MAT257 Assignment 15 (d)

(Author's name here)

March 4, 2022

1. For all vector fields $F = F_1e_1 + F_2e_2 + F_3e_3$ on $\mathbb{R}^3$, we define:

$$\omega^1_F := F_1dx + F_2dy + F_3dz,$$

$$\omega^2_F := F_1dy \wedge dz + F_2dz \wedge dx + F_3dx \wedge dy.$$

(a) We are given a function $f : \mathbb{R}^3 \to \mathbb{R}$ and a vector field $F = F_1e_1+F_2e_2+F_3e_3$ on $\mathbb{R}^3$. First, we will prove that $df = \omega^1_{\operatorname{grad} f}$. To begin, we know by definition that $\operatorname{grad} f = \frac{\partial f}{\partial x}e_1 + \frac{\partial f}{\partial y}e_2 + \frac{\partial f}{\partial z}e_3$. As a result,

$$\begin{aligned}\omega^1_{\operatorname{grad} f} &= \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \\ &= dx \wedge \frac{\partial}{\partial x}f + dy \wedge \frac{\partial}{\partial y}f + dz \wedge \frac{\partial}{\partial z}f \\ &= df,\end{aligned}$$

so $df = \omega^1_{\operatorname{grad} f}$, as required.

Additionally, we will prove that $d(\omega^1_F) = \omega^2_{\operatorname{curl} F}$. To begin, we are given by definition that $\operatorname{curl} F = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})e_1 + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})e_2 + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})e_3$. As a result:

$$\omega^2_{\operatorname{curl} F} = (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})dy \wedge dz + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})dz \wedge dx + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dx \wedge dy.$$

Moreover, we can compute $d(\omega^1_F)$ as follows:

$$\begin{aligned}d(\omega^1_F) &= d(F_1dx + F_2dy + F_3dz) \\ &= dx \wedge \frac{\partial}{\partial x}(F_1dx + F_2dy + F_3dz) + dy \wedge \frac{\partial}{\partial y}(F_1dx + F_2dy + F_3dz) \\ &\quad + dz \wedge \frac{\partial}{\partial z}(F_1dx + F_2dy + F_3dz) \\ &= \frac{\partial F_1}{\partial x}dx \wedge dx + \frac{\partial F_2}{\partial x}dx \wedge dy + \frac{\partial F_3}{\partial x}dx \wedge dz + \frac{\partial F_1}{\partial y}dy \wedge dx + \frac{\partial F_2}{\partial y}dy \wedge dy + \frac{\partial F_3}{\partial y}dy \wedge dz \\ &\quad + \frac{\partial F_1}{\partial z}dz \wedge dx + \frac{\partial F_2}{\partial z}dz \wedge dy + \frac{\partial F_3}{\partial z}dz \wedge dz \\ &= 0 + \frac{\partial F_2}{\partial x}dx \wedge dy - \frac{\partial F_3}{\partial x}dz \wedge dx - \frac{\partial F_1}{\partial y}dx \wedge dy + 0 + \frac{\partial F_3}{\partial y}dy \wedge dz + \frac{\partial F_1}{\partial z}dz \wedge dx - \frac{\partial F_2}{\partial z}dy \wedge dz + 0 \\ &= (\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z})dy \wedge dz + (\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x})dz \wedge dx + (\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y})dx \wedge dy \\ &= \omega^2_{\operatorname{curl} F}.\end{aligned}$$

Therefore, $d(\omega^1_F) = \omega^2_{\operatorname{curl} F}$, as required.

Finally, we will prove that $d(\omega^2_F) = (\operatorname{div} F)dx \wedge dy \wedge dz$. To begin, we know by definition that

$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$. Moreover, we can compute $d(\omega_F^2)$ as follows:

$$\begin{aligned}
d(\omega_F^2) &= d(F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy) \\
&= dx \wedge \frac{\partial}{\partial x}(F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy) \\
&\quad + dy \wedge \frac{\partial}{\partial y}(F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy) \\
&\quad + dz \wedge \frac{\partial}{\partial z}(F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy) \\
&= \frac{\partial F_1}{\partial x} dx \wedge dy \wedge dz + \frac{\partial F_2}{\partial x} dx \wedge dz \wedge dx + \frac{\partial F_3}{\partial x} dx \wedge dx \wedge dy \\
&\quad + \frac{\partial F_1}{\partial y} dy \wedge dy \wedge dz + \frac{\partial F_2}{\partial y} dy \wedge dz \wedge dx + \frac{\partial F_3}{\partial y} dy \wedge dx \wedge dy \\
&\quad + \frac{\partial F_1}{\partial z} dz \wedge dy \wedge dz + \frac{\partial F_2}{\partial z} dz \wedge dz \wedge dx + \frac{\partial F_3}{\partial z} dz \wedge dx \wedge dy \\
&= \frac{\partial F_1}{\partial x} dx \wedge dy \wedge dz + 0 + 0 + 0 + \frac{\partial F_2}{\partial y} dx \wedge dy \wedge dz + 0 + 0 + 0 + \frac{\partial F_3}{\partial z} dx \wedge dy \wedge dz \\
&= (\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}) dx \wedge dy \wedge dz \\
&= (\operatorname{div} F) dx \wedge dy \wedge dz.
\end{aligned}$$

Therefore, $d(\omega_F^2) = (\operatorname{div} F) dx \wedge dy \wedge dz$, as required. $\square$

(b) Next, we will use part (a) to prove that $\operatorname{curl} \operatorname{grad} f = 0$.
Applying part (a) twice, we have $\omega^2_{\operatorname{curl} \operatorname{grad} f} = d(\omega^1_{\operatorname{grad} f}) = d(df) = 0$. Next, if we write $\operatorname{curl} \operatorname{grad} f = F_1 e_1 + F_2 e_2 + F_3 e_3$, then we obtain:

$$0 = \omega^2_{\operatorname{curl} \operatorname{grad} f} = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy.$$

Since $dy \wedge dz, dz \wedge dx, dx \wedge dy$ are linearly independent, it follows that $F_1 = F_2 = F_3 = 0$, so $\operatorname{curl} \operatorname{grad} f = 0$, as required.
Finally, we will use part (a) to prove that $\operatorname{div} \operatorname{curl} F = 0$.
Applying part (a) twice, we have $(\operatorname{div} \operatorname{curl} F) dx \wedge dy \wedge dz = d(\omega^2_{\operatorname{curl} F}) = d(d(\omega^1_F)) = 0$. Since $dx \wedge dy \wedge dz$ is nonzero, it follows that $\operatorname{div} \operatorname{curl} F = 0$, as required. $\square$

2. We are given two open sets $U, V \subseteq \mathbb{R}^n$ which are diffeomorphic, so there exists a differentiable function $g : U \to V$ with a differentiable inverse $g^{-1} : V \to U$. We are also given that every closed form on U is exact. Then, we will show that every closed form on V is also exact.
   Let ω be any arbitrary closed form on V. Then, $d\omega = 0$. Next, consider the form $g^*\omega$ on U. Applying Spivak's Theorem 4-10(4), we obtain $d(g^*\omega) = g^*(d\omega) = g^*(0) = 0$, so $g^*\omega$ is a closed form on U. Since we are given that every closed form on U is exact, it follows that $g^*\omega = d\eta$ for some form η on U. Then, we apply $(g^{-1})^*$ to both sides to obtain:

$$\begin{aligned}
(g^{-1})^*(g^*\omega) &= (g^{-1})^*(d\eta) \\
((g^{-1})^* \circ g^*)\omega &= d((g^{-1})^*\eta) \qquad \text{(Applying Spivak's Theorem 4-10(4))} \\
(g \circ g^{-1})^*\omega &= d((g^{-1})^*\eta) \\
\omega &= d((g^{-1})^*\eta).
\end{aligned}$$

   Since $(g^{-1})^*\eta$ is a form on V, this proves that ω is exact. Therefore, every closed form on V is exact, as required. $\square$

3. Given the polar coordinates $x = r\cos\theta, y = r\sin\theta$ in $\mathbb{R}^2_{(x,y)}$, we will prove that:

$$d\theta = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$$

where θ is defined.
First, we can express dx in terms of dr and $d\theta$ as follows:

$$\begin{aligned}
dx &= d(r\cos\theta) \\
&= dr \wedge \cos\theta + r \wedge d(\cos\theta) && \text{(Leibniz rule)} \\
&= \cos\theta dr + r \wedge (\frac{\partial \cos\theta}{\partial x}dx + \frac{\partial \cos\theta}{\partial y}dy) \\
&= \cos\theta dr + r \wedge (-\sin\theta\frac{\partial \theta}{\partial x}dx - \sin\theta\frac{\partial \theta}{\partial y}dy) && \text{(Chain rule)} \\
&= \cos\theta dr - r\sin\theta(\frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy) \\
&= \cos\theta dr - r\sin\theta d\theta. && (1)
\end{aligned}$$

Similarly, we can express dy in terms of dr and $d\theta$ as follows:

$$\begin{aligned}
dy &= d(r\sin\theta) \\
&= dr \wedge \sin\theta + r \wedge d(\sin\theta) && \text{(Leibniz rule)} \\
&= \sin\theta dr + r \wedge (\frac{\partial \sin\theta}{\partial x}dx + \frac{\partial \sin\theta}{\partial y}dy) \\
&= \sin\theta dr + r \wedge (\cos\theta\frac{\partial \theta}{\partial x}dx + \cos\theta\frac{\partial \theta}{\partial y}dy) && \text{(Chain rule)} \\
&= \sin\theta dr + r\cos\theta(\frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy) \\
&= \sin\theta dr + r\cos\theta d\theta. && (2)
\end{aligned}$$

Next, if we subtract $\sin\theta$ times (1) from $\cos\theta$ times (2), we obtain:

$$\begin{aligned}
\cos\theta dy - \sin\theta dx &= \cos\theta\sin\theta dr + r\cos^2\theta d\theta - \sin\theta\cos\theta dr + r\sin^2\theta d\theta \\
-\sin\theta dx + \cos\theta dy &= r(\cos^2\theta + \sin^2\theta)d\theta \\
-\sin\theta dx + \cos\theta dy &= r d\theta \\
\frac{-r\sin\theta}{r^2}dx + \frac{r\cos\theta}{r^2}dy &= d\theta \\
\frac{-y}{r^2(\cos^2\theta+\sin^2\theta)}dx + \frac{x}{r^2(\cos^2\theta+\sin^2\theta)}dy &= d\theta \\
\frac{-y}{r^2\cos^2\theta+r^2\sin^2\theta}dx + \frac{x}{r^2\cos^2\theta+r^2\sin^2\theta}dy &= d\theta \\
\frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy &= d\theta.
\end{aligned}$$

Therefore, $d\theta = \frac{-y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy$, as required. □

4. We are given the following forms in $\mathbb{R}^3$:

$$\omega = xydx + 3dy - yzdz,$$

$$\eta = xdx - yz^2dy + 2xdz.$$

Then, we will verify by direct computations that $d(d\omega) = 0$ and that $d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge (d\eta)$. First, we can compute $d\omega$ as follows:

$$\begin{aligned}
d\omega &= d(xydx + 3dy - yzdz) \\
&= dx \wedge \frac{\partial}{\partial x}(xydx + 3dy - yzdz) + dy \wedge \frac{\partial}{\partial y}(xydx + 3dy - yzdz) + dz \wedge \frac{\partial}{\partial z}(xydx + 3dy - yzdz) \\
&= dx \wedge (ydx + 0dy + 0dz) + dy \wedge (xdx + 0dy - zdz) + dz \wedge (0dx + 0dy - ydz) \\
&= ydx \wedge dx + xdy \wedge dx - zdy \wedge dz - ydz \wedge dz \\
&= 0 - xdx \wedge dy - zdy \wedge dz + 0 \\
&= -xdx \wedge dy - zdy \wedge dz.
\end{aligned}$$

Next, we can compute $d(d\omega)$ as follows:

$$\begin{aligned}
d(d\omega) &= d(-xdx \wedge dy - zdy \wedge dz) \\
&= dx \wedge \frac{\partial}{\partial x}(-xdx \wedge dy - zdy \wedge dz) + dy \wedge \frac{\partial}{\partial y}(-xdx \wedge dy - zdy \wedge dz) \\
&\quad + dz \wedge \frac{\partial}{\partial z}(-xdx \wedge dy - zdy \wedge dz) \\
&= dx \wedge (-dx \wedge dy) + dy \wedge 0 + dz \wedge (-dy \wedge dz) \\
&= -dx \wedge dx \wedge dy - dz \wedge dy \wedge dz \\
&= 0.
\end{aligned}$$

Therefore, $d(d\omega) = 0$, as required.

Next, we can compute $\omega \wedge \eta$ as follows:

$$\begin{aligned}
\omega \wedge \eta &= (xydx + 3dy - yzdz) \wedge (xdx - yz^2dy + 2xdz) \\
&= x^2ydx \wedge dx - xy^2z^2dx \wedge dy + 2x^2ydx \wedge dz + 3xdy \wedge dx - 3yz^2dy \wedge dy + 6xdy \wedge dz \\
&\quad - xyzdz \wedge dx + y^2z^3dz \wedge dy - 2xyzdz \wedge dz \\
&= 0 - xy^2z^2dx \wedge dy + 2x^2ydx \wedge dz - 3xdx \wedge dy + 0 + 6xdy \wedge dz + xyzdx \wedge dz - y^2z^3dy \wedge dz + 0 \\
&= (-xy^2z^2 - 3x)dx \wedge dy + (2x^2y + xyz)dx \wedge dz + (6x - y^2z^3)dy \wedge dz.
\end{aligned}$$

Next, we can compute $d(\omega \wedge \eta)$ as follows:

$$\begin{aligned}
d(\omega \wedge \eta) &= d((-xy^2z^2 - 3x)dx \wedge dy + (2x^2y + xyz)dx \wedge dz + (6x - y^2z^3)dy \wedge dz) \\
&= dx \wedge \frac{\partial}{\partial x}((-xy^2z^2 - 3x)dx \wedge dy + (2x^2y + xyz)dx \wedge dz + (6x - y^2z^3)dy \wedge dz) \\
&\quad + dy \wedge \frac{\partial}{\partial y}((-xy^2z^2 - 3x)dx \wedge dy + (2x^2y + xyz)dx \wedge dz + (6x - y^2z^3)dy \wedge dz) \\
&\quad + dz \wedge \frac{\partial}{\partial z}((-xy^2z^2 - 3x)dx \wedge dy + (2x^2y + xyz)dx \wedge dz + (6x - y^2z^3)dy \wedge dz) \\
&= dx \wedge ((-y^2z^2 - 3)dx \wedge dy + (4xy + yz)dx \wedge dz + 6dy \wedge dz) \\
&\quad + dy \wedge (-2xyz^2dx \wedge dy + (2x^2 + xz)dx \wedge dz - 2yz^3dy \wedge dz) \\
&\quad + dz \wedge (-2xy^2zdx \wedge dy + xydx \wedge dz - 3y^2z^2dy \wedge dz) \\
&= (-y^2z^2 - 3)dx \wedge dx \wedge dy + (4xy + yz)dx \wedge dx \wedge dz + 6dx \wedge dy \wedge dz \\
&\quad - 2xyz^2dy \wedge dx \wedge dy + (2x^2 + xz)dy \wedge dx \wedge dz - 2yz^3dy \wedge dy \wedge dz \\
&\quad - 2xy^2zdz \wedge dx \wedge dy + xydz \wedge dx \wedge dz - 3y^2z^2dz \wedge dy \wedge dz \\
&= 0 + 0 + 6dx \wedge dy \wedge dz + 0 - (2x^2 + xz)dx \wedge dy \wedge dz + 0 - 2xy^2zdx \wedge dy \wedge dz + 0 + 0 \\
&= (-2xy^2z - 2x^2 - xz + 6)dx \wedge dy \wedge dz.
\end{aligned}$$

Next, we can compute $(d\omega) \wedge \eta$ as follows:

$$\begin{aligned}
(d\omega) \wedge \eta &= (-xdx \wedge dy - zdy \wedge dz) \wedge (xdx - yz^2dy + 2xdz) \\
&= -x^2dx \wedge dy \wedge dx + xyz^2dx \wedge dy \wedge dy - 2x^2dx \wedge dy \wedge dz \\
&\quad - xzdy \wedge dz \wedge dx + yz^3dy \wedge dz \wedge dy - 2xzdy \wedge dz \wedge dz \\
&= 0 + 0 - 2x^2dx \wedge dy \wedge dz - xzdx \wedge dy \wedge dz + 0 + 0 \\
&= (-2x^2 - xz)dx \wedge dy \wedge dz.
\end{aligned}$$

Next, we can compute $d\eta$ as follows:

$$\begin{aligned}
d\eta &= d(xdx - yz^2dy + 2xdz) \\
&= dx \wedge \frac{\partial}{\partial x}(xdx - yz^2dy + 2xdz) + dy \wedge \frac{\partial}{\partial y}(xdx - yz^2dy + 2xdz) + dz \wedge \frac{\partial}{\partial z}(xdx - yz^2dy + 2xdz) \\
&= dx \wedge (dx + 2dz) + dy \wedge (-z^2dy) + dz \wedge (-2yzdy) \\
&= dx \wedge dx + 2dx \wedge dz - z^2dy \wedge dy - 2yzdz \wedge dy \\
&= 0 + 2dx \wedge dz + 0 + 2yzdy \wedge dz \\
&= 2dx \wedge dz + 2yzdy \wedge dz.
\end{aligned}$$

Next, we can compute $\omega \wedge d\eta$ as follows:

$$\begin{aligned}
\omega \wedge d\eta &= (xydx + 3dy - yzdz) \wedge (2dx \wedge dz + 2yzdy \wedge dz) \\
&= 2xydx \wedge dx \wedge dz + 2xy^2zdx \wedge dy \wedge dz + 6dy \wedge dx \wedge dz + 6yzdy \wedge dy \wedge dz \\
&\quad - 2yzdz \wedge dx \wedge dz - 2y^2z^2dz \wedge dy \wedge dz \\
&= 0 + 2xy^2zdx \wedge dy \wedge dz - 6dx \wedge dy \wedge dz + 0 + 0 + 0 \\
&= (2xy^2z - 6)dx \wedge dy \wedge dz.
\end{aligned}$$

Finally, we can compute $d\omega \wedge \eta - \omega \wedge d\eta$ as follows:

$$\begin{aligned} d\omega \wedge \eta - \omega \wedge d\eta &= (-2x^2 - xz)dx \wedge dy \wedge dz - (2xy^2z - 6)dx \wedge dy \wedge dz \\ &= (-2xy^2z - 2x^2 - xz + 6)dx \wedge dy \wedge dz \\ &= d(\omega \wedge \eta). \end{aligned}$$

Therefore, $d(\omega \wedge \eta) = d\omega \wedge \eta - \omega \wedge d\eta$, as required. $\square$

5. We are given the form:

$$\omega := \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{|x|^p} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$$

in $\Omega^{n-1}(\mathbb{R}^n - \{0\})$, where p is some positive real number.

(a) We will compute $d\omega$ as follows:

$$\begin{aligned}
d\omega &= \sum_{j=1}^{n} dx_j \wedge \frac{\partial}{\partial x_j} \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{|x|^p} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \sum_{j=1}^{n} \sum_{i=1}^{n} (\frac{\partial}{\partial x_j} (-1)^{i-1} \frac{x_i}{|x|^p}) dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n.
\end{aligned}$$

If $i \neq j$, then $dx_j \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$ contains a repeating dx_j, so it equals zero. Then, the only non-vanishing terms in the above summation occur when $i = j$, so:

$$\begin{aligned}
d\omega &= \sum_{i=1}^{n} (\frac{\partial}{\partial x_i} (-1)^{i-1} \frac{x_i}{|x|^p}) dx_i \wedge dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} (\frac{\partial}{\partial x_i} (-1)^{i-1} \frac{x_i}{|x|^p})(-1) dx_1 \wedge dx_i \wedge dx_2 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n \\
&= \cdots \qquad\qquad \text{(Apply } i-2 \text{ remaining transpositions)} \\
&= \sum_{i=1}^{n} (\frac{\partial}{\partial x_i} (-1)^{i-1} \frac{x_i}{|x|^p})(-1)^{i-1} dx_1 \wedge \cdots \wedge dx_i \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \frac{x_i}{|x|^p} dx_1 \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} \frac{(\frac{\partial}{\partial x_i} x_i)|x|^p - x_i(\frac{\partial}{\partial x_i}|x|^p)}{|x|^{2p}} dx_1 \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} \frac{|x|^p - p x_i |x|^{p-1} \frac{\partial}{\partial x_i} \sqrt{x_1^2 + \cdots + x_i^2 + \cdots + x_n^2}}{|x|^{2p}} dx_1 \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} \frac{|x|^p - p x_i |x|^{p-1} \frac{2x_i}{2\sqrt{x_1^2 + \cdots + x_i^2 + \cdots + x_n^2}}}{|x|^{2p}} dx_1 \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} \frac{|x|^p - p x_i^2 |x|^{p-2}}{|x|^{2p}} dx_1 \wedge \cdots \wedge dx_n \\
&= \sum_{i=1}^{n} \frac{|x|^2 - p x_i^2}{|x|^{p+2}} dx_1 \wedge \cdots \wedge dx_n \\
&= \frac{1}{|x|^{p+2}} (\sum_{i=1}^{n} |x|^2 - p \sum_{i=1}^{n} x_i^2) dx_1 \wedge \cdots \wedge dx_n \\
&= \frac{1}{|x|^{p+2}} (n|x|^2 - p|x|^2) dx_1 \wedge \cdots \wedge dx_n \\
&= \frac{n-p}{|x|^p} dx_1 \wedge \cdots \wedge dx_n.
\end{aligned}$$

Therefore, $\boxed{d\omega = \frac{n-p}{|x|^p} dx_1 \wedge \cdots \wedge dx_n}$, as required. $\square$

(b) We will determine which values of p give $d\omega = 0$.
From part (a), we have $d\omega = \frac{n-p}{|x|^p} dx_1 \wedge \cdots \wedge dx_n$, so $d\omega = 0$ if and only if $\frac{n-p}{|x|^p} = 0$, so $\boxed{p = n}$, as required. $\square$

Notes on intuition

Now, let us develop some intuition for how to approach these problems and motivate these solutions. (Note: This section was not submitted for grading.)

1. This was a computational problem, not much to say here.
2. The main idea for this problem was to use g^* and $(g^{-1})^*$ to convert between forms on U and forms on V. As mentioned by the grader, there was some confusion between g^* and $(g^{-1})^*$. As a mnemonic/rule of thumb, pullbacks will pull in the opposite direction of the original object; in this case, g maps from U to V, so g^* maps in the opposite direction, from forms on V to forms on U.
3. As explained in the grader's comments, we cannot assume that $\theta = \arctan(\frac{y}{x})$ because $\tan$ is not injective on $[0, 2\pi)$. Instead, we have explicit formulas for x and y in terms of r and θ, which motivates us to directly compute dx and dy in terms of dr and $d\theta$. This gives us a system of linear equations where dr and $d\theta$ are the "variables". Then, we can solve this system of linear equations to find $d\theta$ without an explicit formula for $d\theta$.
4. This was a computational problem, not much to say here.
5. This was a computational problem, not much to say here.