Q1a: We claim that $\gamma = (\iota(v_1) \ldots \iota(v_n))$ is a basis for V^{**} and that it is the dual basis of $(\phi_1 \ldots \phi_n)$, the dual basis of V^* We will prove that γ is linearly independant and spans V^{**}. First suppose that for some scalar $\alpha_1, \ldots, \alpha_n$ we have that

$$\alpha_1 \iota(v_1) + \ldots \alpha_n \iota(v_n) = 0$$

From the definition of ι, we have

$$\alpha_1 \phi(v_1) + \ldots \alpha_n \phi(v_n) = 0, \forall \phi \in V^*$$

Now write $\phi = \beta_1 \phi_q + \ldots \beta_n \phi_n$ for scalars $\beta_1, \ldots \beta_n$. We see that

$$\alpha_1(\beta_1 \phi_1(v_1) + \cdots + \beta_n \phi_n(v_1)) + \ldots \alpha_n(\beta_1 \phi_1(v_n) + \ldots \beta_n \phi_n(v_n)) = 0$$

From the definition of the dual basis this gives us:

$$\alpha_1 \beta_1 + \cdots + \alpha_n \beta_n = 0$$

Since this is true for every β_i, we must have that $\alpha_1 = \cdots = \alpha_n = 0$. Hence γ is a linearly independant set. We now claim that it spans V^{**}. Now suppose that $\psi \in V^{**}$, and $\psi(\phi_i) = k_i$. Let $\phi = \beta_1 \phi_1 + \ldots \beta_n \phi_n$. We see that

$$\begin{aligned} \psi(\phi) &= \psi(\beta_1 \phi_1 + \cdots + \beta_n \phi_n) \\ &= \beta_1 k_1 + \cdots + \beta_n k_n \\ &= k_1 \phi(v_1) + \ldots k_n \phi(v_n) \\ &= k_1 \iota_{v_1}(\phi) + \cdots + k_n \iota_{v_n}(\phi) \end{aligned}$$

Thus γ spans V^{**} and we conclude it is a basis. We now want to show that γ is dual to $(\phi_1, \ldots, \phi_n)$. Notice that

$$\iota(v_i)(\phi_j) = \phi_j(v_i) = \delta_{ij}$$

We conclude that γ is indeed the dual of $(\phi_1, \ldots \phi_n)$

Q1b: First, observe that $\iota(v)(\alpha\phi + \psi) = \alpha\phi + \psi(v) = \alpha\phi(v) + \phi(v) = \alpha\iota(v)(\phi) + \iota(v)(\psi)$, so ι is linear. Note that by 1b, we see that the image of ι is n dimensional, and the domain is as well n dimensional. Hence by the Rank-Nullity theorem we conclude it is a bijection . Thus it is a linear isomorphism between V and V^{**}

Q2: Since V a 3 dimensional vector space we have from basic linear algebra that V^* will also be 3 dimensional. It suffices to check that $\phi_{-1}, \phi_0, \phi_1$ either span V^* or are linearly independant. We will show linear independance. We will denote $p \in V$ as $p(x) = ax^2 + bx + c$. Suppose that for some scalars $\alpha_1, \alpha_2, \alpha_3$,

$$\alpha_1\phi_{-1}(p) + \alpha_2\phi_0(p) + \alpha_3\phi_1(p) = 0, \forall p \in V$$

Then we have that

$$\alpha_1(a - b + c) + \alpha_2(c) + \alpha_3(a + b + c) = 0$$

Re writing this expression get that

$$a(\alpha_1 + \alpha_4) + b(\alpha_3 - \alpha_3) + c(\alpha_2 + \alpha_3)$$

Since this is true for all polynomials, we, taking $b = 1 = a, c = 0$ we see that $\alpha_3 = 0$, taking $a = c = 0, b = 1$ gives us that $\alpha_1 = \alpha_3 = 0$. Finally, if we take $a = 1$, we see that $\alpha_2 = -\alpha_1 = 0$. Thus we conclude this is a linearly independant list, and so it is a basis of V^*. We now will find a basis $\beta = (p_{-1}, p_0, p_1)$ of V so that $\beta^* = \gamma$. In other words, for each ϕ_i, $\phi_i(p_j) = \delta_{ij}$. First consider p_{-1}. We require that $p_{-1}(-1) = 1, p_{-1}(0) = p_{-1}(1) = 0$. Choosing $p_{-1}(x) = \frac{1}{2}x^2 - \frac{1}{2}x$ will satisfy these properties. Setting $p_0(x) = -x^2 + 1$, we see that $p_0(-1) = p_0(1) = 0$ and $p_0(0) = 1$. Finally, setting $p_1(x) = \frac{1}{2}x^2 + \frac{1}{2}$ will give us the desired properties. Hence the basis $\beta = (\frac{1}{2}x^2 - \frac{1}{2}x, -x^2 + 1, \frac{1}{2}x^2 + \frac{1}{2}x)$ satisifes $\beta^* = \gamma$.

Q3: To show that $B \in \mathcal{T}^2(\mathcal{T}^k(V))$, we must show that it is 2-linear. Thus, for $T_1, T_2, T_3 \in \mathcal{T}^k(V)$ and $\alpha \in \mathbb{R}$, we evaluate $B(T_1 + \alpha T_2, T_3)$ and $B(T_1, T_2 + \alpha T_3)$. We see the following:

$$\begin{aligned}
B(T_1 + \alpha T_2, T_3) &= \sum_{i_1,\ldots,i_k=1}^{n} (T_1 + \alpha T_2)(v_{i_1} \ldots v_{i_k}) T_3(v_{i_1} \ldots v_{i_k}) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} [T_1(v_{i_1} \ldots v_{i_k}) + \alpha T_2(v_{i_1} \ldots v_{i_k})] T_3(v_{i_1} \ldots v_{i_k}) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} T_1(v_{i_1} \ldots v_{i_k}) T_3(v_{i_1} \ldots v_{i_k}) + \alpha T_2(v_{i_1} \ldots v_{i_k}) T_3(v_{i_1} \ldots v_{i_k}) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} T_1(v_{i_1} \ldots v_{i_k}) T_3(v_{i_1} \ldots v_{i_k}) + \alpha \sum_{i_1,\ldots,i_k}^{n} T_2(v_{i_1} \ldots v_{i_k}) T_3(v_{i_1} \ldots v_{i_k}) \\
&= B(T_1, T_3) + \alpha B(T_2, T_3)
\end{aligned}$$

By almost exactly the same computation, we see that $B(T_1, T_2 + \alpha T_3) = B(T_1, T_2) + \alpha B(T_1, T_3)$. B is bilinear and hence belongs to $\mathcal{T}^2(\mathcal{T}^k(V))$

Q3b: We now wish to show that B is an inner product on $\mathcal{T}^k(V)$. We have shown above that B is bilinear. It remains to prove it is symmetric and positive definite. First, observe the following:

$$\begin{aligned}
B(T_1, T_2) &= \sum_{i_1,\ldots,i_k=1}^{n} T_1(v_{i_1} \ldots v_{i_k}) T_2(v_{i_1} \ldots v_{i_k}) \\
&= \sum_{i_1,\ldots,i_k=1}^{n} T_2(v_{i_1} \ldots v_{i_k}) T_1(v_{i_1} \ldots v_{i_k}) \\
&= B(T_2, T_1)
\end{aligned}$$

Hence B is symmetric. We will now show that for any $T \in \mathcal{T}^k(V)$, $B(T, T) \geq 0$ with equality holding if and only if $T = 0$. Observe:

$$\begin{aligned}
B(T, T) &= \sum_{i_1,\ldots,i_k=1}^{n} T(v_{i_1} \ldots v_{i_k}) T(v_{i_1} \ldots v_{i_k}) \\
&= \sum_{i_1,\ldots,i_k}^{n} [T(v_{i_1} \ldots v_{i_k})]^2 \geq 0
\end{aligned}$$

We note that equality holds if and only iff for each v_{i_j}, $T(v_{i_1} \ldots v_{i_k}) = 0$, meaning that on any k-tuple of basis vectors, $T = 0$. This is equivalent to saying that T is the 0-mapping.