwritten solution contains several distracting justifications that "we are close enough to a to do this and that". It should be safe to ignore such remarks while reading the solution.

3. This question is Question 5 from Assignment 1. The solution is available in the Assignment 1 Solution Set on the class website, and my handwritten solution here is very similar, so I will simply give a quick review.
First, for all $x \notin [0,1] - A$, there are two ways for a point x to be outside $[0,1] - A$: Either x is outside $[0,1]$, or x is inside A. If x is outside $[0,1]$, then x is far away from the set $A \subseteq [0,1]$, so x is in the exterior of A. If x is inside A, then it is inside one of the open intervals that comprise A, so x is in the interior of A. Either way, x is not in the boundary of A.
Next, for all $x \in [0,1] - A$, every open interval around x contains the point x outside A. Since the rationals are dense, we can argue that every open interval around x also contains a rational number in $[0,1]$. Such a number must be in A, so every open interval contains points inside and outside A. Thus, x is in the boundary of A.
These steps, when combined, prove that $[0,1] - A$ is the boundary of A, as required.

4. This question is new. To prove that f is integrable, we could find partitions whose lower sums and upper sums approach each other so that the lower integral and upper integral of f are equal. When in doubt, it is a good idea to pick the partition using uniformly spaced cutpoints $\{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\}$ to obtain a simple partition. This partition has subrectangles of the form:

$$S_{i,j} = [\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}].$$

Note that all subrectangles have the same volume of $\frac{1}{n^2}$.
Since f is also relatively simple, we obtain the following cases for $S_{i,j}$:

   i) $S_{i,j}$ could be completely above the diagonal $y = x$, so $f(x,y) = 0$ for all $(x,y) \in S_{i,j}$. Then, $m_{S_{i,j}}(f) = M_{S_{i,j}}(f) = 0$. There turn out to be $\frac{(n-1)n}{2}$ such rectangles. (This is Case 1 in the handwritten solution.)

ii) $S_{i,j}$ could touch the diagonal $y = x$, so $f(x, y)$ takes values of both 0 and 1 as (x, y) ranges across $S_{i,j}$. Then, $m_{S_{i,j}}(f) = 0$ and $M_{S_{i,j}}(f) = 1$. There turn out to be $n+(n-1) = 2n-1$ such rectangles, so the proportion of all n^2 rectangles covered by this case is $\frac{2n-1}{n^2} = \frac{2}{n} - \frac{1}{n^2}$. This case is the only source of discrepancy between the lower and upper sums of f, so it is good news that it contains few rectangles. (This is Cases 2 and 3 in the handwritten solution.)

iii) $S_{i,j}$ could be completely below the diagonal $y = x$, so $f(x, y) = 1$ for all $(x, y) \in S_{i,j}$. Then, $m_{S_{i,j}}(f) = M_{S_{i,j}}(f) = 1$. There turn out to be $\frac{(n-2)(n-1)}{2}$ such rectangles. (This is Case 4 in the handwritten solution.)

Combining these cases, the rest of the solution computes the upper and lower sums:

$$U(f, P) = \sum_{S \in P} \mathrm{vol}(S) M_S(f) = \frac{1}{n^2} \sum_{S \in P} M_S(f), \qquad L(f, P) = \sum_{S \in P} \mathrm{vol}(S) m_S(f) = \frac{1}{n^2} \sum_{S \in P} m_S(f).$$

Then, it finds that both the upper and lower sums converge to $\frac{1}{2}$ as $n \to \infty$, giving us the chain of inequalities:

$$\frac{1}{2} \leq L(f) \leq U(f) \leq \frac{1}{2}.$$

We conclude that $L(f) = U(f) = \frac{1}{2}$, so f is integrable (with $\int_{[0,1]\times[0,1]} f = \frac{1}{2}$)), as required.

Q1 25

Solve all 4 problems. Write your solutions only where indicated, or write explicitly, "continued on page X".
Neatness counts! Language counts!

1BE1BA34-8A04-4741-BBAC-322EE5B5F950
term-test-1-90fcc
#45 3 of 18

Problem 1. A continuous function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$|(x_1 - x_2) - (f(x_1) - f(x_2))| \leq \frac{1}{7}|x_1 - x_2|$$ ← We will call this (*).

for every x_1 and x_2 in $\mathbb{R}^n$. Prove that for every $y \in \mathbb{R}^n$ there is an $x \in \mathbb{R}^n$ such that $f(x) = y$.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Tip. You may want to start by writing "draft solutions" on the last pages of this notebook and only then write the perfected versions in the space left here for solutions.

Your solution of Problem 1.

Let any $y \in \mathbb{R}^n$ be given. Then, define the sequence $\{x_n\}_{n\geq 0}$ by the following recursive formula:

$$x_n := \begin{cases} 0, & \text{if } n=0, \\ x_{n-1} + (y - f(x_{n-1})), & \text{if } n \in \mathbb{N}. \end{cases}$$

Next, for all $n \geq 2$, we have the following two equations, by definition:

$$x_n = x_{n-1} + (y - f(x_{n-1})) \quad ①$$
$$x_{n-1} = x_{n-2} + (y - f(x_{n-2})) \quad ②$$

Subtracting ② from ① yields:

$x_n - x_{n-1} = (x_{n-1} - x_{n-2}) - (f(x_{n-1}) - f(x_{n-2}))$, so:

$$|x_n - x_{n-1}| = |(x_{n-1} - x_{n-2}) - (f(x_{n-1}) - f(x_{n-2}))|$$
$$\leq \frac{1}{7}|x_{n-1} - x_{n-2}|. \quad \text{(Applying (*) for } x_{n-1}, x_{n-2})$$

This is true for all $n \geq 2$, so:

$$|x_n - x_{n-1}| \leq \frac{1}{7}|x_{n-1} - x_{n-2}| \leq \frac{1}{7}\cdot\frac{1}{7}|x_{n-2} - x_{n-3}| \leq \cdots \leq \left(\frac{1}{7}\right)^{n-1}|x_1 - x_0|.$$

(Continued on next page)

Your solution of Problem 1, continued.

If $y = f(x_0)$, we are done. From now on, assume $y \neq f(x_0)$, so $x_1 = x_0 + (y - f(x_0)) \neq x_0$, so $C = |x_1 - x_0| > 0$.

1. (Continued)

Let us define constant $C := |x_1 - x_0|$. Then, we proved $|x_n - x_{n-1}| \leq \frac{1}{7^{n-1}} C$ for all $n \geq 1$.

Next, we will show that $\{x_n\}_{n \geq 0}$ is Cauchy.

For all $\varepsilon > 0$, let us define $M := \log_7\left(\frac{7C}{6\varepsilon}\right)$. Then, for all positive integers $n \geq m > M$, we obtain:

$$\begin{aligned}
|x_n - x_m| &= |(x_n - x_{n-1}) + (x_{n-1} - x_{n-2}) + \cdots + (x_{m+1} - x_m)| \\
&\leq |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m| \\
&\leq \frac{1}{7^{n-1}} C + \frac{1}{7^{n-2}} C + \cdots + \frac{1}{7^m} C \\
&\leq C \cdot \left(\frac{1}{7^m} + \frac{1}{7^{m+1}} + \cdots\right) \\
&= C \cdot \frac{\frac{1}{7^m}}{1 - \frac{1}{7}} \quad \text{(Infinite geometric series, } r = \tfrac{1}{7}) \\
&= \frac{7}{6} C \cdot \frac{1}{7^m} \\
&< \frac{7}{6} C \cdot \frac{1}{7^M} \\
&= \frac{7}{6} C \cdot \frac{6\varepsilon}{7C} \\
&= \varepsilon.
\end{aligned}$$

Thus, for all $\varepsilon > 0$, we found $M \in \mathbb{R}$ such that all positive integers $n \geq m > M$ satisfy $|x_n - x_m| < \varepsilon$, so $\{x_n\}$ is Cauchy.

(Continued on Page 14)

Scratch work — this page will not be read unless you explicitly request it.

1. (Continued)
Therefore, the Cauchy sequence $\{x_n\}$ has a limit; call this limit x.
Next, for all $n \geq 1$, we have $x_n = x_{n-1} + (y - f(x_{n-1}))$,
so: $|y - f(x_{n-1})| = |x_n - x_{n-1}| \leq \frac{1}{7^{n-1}} C$.
Since $\lim_{n\to\infty} \frac{1}{7^{n-1}} C = 0$, this implies that
$\lim_{n\to\infty}(y - f(x_{n-1})) = 0$, so $\lim_{n\to\infty} f(x_n) \overset{\text{shifting index by 1}}{=} \lim_{n\to\infty} f(x_{n-1}) = y$.

Now, since f is continuous, ~~$\lim_{u\to x} f(u)$ exists and equals $f(x)$. Then~~ for all $\varepsilon > 0$, there exists $\delta > 0$ such that all $u \in \mathbb{R}^n$ satisfying $|u - x| < \delta$ also satisfy $|f(u) - f(x)| < \varepsilon$. Since sequence $\{x_n\}$ approaches x, we can pick large enough n such that $|x_n - x| < \delta$ and such that $|f(x_n) - y| < \varepsilon$ (since $\lim_{n\to\infty} f(x_n) = y$). Then,
$|f(x) - y| \leq |f(x) - f(x_n)| + |f(x_n) - y|$ (Triangle inequality)
$< \varepsilon + \varepsilon$
$= 2\varepsilon$.

(Continued on Page 15)

Scratch work — this page will not be read unless you explicitly request it.

B4078F12-35FB-4513-A4D7-B935EBBFF828
term-test-1-90fcc
#45 15 of 18

1. (Continued)

This is true for all $\varepsilon>0$. letting ε shrink to 0, we conclude that $|f(x)-y|=0$, so $f(x)=y$ for our choice of x, as required.

Q2 25

Solve all 4 problems. Write your solutions only where indicated, or write explicitly, "continued on page X".
Neatness counts! Language counts!

Problem 2. Show that if a function f is defined near a point $a \in \mathbb{R}^n$ and has continuous partial derivatives near a, then it is differentiable at a. ← We will assume $f: \mathbb{R}^n \to \mathbb{R}$.

Tip. In math exams, "show" means "prove".

Your solution of Problem 2.

f is defined and has continuous partial derivatives near a, so there exists some radius r such that for all $h \in \mathbb{R}^n$ satisfying $|h| < r$, $f(a+h)$ is defined and $\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}$ are continuous at $a+h$.

Now, let any $h \in \mathbb{R}^n$ be given such that $|h| < r$.

Then, let us define the following points in $\mathbb{R}^n$:

$b_0 := (a_1, a_2, \dots, a_n) = a$

$b_1 := (a_1 + h_1, a_2, \dots, a_n)$

$\vdots$

$b_k := (a_1 + h_1, \dots, a_k + h_k, a_{k+1}, \dots, a_n)$

$\vdots$

$b_n := (a_1 + h_1, \dots, a_n + h_n) = a + h.$

For all $0 \le k \le n$, we have:

$$\begin{aligned} |b_k - a| &= |(h_1, \dots, h_k, 0, \dots, 0)| \\ &= \sqrt{\sum_{i=1}^{k} h_i^2} \\ &\le \sqrt{\sum_{i=1}^{n} h_i^2} \\ &= |h| \\ &< r, \end{aligned}$$

So $f(b_k)$ is defined.

(Continued on next page)

Your solution of Problem 2, continued.

2. (Continued)
Now, for all $1 \le k \le n$, (Because b_{k-1}, b_k are near a) we have that $\frac{\partial f}{\partial x_k}$ exists between b_{k-1} and b_k. (Here, "between b_{k-1} and b_k" means a point of the form $(a_1+h_1, \dots, a_{k-1}+h_{k-1}, t, a_{k+1}, \dots, a_n)$, where t is between a_k and a_k+h_k.) Then, by the MAT157 Mean Value Theorem, we obtain:

$$f(b_k)-f(b_{k-1}) = h_k \frac{\partial f(c_k)}{\partial x_k}$$

for some c_k between b_{k-1} and b_k.

Next, since we showed above that $|b_k - a| \le |h|$ for all k, we obtain $|b_{k-1}-a|, |b_k - a| \le |h|$. Then, since c_k is between b_k and b_{k-1}, we can write c_k as $\lambda b_k + (1-\lambda) b_{k-1}$, where $\lambda \in [0,1]$. As a result:

$$\begin{aligned} |c_k - a| &= |\lambda b_k + (1-\lambda) b_{k-1} - a| \\ &= |\lambda (b_k - a) + (1-\lambda)(b_{k-1}-a)| \\ &\le \lambda |b_k - a| + (1-\lambda)|b_{k-1} - a| \quad \text{(Triangle inequality)} \\ &\le \lambda h + (1-\lambda) h \\ &= h. \end{aligned}$$

(Continued on Page 16)

Scratch work — this page will not be read unless you explicitly request it.

2. (Continued)

We showed that $|c_k - a| < |h|$. Since f has continuous partial derivatives near a, this implies that:

$$\lim_{h\to 0} \frac{\partial f(c_k)}{\partial x_k} = \lim_{c_k \to a} \frac{\partial f(c_k)}{\partial x_k} = \frac{\partial f(a)}{\partial x_k}.$$

Now, define the linear map $L: \mathbb{R}^n \to \mathbb{R}$ by $L(h) = \frac{\partial f(a)}{\partial x_1} h_1 + \frac{\partial f(a)}{\partial x_2} h_2 + \cdots + \frac{\partial f(a)}{\partial x_n} h_n$.

Then, we obtain:

$$\lim_{h\to 0} \frac{|f(a+h) - f(a) - L(h)|}{|h|}$$

$$= \lim_{h\to 0} \frac{|f(b_n) - f(b_0) - L(h)|}{|h|}$$

$$= \lim_{h\to 0} \frac{|(f(b_n) - f(b_{n-1}) - L(0,\dots,h_n)) + (f(b_{n-1}) - f(b_{n-2}) - L(0,\dots,h_{n-1},0)) + \cdots + (f(b_1) - f(b_0) - L(h_1,\dots,0))|}{|h|}$$

$$\leq \lim_{h\to 0}\left(\frac{|f(b_n) - f(b_{n-1}) - L(0,\dots,h_n)|}{|h|} + \cdots + \frac{|f(b_1) - f(b_0) - L(h_1,\dots,0)|}{|h|}\right)$$

Applying Triangle inequality

(Continued on page 17)

Scratch work — this page will not be read unless you explicitly request it.

2. (Continued)

$$= \lim_{h\to 0} \sum_{k=1}^{n} \frac{\left| h_k \frac{\partial f(c_k)}{\partial x_k} - h_k \frac{\partial f(a)}{\partial x_k} \right|}{|h|}$$

$$= \lim_{h\to 0} \sum_{k=1}^{n} \frac{|h_k|}{|h|} \left| \frac{\partial f(c_k)}{\partial x_k} - \frac{\partial f(a)}{\partial x_k} \right|$$

$$\leq \lim_{h\to 0} \sum_{k=1}^{n} 1 \cdot \left| \frac{\partial f(c_k)}{\partial x_k} - \frac{\partial f(a)}{\partial x_k} \right|$$

$= 0$ (Since $\lim_{h\to 0} \frac{\partial f(c_k)}{\partial x_k} = \frac{\partial f(a)}{\partial x_k}$)

Therefore, since $\lim_{h\to 0} \frac{|f(a+h) - f(a) - L(h)|}{|h|} = 0$, f is differentiable at a with $Df(a) = L$, as required.

Q3 25

Solve all 4 problems. Write your solutions only where indicated, or write explicitly, "continued on page *X*".
Neatness counts! Language counts!

10B58284-DED8-4E21-A6A5-9CA817D5F8BD
term-test-1-90fcc
#45 7 of 18

Problem 3. A set $A \subset [0, 1]$ is a (possibly infinite) union of open intervals and it contains all the rational numbers in $[0, 1]$. Show that the boundary of A is $[0, 1] \setminus A$.

Your solution of Problem 3.

Step 1: We will prove that all $x \in A$ are not in the boundary of A.
Let any $x \in A$ be given. Then, since A is a union of open intervals, x is inside some open interval $(a,b) \subseteq A$. In other words, there exists an open rectangle around x contained in A, so x is in the interior of A, not the boundary of A, as required.

Step 2: We will prove that all $x \in (-\infty,0) \cup (1,\infty)$ are not in the boundary of A.
If $x \in (-\infty, 0)$, then x is in the open rectangle $(x-1, 0)$, and this rectangle does not intersect A because $A \subseteq [0,1] \subseteq (\mathbb{R} - (x-1,0))$.
If $x \in (1,\infty)$, then x is in the open rectangle $(1, x+1)$, and this rectangle does not intersect A because $A \subseteq [0,1] \subseteq (\mathbb{R} - (1,x+1))$.
Either way, there exists an open rectangle around x contained in $\mathbb{R} - A$, so x is in the exterior of A, not the boundary, as required.

(Continued on next page)

Your solution of Problem 3, continued.

3. (Continued)

Step 3: We will prove that all $x \in [0,1]-A$ are in the boundary of A. Consider the following cases:

Case 1: $x=1$. Then, for all open rectangles (a,b) containing x, the open rectangle $(\max(0,a),b)$ also contains x. Since $\mathbb{Q}$ is dense, $(\max(0,a),b)$ contains a rational number strictly between $\max(0,a)$ and x (thus, inside $[0,1]$). A must contain this rational number since it is inside $[0,1]$.

Case 2: $0 \leq x < 1$. Then, for all open rectangles (a,b) containing x, the open rectangle $(a, \min(1,b))$ also contains x. Since $\mathbb{Q}$ is dense, $(a,\min(1,b))$ contains a rational number strictly between x and $\min(1,b)$ (thus, inside $(x,1) \subseteq [0,1]$). A must contain this rational number since it is inside $[0,1]$.

In either case, all open rectangles containing x must intersect A. All such rectangles also contain $x \in [0,1]-A$, so all such rectangles intersect $\mathbb{R}-A$.

(Continued on Page 11)

Scratch work — this page will not be read unless you explicitly request it.

3. (Continued)
Thus, we proved that all $x \in [0,1] - A$ are in the boundary of A.

Overall, Steps 1 and 2 showed that all x outside $[0,1] - A$ are not in the boundary of A (since all such x must be outside $[0,1]$ or inside A). Moreover, Step 3 showed that all $x \in [0,1] - A$ are in the boundary of A. We conclude that $[0,1] - A$ equals the boundary of A, as required.

Q4 25

Solve all 4 problems. Write your solutions only where indicated, or write explicitly, "continued on page X".
Neatness counts! Language counts!

63119921-4AD4-4379-8693-DB6E81AD3111
term-test-1-90fcc
#45 9 of 18

Problem 4. Let $f : [0,1]\times[0,1] \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} 1 & x > y \\ 0 & x \le y \end{cases}.$$

Prove that f is integrable on $[0,1]\times[0,1]$ directly by using partitions (namely, without using theorems about continuity and integrability).

Your solution of Problem 4.

For all $n \in \mathbb{N}$ let us define the partition P_n by using the cutpoints $\{0, \frac{1}{n}, \frac{2}{n}, \dots, 1\}$ for the x- and y-directions. In other words, every subrectangle in P will be of the form:

$S_{i,j} = [\frac{i}{n}, \frac{i+1}{n}] \times [\frac{j}{n}, \frac{j+1}{n}] \subseteq \mathbb{R}^2$, $0 \le i, j \le n-1$.

Consider the following 4 cases:

Case 1: $i \le j-1$. Then, for all $(x,y) \in S_{i,j}$, we have $x \le \frac{i+1}{n} \le \frac{j}{n} \le y$, so $f(x,y)=0$. This is true for all $(x,y) \in S_{i,j}$, so $m_{S_{i,j}}(f) = M_{S_{i,j}}(f) = 0$.

The number of pairs (i,j) for which this happens is $0 + 1 + 2 + \dots + (n-1) = \frac{(n-1)n}{2}$.

For $j=0$, For $j=1$, ..., For $j=n-1$

Case 2: $i=j$. Then, $(\frac{i}{n}, \frac{j}{n}) \in S_{i,j}$ satisfies $\frac{i}{n} = \frac{j}{n}$, so $f(\frac{i}{n}, \frac{j}{n}) = 0$. Moreover, $(\frac{i+1}{n}, \frac{j}{n}) \in S_{i,j}$ satisfies $\frac{i+1}{n} = \frac{j+1}{n} > \frac{j}{n}$, so $f(\frac{i+1}{n}, \frac{j}{n}) = 1$.

(Continued on next page)

4. (Continued)

Since f is bounded between 0 and 1, this shows that $m_{S_{i,j}}(f)=0$, $M_{S_{i,j}}(f)=1$. This happens for n pairs (i,j) since $i=j\in\underbrace{\{0,1,\dots,n-1\}}_{n \text{ elements}}$.

Case 3: $i=j+1$. Then, $(\frac{i}{n},\frac{j+1}{n})\in S_{i,j}$ satisfies $\frac{i}{n}=\frac{j+1}{n}$, so $f(\frac{i}{n},\frac{j+1}{n})=0$. Also, $(\frac{i}{n},\frac{j}{n})\in S_{i,j}$ satisfies $\frac{i}{n}=\frac{j+1}{n}>\frac{j}{n}$, so $f(\frac{i}{n},\frac{j}{n})=1$. Since f is bounded between 0 and 1, this shows that $m_{S_{i,j}}(f)=0$, $M_{S_{i,j}}(f)=1$.
This happens for $n-1$ pairs (i,j) since $j\in\{0,1,\dots,n-2\}$. (If $j=n-1$, then $i=n>n-1$.)

Case 4: $i\geq j+2$. Then, all $(x,y)\in S_{i,j}$ satisfy $y\leq\frac{j+1}{n}<\frac{j+2}{n}\leq\frac{i}{n}\leq x$, so $f(x,y)=1$. As a result, $m_{S_{i,j}}(f)=M_{S_{i,j}}(f)=1$. This happens for $\underset{j=n-1}{0}+\underset{j=n-2}{0}+\underset{j=n-3}{1}+2+\cdots+\underset{j=0}{(n-2)}=\frac{(n-2)(n-1)}{2}$ pairs (i,j).

(Continued on Page 12)

Scratch work — this page will not be read unless you explicitly request it.

4. (Continued)

Then, the lower sum is:

$$\begin{aligned}
L(f, P_n) &= \sum_{S_{i,j}} m_{S_{i,j}}(f)\, \mathrm{vol}(S_{i,j}) \\
&= \sum_{S_{i,j}} m_{S_{i,j}}(f) \left(\tfrac{i+1}{n} - \tfrac{i}{n}\right)\left(\tfrac{j+1}{n} - \tfrac{j}{n}\right) \\
&= \frac{1}{n^2} \sum_{S_{i,j}} m_{S_{i,j}}(f) \\
&= \frac{1}{n^2}\Big(\sum_{S_{i,j} \in \text{Case 1}} m_{S_{i,j}}(f) + \sum_{S_{i,j} \in \text{Case 2}} m_{S_{i,j}}(f) \\
&\qquad + \sum_{S_{i,j} \in \text{Case 3}} m_{S_{i,j}}(f) + \sum_{S_{i,j} \in \text{Case 4}} m_{S_{i,j}}(f) \Big) \\
&= \frac{1}{n^2}\left(\frac{(n-1)n}{2} \cdot 0 + n \cdot 0 + (n-1) \cdot 0 + \frac{(n-2)(n-1)}{2} \cdot 1 \right) \\
&= \frac{(n-2)(n-1)}{2n^2} \\
&= \frac{1}{2} - \frac{3}{2n} + \frac{2}{2n^2}
\end{aligned}$$

Thus, $\underline{L}(f) \geq L(f, P_n) = \frac{1}{2} - \frac{3}{2n} + \frac{1}{n^2}$ for all n.
Since $\lim_{n\to\infty} \left(\frac{1}{2} - \frac{3}{2n} + \frac{1}{n^2}\right) = \frac{1}{2}$, this yields $\underline{L}(f) \geq \frac{1}{2}$.

(Continued on Page 13)

Scratch work — this page will not be read unless you explicitly request it.

4. (Continued)

The upper sum is:

$$U(f,P_n) = \sum_{S_{ij}} M_{S_{ij}}(f)\,\mathrm{vol}(S_{ij})$$

$$= \frac{1}{n^2}\sum_{S_{ij}} M_{S_{ij}}(f) \quad (\text{Since } \mathrm{vol}(S_{ij}) = \tfrac{1}{n^2})$$

$$= \frac{1}{n^2}\Big(\sum_{S_{ij}\in \text{Case }1} M_{S_{ij}}(f) + \sum_{S_{ij}\in\text{Case }2} M_{S_{ij}}(f) + \sum_{S_{ij}\in\text{Case }3} M_{S_{ij}}(f) + \sum_{S_{ij}\in\text{Case }4} M_{S_{ij}}(f)\Big)$$

$$= \frac{1}{n^2}\Big(\frac{(n-1)n}{2}\cdot 0 + n\cdot 1 + (n-1)\cdot 1 + \frac{(n-1)(n-2)}{2}\cdot 1\Big)$$

$$= \frac{n}{n^2} + \frac{n-1}{n^2} + \frac{(n-1)(n-2)}{2n^2}$$

$$= \frac{1}{n} + \Big(\frac{1}{n} - \frac{1}{n^2}\Big) + \Big(\frac{1}{2} - \frac{3}{2n} + \frac{1}{n^2}\Big)$$

$$= \frac{1}{2} + \frac{1}{2n}$$

Thus, $U(f) \leq U(f,P_n) = \frac{1}{2} + \frac{1}{2n}$ for all n.
Since $\lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{2n}\right) = \frac{1}{2}$, this shows $U(f) \leq \frac{1}{2}$.
Overall, $U(f) \leq \frac{1}{2} \leq L(f)$. This, combined with $L(f) \leq U(f)$ (theorem in class), proves that $U(f) = L(f)$, so f is integrable by definition, as required.