Q1:
First letting $A = (0, a) \times (0, \frac{\pi}{2}) \times (0, 2\pi)$ we see that $g(A) = V \setminus C$, for some content 0 set C. Thus by COV $\int_{g(A)} z = \int_A z \circ g \cdot |detg'|$. We see that $z \circ g = rsin\phi$ Computing g' get

$$g' = \begin{bmatrix} \cos\theta\cos\phi & -r\sin\phi\cos\theta & -r\cos\phi\sin\theta \\ \cos\phi\sin\theta & -r\sin\phi\sin\theta & r\cos\phi\cos\theta \\ \sin\phi & r\cos\phi & 0 \end{bmatrix}$$

We have that $|\det g'| = r^2\cos\phi$. This will be nonzero on the domain of g, so we can apply COV. We evaluate:

$$\begin{aligned}
\int_{g(A)} z &= \int_A z \circ g |\det g'| && \text{(by COV)} \\
&= \int_A r^3 sin\phi cos\phi \\
&= \int_0^a \int_0^{2\pi} \int_0^{\frac{\pi}{2}} r^3 \sin\phi\cos\phi \quad d\phi d\theta dr && \text{(by Fubini's Theorem)} \\
&= \int_0^a \int_0^{2\pi} r^3 \frac{\sin^2\phi}{2}\Big|_0^{\frac{\pi}{2}} \quad d\theta dr \\
&= \int_0^a \int_0^{2\pi} \frac{r^3}{2} \quad d\theta dr \\
&= \int_0^a \pi r^3 dr \\
&= \frac{\pi a^4}{4}
\end{aligned}$$

Q2:
By changing to polar coordinates we can rewrite define $V_1 = \{(r,\theta) : r \in (0,1), \theta \in (0,2\pi)\}$ and $V_2 = \{(r,\theta) : r > 1, \theta \in (0,2\pi)\}$. We see that $g(V_1) = U_1$ and $g(V_2) = U_2$. We know $|\det g'| = r$, and g injective on V_1 and V_2 so by the COV theorem we evaluate:

$$\begin{aligned}
\int_{g(V_1)} f &= \int_{V_1} f \circ g |\det g'| \\
&= \lim_{t\to 0} \int_t^1 \int_0^{2\pi} \frac{1}{r^2} r d\theta dr && \text{(by Fubini's Theorem and discussion in class)} \\
&= \lim_{t\to 0} 2\pi \int_0^1 \frac{1}{r} dr \\
&= \lim_{t\to 0} 2\pi [log(r)]\Big|_t^1 \\
&= -\infty
\end{aligned}$$

In other words, for some PO1, $\{\phi_i\}$, $\sum \int \phi_i f$ diverges so f is not integrable. We now evaluate $\int_{g(V_2)} f$ using the COV theorem.

$$\begin{aligned}
\int_{g(V_2)} f &= \int_{V_2} f \circ g |\det g'| \\
&= \lim_{t\to \infty} \int_1^t \int_0^{2\pi} \frac{1}{r^2} r d\theta dr && \text{(by Fubini's Theorem, and discussion in class)} \\
&= \lim_{t\to \infty} 2\pi \int_1^t \frac{1}{r} dr \\
&= \lim_{t\to \infty} 2\pi [\log(r)]\Big|_1^t \\
&= \infty
\end{aligned}$$

So, for some PO1 of V_2, $\sum \int \phi_i f$ diverges, so f is not integrable.

Q3:
Define $A = \{(u,v) : 1 < u < \sqrt{2}, 1 < v < 2\}$. Let $g(u,v) = (\frac{u}{v}, vu)$. On the set A, g will be 1-1 and onto on $g(A)$. Note that $g(A) = B$ so we can apply the COV theorem to compute $\int_B f$. We have that $g' = \begin{bmatrix} \frac{1}{v} & v \\ \frac{-u}{v^2} & u \end{bmatrix}$. Therefore, $\det g' = \frac{2u}{v}$. This will always be nonzero in our domain, so by the COV theorem

$$\begin{aligned}
\int_B f &= \int_A f \circ g |\det g'| \\
&= \int_A u^5 v \cdot \frac{2u}{v} \\
&= \int_1^4 \int_1^{\sqrt{2}} 2u^6 du dv && \text{(by Fubini's Theorem)} \\
&= \int_1^4 [\frac{2u^7}{7}]\Big|_1^{\sqrt{2}} dv \\
&= \frac{2}{7} \int_1^2 (8\sqrt{2} - 1) dv \\
&= \frac{2}{7}(8\sqrt{2} - 1)
\end{aligned}$$

Q4:
Define $A = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y\}$. Choose

$$g(x, y, z) = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

. Notice that $g(A) = T$, and g is invertible hence we can apply the COV theorem. Evaluate $\int_{g(A)} f$ as

$$\begin{aligned}
\int_{g(A)} f &= \int_A f \circ g |\det g'| \\
&= \int_A 4x \\
&= \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 4x \quad dzdydx && \text{(by Fubini's Theorem)} \\
&= 4 \int_0^1 \int_0^{1-x} x - x^2 - xy \quad dydx \\
&= 4 \int_0^1 x(1-x) - x^2(1-x) - \frac{1}{2}x(1-x)^2 dx \\
&= \frac{1}{6}
\end{aligned}$$

Q5:
First notice that $|\det g'| = r > 0$. We can apply the COV Theorem to compute the value of the integral.

$$\begin{aligned}
\int_{T(A)} 1 &= \int_A 1|\det g'| \\
&= \int_0^{2\pi}\int_{b-a}^{b+a}\int_{-\sqrt{a^2-(r-b)^2}}^{\sqrt{a^2-(r-b)^2}} r \quad dzdrd\theta && \text{(by Fubini's Theorem)} \\
&= 2\int_0^{2\pi}\int_{b-a}^{b+a} r\sqrt{a^2-(r-b)^2} \quad drd\theta \\
&= 2\int_0^{2\pi}\int_{-a}^{a} (u+b)\sqrt{a^2-u^2} \quad dud\theta && \text{(substitution u = r-b)} \\
&= 2\int_0^{2\pi}\int_{-a}^{a} u\sqrt{a^2-u^2} \quad dud\theta + 2\int_0^{2\pi}\int_{-a}^{a} b\sqrt{a^2-u^2} \quad dud\theta \\
&= 2\int_0^{2\pi}\int_{-a}^{a} b\sqrt{a^2-u^2} \quad dud\theta && \text{(since first integral is of an odd function)} \\
&= 2b\int_0^{2\pi} \frac{\pi}{2}a^2 d\theta \\
&= 2\pi^2 a^2 b
\end{aligned}$$