MAT257 Assignment 16 (Cubes and Chains)

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1. (a) Given that $\partial^2 = 0$, we will show that if $b = \partial c$ for some chains b and c, then $\partial b = 0$. Indeed, we have $\partial b = \partial(\partial c) = \partial^2 c = 0$, as required. $\square$

(b) We will show that the 1-cube $b(t) = (\cos 2\pi t, \sin 2\pi t)$ in $\mathbb{R}^2 - \{0\}$ has $\partial b = 0$, yet it is not the boundary of any 2-chain $c \in C_2(\mathbb{R}^2 - \{0\})$.
First, let a be the 0-cube $a = (1,0)$ in $\mathbb{R}^2 - \{0\}$. Then, $b \circ I^1_{(1,0)} = b(0) = (\cos 0, \sin 0) = (1,0) = a$ and $b \circ I^1_{(1,1)} = b(1) = (\cos 2\pi, \sin 2\pi) = (1,0) = a$. Thus, we can compute ∂b as follows:

$$\begin{aligned}
\partial b &= \sum_{i=1}^{1} \sum_{\alpha \in \{0,1\}} (-1)^{i+\alpha} b \circ I^1_{(i,\alpha)} \\
&= (-1)^{1+0} b \circ I^1_{(1,0)} + (-1)^{1+1} b \circ I^1_{(1,1)} \\
&= -a + a \\
&= 0,
\end{aligned}$$

as required.
Next, assume for contradiction that there exists a 2-chain c such that $b = \partial c$. Let us define the 1-form $\omega := \frac{-ydx + xdy}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 - \{0\})$. Then, we plan to prove that $\int_c d\omega \neq \int_{\partial c} \omega$, contradicting Stokes' Theorem.
First, we can compute $d\omega$ as follows:

$$\begin{aligned}
d\omega &= dx \wedge \frac{\partial}{\partial x}\omega + dy \wedge \frac{\partial}{\partial y}\omega \\
&= dx \wedge \frac{\partial}{\partial x} \frac{-ydx + xdy}{x^2+y^2} + dy \wedge \frac{\partial}{\partial y} \frac{-ydx + xdy}{x^2+y^2} \\
&= dx \wedge \frac{\partial}{\partial x} \frac{-y}{x^2+y^2} dx + dx \wedge \frac{\partial}{\partial x} \frac{x}{x^2+y^2} dy + dy \wedge \frac{\partial}{\partial y} \frac{-y}{x^2+y^2} dx + dy \wedge \frac{\partial}{\partial y} \frac{x}{x^2+y^2} dy \\
&= 0 + \frac{\frac{\partial}{\partial x} x \cdot (x^2+y^2) - x \cdot \frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} dx \wedge dy - \frac{\frac{\partial}{\partial y}(-y) \cdot (x^2+y^2) - (-y) \cdot \frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} dx \wedge dy + 0 \\
&= \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} dx \wedge dy - \frac{-x^2-y^2+2y^2}{(x^2+y^2)^2} dx \wedge dy \\
&= \frac{-x^2+y^2}{(x^2+y^2)^2} dx \wedge dy - \frac{-x^2+y^2}{(x^2+y^2)^2} dx \wedge dy \\
&= 0.
\end{aligned}$$

Since $d\omega = 0$, it follows that $\int_c d\omega = 0$.

Next, we can compute $\int_{\partial c} \omega$ as follows:

$$\begin{aligned}
\int_{\partial c} \omega &= \int_b \omega \\
&= \int_{I^1} b^* \omega \\
&= \int_{I^1} b^* \frac{-ydx + xdy}{x^2 + y^2} \\
&= \int_{I^1} \frac{-\sin 2\pi t d(\cos 2\pi t) + \cos 2\pi t d(\sin 2\pi t)}{\cos^2 2\pi t + \sin^2 2\pi t} \\
&= \int_{I^1} \frac{-\sin 2\pi t \cdot \frac{d}{dt} \cos 2\pi t dt + \cos 2\pi t \cdot \frac{d}{dt} \sin 2\pi t dt}{\cos^2 2\pi t + \sin^2 2\pi t} \\
&= \int_{I^1} \frac{2\pi \sin^2 2\pi t dt + 2\pi \cos^2 2\pi t dt}{\cos^2 2\pi t + \sin^2 2\pi t} \\
&= \int_{I^1} \frac{2\pi dt}{1} \\
&= \int_0^1 2\pi dt \\
&= 2\pi t \Big|_0^1 \\
&= 2\pi \\
&\neq \int_c d\omega.
\end{aligned}$$

Thus, $\int_c d\omega \neq \int_{\partial c} \omega$, contradicting Stokes' Theorem.
Therefore, by contradiction, b cannot be the boundary of any 2-chain on $\mathbb{R}^2 - \{0\}$, as required. $\square$

2. We are given a 1-form $\omega = f dx \in \Omega^1([0,1])$ such that f is smooth and $f(0) = f(1)$. Then, we will show that there is a unique real number λ such that $\omega - \lambda dx$ is of the form dg for some smooth $g : [0,1] \to \mathbb{R}$ for which $g(0) = g(1)$. In fact, we will show that $\omega - \lambda dx$ is of that form if and only if $\lambda = \int_0^1 f(x)dx$.
First, for the "$\Rightarrow$" direction, assume that $\omega - \lambda dx = dg$ for some smooth $g : [0,1] \to \mathbb{R}$ for which $g(0) = g(1)$. By definition of d, we have $dg = \frac{d}{dx} g(x)dx = g'(x)dx$. Then, we can integrate $\omega - \lambda dx = dg$ over the interval $I = [0,1]$ to obtain:

$$\begin{aligned}
\int_I (\omega - \lambda dx) &= \int_I dg \\
\int_I f(x)dx - \int_I \lambda dx &= \int_I g'(x)dx \\
\int_I f(x)dx - \lambda x\Big|_0^1 &= \int_0^1 g'(x)dx \\
\int_I f(x)dx - \lambda &= \int_0^1 g'(x)dx \\
\int_I f(x)dx - \lambda &= g(1) - g(0) \qquad \text{(Fundamental Theorem of Calculus)} \\
\int_I f(x)dx - \lambda &= 0. \qquad \text{(Since } g(1) = g(0)\text{)}
\end{aligned}$$

Thus, $\lambda = \int_I f(x)dx$, as required for the "$\Rightarrow$" direction.
Next, for the "$\Leftarrow$" direction, assume that $\lambda = \int_0^1 f(x)dx$. Then, let us define the smooth function $g : [0,1] \to \mathbb{R}$ by $g(x) := \int_0^x f(t)dt - \lambda x$. Then, we can show that $g(0) = g(1)$ using the following LS-RS proof:

$$\begin{aligned}
\text{LS} &= g(0) & \text{RS} &= g(1) \\
&= \int_0^0 f(t)dt - \lambda \cdot 0 & \text{RS} &= \int_0^1 f(t)dt - \lambda \\
&= 0 - 0 & &= \lambda - \lambda \\
&= 0 & &= 0 \\
&= \text{RS}.
\end{aligned}$$

Thus, LS $=$ RS, so $g(0) = g(1)$, as desired.
Next, we can show that $\omega - \lambda dx = dg$ using the following LS-RS proof:

$$\begin{aligned}
\text{LS} &= \omega - \lambda dx & \text{RS} &= dg \\
&= f(x)dx - \lambda dx & &= \frac{d}{dx} g(x)dx \\
&= (f(x) - \lambda)dx & &= \frac{d}{dx}\left(\int_0^x f(t)dt - \lambda x\right)dx \\
& & &= (f(x) - \lambda)dx \qquad \text{(Fundamental Theorem of Calculus)} \\
& & &= \text{LS}.
\end{aligned}$$

Thus, LS $=$ RS, so $\omega - \lambda dx = dg$, as desired.
Overall, we found a smooth function $g : [0,1] \to \mathbb{R}$ satisfying $g(0) = g(1)$ such that $\omega - \lambda dx = dg$, as required for the "$\Leftarrow$" direction.

Since we proved both directions, we conclude that there is a unique $\lambda \in \mathbb{R}$ such that $\omega - \lambda dx = dg$ for some smooth $g : [0, 1] \to \mathbb{R}$ for which $g(0) = g(1)$, and this value of λ is $\int_0^1 f(x)dx$, as required. $\square$

3. We are given a closed form $\omega \in \Omega^1(\mathbb{R}^2 - \{0\})$, as well as $\eta := \frac{-ydx+xdy}{x^2+y^2}$. Then, we will prove that there is a unique real number λ such that $\omega - \lambda\eta$ is exact.
First, for all $r > 0$, let us define the 1-cube $b_r(t) = (r\cos 2\pi t, r\sin 2\pi t)$ in $\mathbb{R}^2 - \{0\}$. Then, we will prove that $\omega - \lambda\eta$ is exact if and only if $\lambda = \frac{1}{2\pi}\int_{b_1}\omega$.
First, for the "$\Rightarrow$" direction, suppose that $\omega - \lambda\eta$ is exact. Then, $\omega - \lambda\eta = dg$ for some $g \in \Omega^0(\mathbb{R}^2 - \{0\})$.
Next, for all $r > 0$, define the 0-cube $a_r = (r, 0)$. Then, $b_r \circ I^1_{(1,0)} = b_r(0) = (r\cos 0, r\sin 0) = a_r$ and $b_r \circ I^1_{(1,1)} = b_r(1) = (r\cos 2\pi, r\sin 2\pi) = a_r$. As a result, we can compute ∂b_r as follows:

$$\begin{aligned}
\partial b_r &= \sum_{i=1}^{1}\sum_{\alpha\in\{0,1\}}(-1)^{i+\alpha} b_r \circ I^1_{(i,\alpha)} \\
&= (-1)^1 b_r \circ I^1_{(1,0)} + (-1)^2 b_r \circ I^1_{(1,1)} \\
&= -a_r + a_r \\
&= 0.
\end{aligned}$$

It follows that $\int_{\partial b_r} g = 0$. Then, by Stokes' Theorem, we obtain $\int_{b_r} dg = \int_{\partial b_r} g = 0$, so:

$$\begin{aligned}
\int_{b_r} dg &= 0 \\
\int_{b_r}(\omega - \lambda\eta) &= 0 \\
\int_{b_r}\omega - \lambda\int_{b_r}\eta &= 0 \\
\lambda\int_{b_r}\eta &= \int_{b_r}\omega \\
\lambda\int_{b_r}\frac{-ydx+xdy}{x^2+y^2} &= \int_{b_r}\omega \\
\lambda\int_{I^1} b_r^*(\frac{-ydx+xdy}{x^2+y^2}) &= \int_{b_r}\omega \\
\lambda\int_{I^1}\frac{-r\sin 2\pi t d(r\cos 2\pi t) + r\cos 2\pi t d(r\sin 2\pi t)}{r^2\cos^2 2\pi t + r^2\sin^2 2\pi t} &= \int_{b_r}\omega \\
\lambda\int_{I^1}\frac{-r\sin 2\pi t\cdot\frac{d}{dt}r\cos 2\pi t dt + r\cos 2\pi t\cdot\frac{d}{dt}r\sin 2\pi t dt}{r^2\cos^2 2\pi t + r^2\sin^2 2\pi t} &= \int_{b_r}\omega \\
\lambda\int_{I^1}\frac{r^2}{r^2}\frac{2\pi\sin^2 2\pi t dt + 2\pi\cos^2 2\pi t dt}{\cos^2 2\pi t + \sin^2 2\pi t} &= \int_{b_r}\omega \\
\lambda\int_{I^1}\frac{2\pi}{1}dt &= \int_{b_r}\omega \\
\lambda\cdot 2\pi &= \int_{b_r}\omega \\
\lambda &= \frac{1}{2\pi}\int_{b_r}\omega.
\end{aligned}$$

Substituting $r = 1$, we obtain $\lambda = \frac{1}{2\pi}\int_{b_1}\omega$, as required for the "$\Rightarrow$" direction.
Next, for the "$\Leftarrow$" direction, suppose that $\lambda = \frac{1}{2\pi}\int_{b_1}\omega$.

Step 1: For all $r > 0$, we will show that $\int_{b_r} \omega = \int_{b_1} \omega = 2\pi\lambda$.
First, let us define the 2-cube $c_r(t_1, t_2) := (((r-1)t_1 + 1)\cos 2\pi t_2, ((r-1)t_1 + 1)\sin 2\pi t_2)$ in $\mathbb{R}^2 - \{0\}$, and let us define the 1-cube $b'_r(t) := ((r-1)t + 1, 0)$ in $\mathbb{R}^2 - \{0\}$. Then:

- We have for all $t \in [0, 1]$ that:

  $$(c_r \circ I^2_{(1,0)})(t) = c_r(0, t) = (((r-1)(0)+1)\cos 2\pi t, ((r-1)(0)+1)\sin 2\pi t) = (\cos 2\pi t, \sin 2\pi t) = b_1(t),$$

  so $c_r \circ I^2_{(1,0)} = b_1$.
- We have for all $t \in [0, 1]$ that:

  $$(c_r \circ I^2_{(1,1)})(t) = c_r(1, t) = (((r-1)(1)+1)\cos 2\pi t, ((r-1)(1)+1)\sin 2\pi t) = (r\cos 2\pi t, r\sin 2\pi t) = b_r(t),$$

  so $c_r \circ I^2_{(1,1)} = b_r$.
- We have for all $t \in [0, 1]$ that:

  $$(c_r \circ I^2_{(2,0)})(t) = c_r(t, 0) = (((r-1)t+1)\cos 0, ((r-1)t+1)\sin 0) = ((r-1)t+1, 0) = b'_r(t),$$

  so $c_r \circ I^2_{(2,0)} = b'_r$.
- We have for all $t \in [0, 1]$ that:

  $$(c_r \circ I^2_{(2,1)})(t) = c_r(t, 1) = (((r-1)t+1)\cos 2\pi, ((r-1)t+1)\sin 2\pi) = ((r-1)t+1, 0) = b'_r(t),$$

  so $c_r \circ I^2_{(2,1)} = b'_r$.

Then, we can compute ∂c_r as follows:

$$\begin{aligned}
\partial c_r &= \sum_{i=1}^{2} \sum_{\alpha \in \{0,1\}} (-1)^{i+\alpha} c_r \circ I^2_{(i,\alpha)} \\
&= (-1)^1 c_r \circ I^2_{(1,0)} + (-1)^2 c_r \circ I^2_{(1,1)} + (-1)^2 c_r \circ I^2_{(2,0)} + (-1)^3 c_r \circ I^2_{(2,1)} \\
&= -b_1 + b_r + b'_r - b'_r \\
&= b_r - b_1.
\end{aligned}$$

Next, we are given that $d\omega = 0$, so $\int_{c_r} d\omega = 0$. Then, by Stokes' Theorem on chains, we obtain:

$$\begin{aligned}
\int_{\partial c_r} \omega &= \int_{c_r} d\omega \\
\int_{b_r - b_1} \omega &= 0 \\
\int_{b_r} \omega - \int_{b_1} \omega &= 0 \\
\int_{b_r} \omega &= \int_{b_1} \omega,
\end{aligned}$$

as required.
Step 2: We will express ω in polar coordinates.
First, let us write ω in terms of basic 1-forms on $(\mathbb{R}^2 - \{0\})_{(x,y)}$ as $\omega = f_1(x, y)dx + f_2(x, y)dy$.

Next, consider the polar coordinate transformation $p : ((0, \infty) \times \mathbb{R})_{(r,t)} \to (\mathbb{R}^2 - \{0\})_{(x,y)}$ defined by $p(r, t) := (r \cos 2\pi t, r \sin 2\pi t)$. Then, we can compute $p^*\omega$ as follows:

$$\begin{aligned}
p^*\omega &= p^*(f_1(x,y)dx + f_2(x,y)dy) \\
&= f_1(r\cos 2\pi t, r\sin 2\pi t)d(r\cos 2\pi t) + f_2(r\cos 2\pi t, r\sin 2\pi t)d(r\sin 2\pi t) \\
&= f_1(r\cos 2\pi t, r\sin 2\pi t)(\frac{\partial}{\partial r} r\cos 2\pi t dr + \frac{\partial}{\partial t} r\cos 2\pi t dt) \\
&\quad + f_2(r\cos 2\pi t, r\sin 2\pi t)(\frac{\partial}{\partial r} r\sin 2\pi t dr + \frac{\partial}{\partial t} r\sin 2\pi t dt) \\
&= f_1(r\cos 2\pi t, r\sin 2\pi t)(\cos 2\pi t dr - 2\pi r\sin 2\pi t dt) + f_2(r\cos 2\pi t, r\sin 2\pi t)(\sin 2\pi t dr + 2\pi r\cos 2\pi t dt) \\
&= (f_1(r\cos 2\pi t, r\sin 2\pi t)\cos 2\pi t + f_2(r\cos 2\pi t, r\sin 2\pi t)\sin 2\pi t)dr \\
&\quad + 2\pi r(f_2(r\cos 2\pi t, r\sin 2\pi t)\cos 2\pi t - f_1(r\cos 2\pi t, r\sin 2\pi t)\sin 2\pi t)dt.
\end{aligned}$$

In other words, if we define the smooth functions $h_1, h_2 : ((0, \infty) \times \mathbb{R})_{(r,t)}$ by:

$$h_1(r,t) := f_1(r\cos 2\pi t, r\sin 2\pi t)\cos 2\pi t + f_2(r\cos 2\pi t, r\sin 2\pi t)\sin 2\pi t,$$

$$h_2(r,t) := 2\pi r(f_2(r\cos 2\pi t, r\sin 2\pi t)\cos 2\pi t - f_1(r\cos 2\pi t, r\sin 2\pi t)\sin 2\pi t),$$

then we obtain:

$$p^*\omega = h_1(r,t)dr + h_2(r,t)dt.$$

Step 3: We will show that $\frac{\partial h_2(r,t)}{\partial r} = \frac{\partial h_1(r,t)}{\partial t}$.
Since we are given that $d\omega = 0$, we obtain:

$$\begin{aligned}
p^*(d\omega) &= 0 \\
d(p^*\omega) &= 0 \\
d(h_1(r,t)dr + h_2(r,t)dt) &= 0 \\
dr \wedge \frac{\partial}{\partial r}(h_1(r,t)dr + h_2(r,t)dt) + dt \wedge \frac{\partial}{\partial t}(h_1(r,t)dr + h_2(r,t)dt) &= 0 \\
\frac{\partial h_1(r,t)}{\partial r} dr \wedge dr + \frac{\partial h_2(r,t)}{\partial r} dr \wedge dt + \frac{\partial h_1(r,t)}{\partial t} dt \wedge dr + \frac{\partial h_2(r,t)}{\partial t} dt \wedge dt &= 0 \\
0 + \frac{\partial h_2(r,t)}{\partial r} dr \wedge dt - \frac{\partial h_1(r,t)}{\partial t} dr \wedge dt + 0 &= 0 \\
\frac{\partial h_2(r,t)}{\partial r} dr \wedge dt &= \frac{\partial h_1(r,t)}{\partial t} dr \wedge dt.
\end{aligned}$$

Thus, $\frac{\partial h_2(r,t)}{\partial r} = \frac{\partial h_1(r,t)}{\partial t}$, as desired.
Step 4: We will expand $\lambda = \frac{1}{2\pi}\int_{b_r} \omega$ as follows:

$$\begin{aligned}
\lambda &= \frac{1}{2\pi}\int_{b_r} \omega \\
&= \frac{1}{2\pi}\int_{I^1} b_r^*\omega \\
&= \frac{1}{2\pi}\int_{I^1} b_r^*(f_1(x,y)dx + f_2(x,y)dy) \\
&= \frac{1}{2\pi}\int_{I^1} f_1(r\cos 2\pi t, r\sin 2\pi t)d(r\cos 2\pi t) + f_2(r\cos 2\pi t, r\sin 2\pi t)d(r\sin 2\pi t)
\end{aligned}$$

Note that we treat r as a constant, so we obtain:

$$\begin{aligned}
\lambda &= \frac{1}{2\pi}\int_{I^1} f_1(r\cos 2\pi t, r\sin 2\pi t)\cdot(-2\pi r\sin 2\pi t)dt + f_2(r\cos 2\pi t, r\sin 2\pi t)\cdot(2\pi r\cos 2\pi t)dt \\
&= \frac{1}{2\pi}\int_0^1 2\pi r(f_2(r\cos 2\pi t, r\sin 2\pi t)\cos 2\pi t - f_1(r\cos 2\pi t, r\sin 2\pi t)\sin 2\pi t)dt. \\
&= \frac{1}{2\pi}\int_0^1 h_2(r,t)dt.
\end{aligned}$$

Step 5: We will construct a candidate for $g \in \Omega^0(\mathbb{R}^2 - \{0\})$ such that $\omega - \lambda\eta = dg$.
Let us define $g : \mathbb{R}^2 - \{0\} \to \mathbb{R}$ as follows: For all $(x,y) \in \mathbb{R}^2 - \{0\}$, write (x,y) in the form $(r\cos 2\pi t, r\sin 2\pi t)$, where $r > 0$, then define:

$$g(r\cos 2\pi t, r\sin 2\pi t) := \int_0^t h_2(r,s)ds - 2\pi\lambda t + \int_1^r h_1(s,0)ds.$$

We must check that g is well-defined. In other words, for all $(r_1,t_1),(r_2,t_2) \in (0,\infty)\times\mathbb{R}$ such that $(r_1\cos 2\pi t_1, r_1\sin 2\pi t_1) = (r_2\cos 2\pi t_2, r_2\sin 2\pi t_2)$, we must show that:

$$\int_0^{t_1} h_2(r_1,s)ds - 2\pi\lambda t_1 + \int_1^{r_1} h_1(s,0)ds = \int_0^{t_2} h_2(r_2,s) - 2\pi\lambda t_2 + \int_1^{r_2} h_1(s,0)ds.$$

First, we have:

$$\begin{aligned}
r_1^2 &= r_1^2(\cos^2 2\pi t_1 + \sin^2 2\pi t_1) \\
&= (r_1\cos 2\pi t_1)^2 + (r_1\sin 2\pi t_1)^2 \\
&= (r_2\cos 2\pi t_2)^2 + (r_2\sin 2\pi t_2)^2 \\
&= r_2^2(\cos^2 2\pi t_2 + \sin^2 2\pi t_2) \\
&= r_2^2.
\end{aligned}$$

Since $r_1^2 = r_2^2$, and since $r_1, r_2 > 0$, it follows that $r_1 = r_2$.
Additionally, from $r_1\cos 2\pi t_1 = r_2\cos 2\pi t_2$, since $r_1 = r_2 > 0$, we obtain $\cos 2\pi t_1 = \cos 2\pi t_2$. Similarly, from $r_1\sin 2\pi t_1 = r_2\sin 2\pi t_2$, we obtain $\sin 2\pi t_1 = \sin 2\pi t_2$. Overall, we obtain that $2\pi t_1 = 2\pi t_2 + 2\pi k = 2\pi(t_2+k)$ for some $k \in \mathbb{Z}$, so $t_1 = t_2 + k$. We may assume without loss of generality that $k \geq 0$; otherwise, we may swap (r_1,t_1) with (r_2,t_2) and proceed in the same way. As a result, we obtain:

$$\begin{aligned}
&\int_0^{t_1} h_2(r_1,s)ds - 2\pi\lambda t_1 + \int_1^{r_1} h_1(s,0)ds \\
&= \int_0^{t_2+k} h_2(r_2,s)ds - 2\pi\lambda(t_2+k) + \int_1^{r_2} h_1(s,0)ds \\
&= \sum_{j=0}^{k-1}\int_j^{j+1} h_2(r_2,s)ds - 2\pi\lambda k + \int_k^{t_2+k} h_2(r_2,s)ds - 2\pi\lambda t_2 + \int_1^{r_2} h_1(s,0)ds
\end{aligned}$$

For the first k integrals, let us apply the u-substitution $u = s - j$, with $ds = du$. With this substitution, we have:

$$\begin{aligned}
&h_2(r_2, s) \\
&= 2\pi r_2(f_2(r_2\cos 2\pi s, r_2\sin 2\pi s)\cos 2\pi s - f_1(r_2\cos 2\pi s, r_2\sin 2\pi s)\sin 2\pi s) \\
&= 2\pi r_2(f_2(r_2\cos(2\pi u + 2\pi j), r_2\sin(2\pi u + 2\pi j))\cos(2\pi u + 2\pi j) \\
&\quad - f_1(r_2\cos(2\pi u + 2\pi j), r_2\sin(2\pi u + 2\pi j))\sin(2\pi u + 2\pi j)) \\
&= 2\pi r_2(f_2(r_2\cos 2\pi u, r_2\sin 2\pi u)\cos 2\pi u - f_1(r_2\cos 2\pi u, r_2\sin 2\pi u)\sin 2\pi u) \\
&= h_2(r_2, u).
\end{aligned}$$

Similarly, for the second-last integral, let us apply the u-substitution $u = s - k$, with $ds = du$ and $h_2(r_2, s) = h_2(r_2, u)$. Then, we obtain:

$$\begin{aligned}
&\int_0^{t_1} h_2(r_1, s)ds - 2\pi\lambda t_1 + \int_1^{r_1} h_1(s, 0)ds \\
&= \sum_{j=0}^{k-1}\int_0^1 h_2(r_2, u)du - 2\pi\lambda k + \int_0^{t_2} h_2(r_2, u)du - 2\pi\lambda t_2 + \int_1^{r_2} h_1(s, 0)ds \\
&= \sum_{j=0}^{k-1} 2\pi\lambda - 2\pi\lambda k + \int_0^{t_2} h_2(r_2, u)du - 2\pi\lambda t_2 + \int_1^{r_2} h_1(s, 0)ds \qquad \text{(Applying Step 4)} \\
&= 2\pi\lambda k - 2\pi\lambda k + \int_0^{t_2} h_2(r_2, u)du - 2\pi\lambda t_2 + \int_1^{r_2} h_1(s, 0)ds \\
&= \int_0^{t_2} h_2(r_2, s)ds - 2\pi\lambda t_2 + \int_1^{r_2} h_1(s, 0)ds.
\end{aligned}$$

Therefore, g is well-defined, as required.
Step 6: We will obtain and solve a system of linear equations for $\frac{\partial g(x,y)}{\partial x}$ and $\frac{\partial g(x,y)}{\partial y}$.
First, given $(x, y) \in \mathbb{R}^2 - \{0\}$, let us write (x, y) in the form $(r\cos 2\pi t, r\sin 2\pi t)$, where $r > 0$. Then, we can compute $\frac{\partial g(r\cos 2\pi t, r\sin 2\pi t)}{\partial t}$ as follows:

$$\begin{aligned}
&\frac{\partial g(r\cos 2\pi t, r\sin 2\pi t)}{\partial t} \\
&= \frac{\partial}{\partial t}(\int_0^t h_2(r, s)ds - 2\pi\lambda t + \int_1^r h_1(s, 0)ds) \\
&= h_2(r, t) - 2\pi\lambda + 0 \qquad \text{(Fundamental Theorem of Calculus)} \\
&= h_2(r, t) - 2\pi\lambda.
\end{aligned}$$

On the other hand, we also have:

$$\begin{aligned}
&\frac{\partial g(r\cos 2\pi t, r\sin 2\pi t)}{\partial t} \\
&= \frac{\partial g(x, y)}{\partial x}\cdot\frac{\partial r\cos 2\pi t}{\partial t} + \frac{\partial g(x, y)}{\partial y}\cdot\frac{\partial r\sin 2\pi t}{\partial t} \qquad \text{(Chain rule)} \\
&= -2\pi r\sin 2\pi t\frac{\partial g(x, y)}{\partial x} + 2\pi r\cos 2\pi t\frac{\partial g(x, y)}{\partial y}.
\end{aligned}$$

As a result, we obtain:

$$-2\pi r\sin 2\pi t\frac{\partial g(x, y)}{\partial x} + 2\pi r\cos 2\pi t\frac{\partial g(x, y)}{\partial y} = h_2(r, t) - 2\pi\lambda. \tag{1}$$

Next, we can compute $\frac{\partial g(r\cos 2\pi t, r\sin 2\pi t)}{\partial r}$ as follows:

$$\begin{aligned}
&\frac{\partial g(r\cos 2\pi t, r\sin 2\pi t)}{\partial r} \\
&= \frac{\partial}{\partial r}\left(\int_0^t h_2(r,s)ds - 2\pi\lambda t + \int_1^r h_1(s,0)ds\right) \\
&= \frac{\partial}{\partial r}\int_0^t h_2(r,s)ds - 0 + h_1(r,0) \qquad \text{(Fundamental Theorem of Calculus)} \\
&= \int_0^t \frac{\partial}{\partial r} h_2(r,s)ds + h_1(r,0) \qquad \text{(Applying Assignment 8 Question 3, since } h_2 \text{ is smooth)} \\
&= \int_0^t \frac{\partial}{\partial s} h_1(r,s)ds + h_1(r,0) \qquad \text{(Applying Step 3)} \\
&= h_1(r,t). \qquad \text{(Fundamental Theorem of Calculus)}
\end{aligned}$$

On the other hand, we also have:

$$\begin{aligned}
&\frac{\partial g(r\cos 2\pi t, r\sin 2\pi t)}{\partial r} \\
&= \frac{\partial g(x,y)}{\partial x}\cdot\frac{\partial r\cos 2\pi t}{\partial r} + \frac{\partial g(x,y)}{\partial y}\cdot\frac{\partial r\sin 2\pi t}{\partial r} \qquad \text{(Chain rule)} \\
&= \cos 2\pi t\frac{\partial g(x,y)}{\partial x} + \sin 2\pi t\frac{\partial g(x,y)}{\partial y}.
\end{aligned}$$

As a result, we obtain:

$$\cos 2\pi t\frac{\partial g(x,y)}{\partial x} + \sin 2\pi t\frac{\partial g(x,y)}{\partial y} = h_1(r,t). \tag{2}$$

Next, to solve for $\frac{\partial g(x,y)}{\partial x}$, let us subtract $\sin 2\pi t$ times (1) from $2\pi r\cos 2\pi t$ times (2). On the left-hand side, we obtain:

$$\begin{aligned}
\text{LS} &= 2\pi r\cos 2\pi t\left(\cos 2\pi t\frac{\partial g(x,y)}{\partial x} + \sin 2\pi t\frac{\partial g(x,y)}{\partial y}\right) - \sin 2\pi t\left(-2\pi r\sin 2\pi t\frac{\partial g(x,y)}{\partial x} + 2\pi r\cos 2\pi t\frac{\partial g(x,y)}{\partial y}\right) \\
&= 2\pi r\left(\cos^2 2\pi t\frac{\partial g(x,y)}{\partial x} + \cos 2\pi t\sin 2\pi t\frac{\partial g(x,y)}{\partial y} + \sin^2 2\pi t\frac{\partial g(x,y)}{\partial x} - \sin 2\pi t\cos 2\pi t\frac{\partial g(x,y)}{\partial y}\right) \\
&= 2\pi r\left((\cos^2 2\pi t + \sin^2 2\pi t)\frac{\partial g(x,y)}{\partial x} + 0\frac{\partial g(x,y)}{\partial y}\right) \\
&= 2\pi r\frac{\partial g(x,y)}{\partial x}.
\end{aligned}$$

On the right-hand side, we obtain:

$$\text{RS} = 2\pi r\cos 2\pi t h_1(r,t) - \sin 2\pi t(h_2(r,t) - 2\pi\lambda).$$

As a result, we obtain:

$$\begin{aligned}
2\pi r\frac{\partial g(x,y)}{\partial x} &= 2\pi r\cos 2\pi t h_1(r,t) - \sin 2\pi t(h_2(r,t) - 2\pi\lambda)\\
\frac{\partial g(x,y)}{\partial x} &= \cos 2\pi t h_1(r,t) - \frac{1}{2\pi r}\sin 2\pi t h_2(r,t) + \frac{\sin 2\pi t}{r}\lambda\\
\frac{\partial g(x,y)}{\partial x} &= \cos 2\pi t(f_1(r\cos 2\pi t, r\sin 2\pi t)\cos 2\pi t + f_2(r\cos 2\pi t, r\sin 2\pi t)\sin 2\pi t)\\
&\qquad - \sin 2\pi t(f_2(r\cos 2\pi t, r\sin 2\pi t)\cos 2\pi t - f_1(r\cos 2\pi t, r\sin 2\pi t)\sin 2\pi t) + \frac{\sin 2\pi t}{r}\lambda\\
\frac{\partial g(x,y)}{\partial x} &= \frac{x}{r}(f_1(x,y)\frac{x}{r} + f_2(x,y)\frac{y}{r}) - \frac{y}{r}(f_2(x,y)\frac{x}{r} - f_1(x,y)\frac{y}{r}) + \frac{y}{r^2}\lambda\\
\frac{\partial g(x,y)}{\partial x} &= \frac{x^2+y^2}{r^2}f_1(x,y) + 0f_2(x,y) + \frac{y}{x^2+y^2}\lambda\\
\frac{\partial g(x,y)}{\partial x} &= f_1(x,y) + \frac{y}{x^2+y^2}\lambda.
\end{aligned}$$

Next, to solve for $\frac{\partial g(x,y)}{\partial y}$, let us add $\cos 2\pi t$ times (1) to $2\pi r\sin 2\pi t$ times (2). On the left-hand side, we obtain:

$$\begin{aligned}
\text{LS} &= 2\pi r\sin 2\pi t(\cos 2\pi t\frac{\partial g(x,y)}{\partial x} + \sin 2\pi t\frac{\partial g(x,y)}{\partial y}) + \cos 2\pi t(-2\pi r\sin 2\pi t\frac{\partial g(x,y)}{\partial x} + 2\pi r\cos 2\pi t\frac{\partial g(x,y)}{\partial y})\\
&= 2\pi r(\sin 2\pi t\cos 2\pi t\frac{\partial g(x,y)}{\partial x} + \sin^2 2\pi t\frac{\partial g(x,y)}{\partial y} - \sin 2\pi t\cos 2\pi t\frac{\partial g(x,y)}{\partial x} + \cos^2 2\pi t\frac{\partial g(x,y)}{\partial y})\\
&= 2\pi r(0\frac{\partial g(x,y)}{\partial x} + (\sin^2 2\pi t + \cos^2 2\pi t)\frac{\partial g(x,y)}{\partial y})\\
&= 2\pi r\frac{\partial g(x,y)}{\partial y}.
\end{aligned}$$

On the right-hand side, we obtain:

$$\text{RS} = 2\pi r\sin 2\pi t h_1(r,t) + \cos 2\pi t(h_2(r,t) - 2\pi\lambda).$$

As a result, we obtain:

$$\begin{aligned}
2\pi r\frac{\partial g(x,y)}{\partial y} &= 2\pi r\sin 2\pi t h_1(r,t) + \cos 2\pi t(h_2(r,t) - 2\pi\lambda)\\
\frac{\partial g(x,y)}{\partial y} &= \sin 2\pi t h_1(r,t) + \frac{1}{2\pi r}\cos 2\pi t h_2(r,t) - \frac{\cos 2\pi t}{r}\lambda\\
\frac{\partial g(x,y)}{\partial y} &= \sin 2\pi t(f_1(r\cos 2\pi t, r\sin 2\pi t)\cos 2\pi t + f_2(r\cos 2\pi t, r\sin 2\pi t)\sin 2\pi t)\\
&\qquad + \cos 2\pi t(f_2(r\cos 2\pi t, r\sin 2\pi t)\cos 2\pi t - f_1(r\cos 2\pi t, r\sin 2\pi t)\sin 2\pi t) - \frac{\cos 2\pi t}{r}\lambda\\
\frac{\partial g(x,y)}{\partial y} &= \frac{y}{r}(f_1(x,y)\frac{x}{r} + f_2(x,y)\frac{y}{r}) + \frac{x}{r}(f_2(x,y)\frac{x}{r} - f_1(x,y)\frac{y}{r}) - \frac{x}{r^2}\lambda\\
\frac{\partial g(x,y)}{\partial y} &= 0f_1(x,y) + \frac{y^2+x^2}{r^2}f_2(x,y) - \frac{x}{x^2+y^2}\lambda\\
\frac{\partial g(x,y)}{\partial y} &= f_2(x,y) - \frac{x}{x^2+y^2}\lambda.
\end{aligned}$$

Step 7: We will conclude that $\omega - \lambda\eta = dg$ as follows:

$$\begin{aligned}
\text{RS} &= dg \\
&= \frac{\partial g(x,y)}{\partial x}dx + \frac{\partial g(x,y)}{\partial y}dy \\
&= (f_1(x,y) + \frac{y}{x^2+y^2}\lambda)dx + (f_2(x,y) - \frac{x}{x^2+y^2}\lambda)dy \\
&= (f_1(x,y)dx + f_2(x,y)dy) + \frac{ydx - xdy}{x^2+y^2}\lambda \\
&= \omega - \lambda\eta.
\end{aligned}$$

Therefore, $\omega - \lambda\eta$ is exact, as required for the "$\Leftarrow$" direction.
Since we proved both directions, we conclude that there is a unique $\lambda \in \mathbb{R}$ such that $\omega - \lambda\eta$ is exact, and this value of λ is $\frac{1}{2\pi}\int_{b_1}\omega$, as required. $\square$

Notes on intuition

Now, let us develop some intuition on how to approach these problems and motivate these solutions. (Note: This section was not submitted for grading.)

1. (a) I really hope we all know how to solve this question.
   (b) The proof that $\partial b = 0$ is mostly computational, so the real challenge was to prove that b cannot be written as ∂c for any $c \in C_2(\mathbb{R}^2 - \{0\})$. Let us assume for contradiction $b = \partial c$ for some c. Then, to apply Stokes' theorem, we first need to pick a useful form $\omega \in \Omega^1(\mathbb{R}^2 - \{0\})$, and then we obtain $\int_b \omega = \int_{\partial c} \omega = \int_c d\omega$. To search for a useful ω, we first observe that we do not know much about c, so it would be nice if ω is closed so that $\int_c d\omega = 0$ regardless of c. Then, to obtain a contradiction, we need $\int_b \omega \neq 0$. Here, a helpful insight is that ω cannot be exact; otherwise, if $\omega = d\eta$ for some $\eta \in \Omega^0(\mathbb{R}^2 - \{0\})$, then Stokes' theorem tells us that $\int_b \omega = \int_b d\eta = \int_{\partial b} \eta = \int_{\partial^2 c} \eta = \int_0 \eta = 0$. This narrows our search further: ω should be closed but not exact.
   The next step is to recall encountering Problem 8 from last year's Test 3 rejects while studying for Test 3. That question asked to prove that the form $\omega = \frac{-ydx+xdy}{x^2+y^2} \in \Omega^1(\mathbb{R}^2 - \{0\})$ is closed but not exact. (Alternatively, Question 3 from this assignment may also have helped you realize that $\frac{-ydx+xdy}{x^2+y^2}$ is closed but not exact, since it was used to adjust a closed-but-not-exact form into a closed-and-exact form.) Then, the discussion above motivates us to pick this form for ω. Indeed, after explicitly computing that $d\omega = 0$ and that $\int_b \omega \neq 0$, we are now done.

2. As stated in the textbook's hint, the first step is to integrate the equation $\omega - \lambda dx = dg$ across $[0,1]$ to find λ. Next, the key insight is to generalize this hint: Integrating across a smaller interval $[0,x]$ gives us $\int_0^x f(t)dt - \lambda x = g(x) - g(0)$, which allows us to find $g(x)$ up to a constant. This helps us make the educated guess that $g(x) = \int_0^x f(t)dt - \lambda x$. After this, we finish with some simple computations.
   Alternatively, we can rewrite the equation $\omega - \lambda dx = dg$ in terms of the elementary 1-form dx to obtain $(f(x) - \lambda)dx = \frac{dg(x)}{dt}dx$, so $f(x) - \lambda = \frac{dg(x)}{dt}$. Then, $g(x)$ should be an antiderivative of $f(x) - \lambda$, which also motivates the formula $g(x) := \int_0^x f(t)dt - \lambda x$.

3. This question was quite difficult, and admittedly, my solution was also rather long and complicated. Let us break down the solution into steps and examine the motivation for each step.
   Step 0: (i.e., proving the "$\Rightarrow$" direction): This question can be solved similarly to Question 2. The main difficulty is determining which "interval" to integrate along. Since η is often written as "$d\theta$", this motivates us to integrate in an "angular direction", so we integrate along a circle centred at the origin. To help us integrate along such a circle, we define the 1-cube b_r to map $[0,1]$ onto the circle. Since we learned in Question 1 that $\partial b_r = 0$, this motivates us to use Stokes' theorem to obtain $\int_{b_r} dg = \int_{\partial b_r} g = 0$. Finally, we follow "simple" computations to expand and evaluate $\int_{b_r} dg$, and then derive our value for λ.
   Step 1: The textbook's hint essentially tells us to do this step. Another motivation for this step is that, in our solution for Step 0, it did not matter whether we use a radius of 1 or any other radius, so we should obtain the same value of λ for any radius, so each $\frac{1}{2\pi}\int_{b_r} \omega$ should equal the same λ. Our technique for proving this using c_r is somewhat motivated by the proof that $\partial b_r = 0$. We obtained $\partial b_r = 0$ because the two ends of b_r closed in on themselves at $(1,0)$; similarly, c_r forms an annulus that closes in on itself on the positive x-axis, and we are left with two faces, b_1 and b_r, so $\partial c_r = b_r - b_1$. Then, we can compare $\int_{b_r} \omega$ with $\int_{b_1} \omega$ using $\int_{\partial c_r} \omega$. Finally, we prove

$\int_{\partial c_r} \omega = 0$ using the standard technique of applying Stokes' theorem.
Step 2: Since we have "$\eta = d\theta$", it makes sense to write ω in terms of polar coordinates as well.
Step 3: We want to use the fact that ω is closed, so we expand $p^*(d\omega) = 0$, and it happens to give us $\frac{\partial h_2(r,t)}{\partial r} = \frac{\partial h_1(r,t)}{\partial t}$. This happens to be useful in Step 6.
Step 4: Similarly to Step 2, we want to re-express λ in terms of the polar coordinates obtained in Step 2.
Step 5: Our definition for g for this problem is motivated by our definition for g in Question 2. In Question 2, we defined g by integrating $\omega - \lambda dx$, so for this problem, we obtain a similar definition for g by "integrating $\omega - \lambda\eta$". This time, there are two directions to integrate along: the angular direction, then the radial direction. Here, we see why it was convenient to find h_1 and h_2 from Step 2: Now, we can easily integrate ω along the angular and radial directions by integrating h_2 and h_1, respectively.
However, a problem could potentially occur if, after integrating in the angular direction along an entire circle, the final value at $t = 1$ does not equal the initial value at $t = 0$, meaning that g is not continuous. Fortunately, applying our work from Step 4, we find that the correctional term $-2\pi\lambda t$ makes g well-defined. Moreover, because of our work from Step 1, the same g is well-defined at all possible radii r, not just on the unit circle.
Step 6: My approach for this step is heavily motivated by Assignment 15 Question 3, where we were also able to find an unknown form by creating a system of linear equations using known forms.
Step 7: We are finally done! Yay!