# MAT257 Assignment 2

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1. a) First, we will find the interior, exterior, and boundary of the set:

$$A_1 = \{x \in \mathbb{R}^n : |x| \le 1\}.$$

For all $x \in \mathbb{R}^n$, we have the following trichotomy: Either $|x| < 1$, or $|x| = 1$, or $|x| > 1$. We will consider each case as follows.

**Case 1**: $|x| < 1$. Then, let us define the positive number $r = 1 - |x|$, and let us consider the open ball $B$ centred at $x$ with radius $r$. For all $y \in B$, we have $|y - x| < r = 1 - |x|$ because $B$ is of radius $r$. As a result, the triangle inequality tells us that:

$$|y| \le |y - x| + |x - 0| < (1 - |x|) + |x| = 1,$$

so $y \in A_1$. In other words, all points inside the open ball $B$ around $x$ are in $A_1$, so $x$ is in the interior of $A_1$.

**Case 2**: $|x| = 1$. Then, for all open sets $U$ around $x$, there is an open rectangle $R = \prod_{i=1}^{n}(a_i, b_i)$ around $x$ contained in $U$. If the coordinates of $x$ are $x = (x_1, \ldots, x_n)$, then we have $a_1 < x_1 < b_1$ because $x \in R$. As a result, we can define the positive number $d = \min(x_1 - a_1, b_1 - x_1)$, and we can define the nonzero point $z \in \mathbb{R}^n$ by $z = (\frac{d}{2}, 0, 0, \ldots, 0)$. Since we have:

$$a_1 = x_1 - (x_1 - a_1) \le x_1 - d < x_1 - \frac{d}{2} < x_1 < b_1,$$

we get $x_1 - \frac{d}{2} \in (a_1, b_1)$, so $x - z \in R \subseteq U$. (Note that subtracting $z$ does not change any of the coordinates of $x$ other than the first coordinate, so they all remain in their respective intervals $(a_i, b_i)$.) Since we also have:

$$a_1 < x_1 < x_1 + \frac{d}{2} < x_1 + d \le x_1 + (b_1 - x_1) = b_1,$$

we get $x_1 + \frac{d}{2} \in (a_1, b_1)$, so $x + z \in R \subseteq U$. (Again, all other coordinates of $x$ remain in their respective intervals $(a_i, b_i)$ after adding $z$.) Now, we can compute that:

$$\begin{aligned}
|x - z|^2 + |x + z|^2 &= \langle x - z, x - z\rangle + \langle x + z, x + z\rangle \\
&= (\langle x, x\rangle - \langle x, z\rangle - \langle z, x\rangle + \langle z, z\rangle) + (\langle x, x\rangle + \langle x, z\rangle + \langle z, x\rangle + \langle z, z\rangle) \\
&= 2\langle x, x\rangle + 2\langle z, z\rangle \\
&= 2|x|^2 + 2|z|^2 \\
&> 2|x|^2 + 0 \qquad\qquad (|z| > 0 \text{ because } z \ne 0) \\
&= 2
\end{aligned}$$

Thus, since the sum of $|x - z|^2$ and $|x + z|^2$ is greater than 2, at least one of these terms must be greater than $\frac{2}{2} = 1$, so at least one of $|x - z|$ or $|x + z|$ must be greater than 1. This gives us that at least one of $x - z$ or $x + z$ is outside $A$. As a result, every open set $U$ around $x$ contains some point outside $A$. All such $U$ also contain $x$, a point inside $A$. Thus, $x$ is in the boundary of $A$ whenever $|x| = 1$.

**Case 3**: $|x| > 1$. Then, let us define the positive number $r = |x| - 1$, and let us consider the open ball $B$ centred at $x$ with radius $r$. For all $y \in B$, we have $|y - x| < r = |x| - 1$ because $B$ is of radius $r$. As a result, the triangle inequality tells us that $|y| + |x - y| \ge |x|$, so:

$$|y| \ge |x| - |x - y| > |x| - (|x| - 1) = 1,$$

so $y \notin A_1$. In other words, all points inside the open ball $B$ around $x$ are outside $A_1$, so $x$ is in the exterior of $A_1$.

Summarizing our results, the interior of $A_1$ is the set $\{x \in \mathbb{R}^n : |x| < 1\}$, the boundary of $A_1$ is the set $\{x \in \mathbb{R}^n : |x| = 1\}$, and the exterior of $A_1$ is the set $\{x \in \mathbb{R}^n : |x| > 1\}$. $\square$

b) Next, we will find the interior, exterior, and boundary of the set:

$$A_2 = \{x \in \mathbb{R}^n : |x| = 1\}.$$

Similarly to part a), we will consider the three following cases:

**Case 1**: $|x| < 1$. Then, similarly to part a), there exists an open ball $B$ around $x$ such that all $y \in B$ satisfy $|y| < 1$, giving us $y \notin A_2$. As a result, $x$ is in the exterior of $A_2$.

**Case 2**: $|x| = 1$. Then, similarly to part a), every open set $U$ around $x$ contains some point $y \in \mathbb{R}^n$ satisfying $|y| > 1$, giving us $y \notin A_2$. Additionally, all such $U$ contains $x$, which is inside $A_2$. As a result, $x$ is in the boundary of $A_2$.

**Case 3**: $|x| > 1$. Then, similarly to part a), there exists an open ball $B$ around $x$ such that all $y \in B$ satisfy $|y| > 1$, giving us $y \notin A_2$. As a result, $x$ is in the exterior of $A_2$.

Summarizing our results, the interior of $A_2$ is empty, the boundary of $A_2$ is $A_2$ itself, and the exterior of $A_2$ is its complement, $A_2^c$. $\square$

c) Finally, we will find the interior, exterior, and boundary of the set:

$$A_3 = \{x \in \mathbb{R}^n : \forall i, x_i \in \mathbb{Q}\}.$$

We claim that all $x \in \mathbb{R}^n$ are in the boundary of $A_3$. Let $U$ be any open set around $x$. Then, there exists an open rectangle $R = \prod_{i=1}^n (a_i, b_i)$ around $x$ contained in $U$. For all $1 \leq i \leq n$, since $\mathbb{Q}$ is dense in $\mathbb{R}$, there exists some rational number $r_i$ inside $(a_i, b_i)$. Then, the point $(r_1, r_2, \ldots, r_n) \in \mathbb{R}^n$ satisfies $(r_1, \ldots, r_n) \in R \subseteq U$ because $r_i \in (a_i, b_i)$ for all $i$. This point is also inside $A_3$ because all of its coordinates are rational. As a result, $U$ contains a point inside $A_3$. In the same way, we can prove that $R \subseteq U$ contains some point whose coordinates are all irrational because the set of all irrational numbers is also dense, and such a point would be outside $A_3$. Since this is true for all open sets $U$ around $x$, we conclude that $x$ is in the boundary of $A_3$ for all $x \in \mathbb{R}^n$. Therefore, the interior and exterior of $A_3$ are empty, and the boundary of $A_3$ is $\mathbb{R}^n$, as required. $\square$

2. a) Given some closed set $A$ and some point $x \notin A$, we will prove that there is a number $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.
First, since $A$ is closed, its complement, $A^c$, is open. Then, since $x \in A^c$, there exists an open rectangle $R = \prod_{i=1}^{n}(a_i, b_i)$ around $x$ contained in $A^c$. In other words, if we denote the $i^{\text{th}}$ coordinate of $x$ by $x_i$, then $a_i < x_i < b_i$ for all $1 \leq i \leq n$. As a result, we can define the positive number $d$ by:

$$d = \min_{1 \leq i \leq n}(\min(x_i - a_i, b_i - x_i)).$$

Now, we claim that $|y - x| \geq d$ for all $y \in A$ for our choice of $d$. Since $y \in A$, and since $R \subseteq A^c$, we have $y \notin R$. As a result, there exists some $1 \leq j \leq n$ such that $y_j \notin (a_j, b_j)$. If $y_j \leq a_j$, we obtain $x_j - y_j \geq x_j - a_j \geq d$, and if $y_j \geq b_j$, we obtain $y_j - x_j \geq b_j - x_j \geq d$. Either way, we get $|x_j - y_j| \geq d$, so:

$$|x - y| = \sqrt{\sum_{i=1}^{n} |x_i - y_i|^2} \geq \sqrt{|x_j - y_j|^2} \geq d.$$

Therefore, our choice of $d > 0$ satisfies $|y - x| \geq d$ for all $y \in A$, as required. $\square$
(Based on the grader's suggestions, here is a cleaner solution for part a) using an open ball instead of an open rectangle. I added this solution after grading was completed.)
First, since $A$ is closed, its complement, $A^c$, is open. Then, since $x \in A^c$, there exists an open ball $B_x(r)$ around $x$ of radius $r > 0$ contained in $A^c$. Now, for all $y \in A$, we must have $|y - x| \geq r$; otherwise, if $|y - x| < r$, we would have $y \in B_x(r) \subseteq A^c$, contradicting $y \in A$. Thus, $|y - x| \geq r$ for all $y \in A$. In other words, if we pick $d = r > 0$, we get $|y - x| \geq d$ for all $y \in A$, as required. $\square$
b) Given two disjoint sets $A, B$ such that $A$ is closed and $B$ is compact, we will prove that there exists $d > 0$ such that $|y - x| \geq d$ for all $x \in A$ and $y \in B$.
First, for all $y \in B$, we know from part a) that there exists a positive number $d_y$ such that $|x - y| \geq d_y$ for all $x \in A$. Then, we can define $B_y$ to be the open ball of radius $\frac{d_y}{2}$ around $y$. Since each $y \in B_y$ for all $y \in B$, these open balls form an open cover of $B$. Thus, since $B$ is compact, there exists a finite subcover $B_{y_1}, \ldots, B_{y_k}$ of $B$ with open balls around $y_1, \ldots, y_k$, respectively. Now, we can define the positive real number:

$$d = \min_{1 \leq i \leq k} \frac{d_{y_i}}{2}.$$

Then, we claim that $|y - x| \geq d$ for all $x \in A$ and all $y \in B$ for our choice of $d$. Since $y \in B$, $y$ is contained in some open ball $B_{y_i}$ in our finite subcover of $B$. Then, by definition of $d_{y_i}$, we have $|x - y_i| \geq d_{y_i}$. Moreover, since $y \in B_{y_i}$, and since $B_{y_i}$ is of radius $\frac{d_{y_i}}{2}$, we have $|y - y_i| < \frac{d_{y_i}}{2}$. As a result, since the triangle inequality gives us $|y - x| + |x - y_i| \geq |y - y_i|$, we obtain:

$$|y - x| \geq |y - y_i| - |x - y_i| \geq d_{y_i} - \frac{d_{y_i}}{2} = \frac{d_{y_i}}{2} \geq d.$$

Therefore, our choice of $d > 0$ satisfies $|y - x| \geq d$ for all $x \in A$ and all $y \in B$, as required. $\square$
c) Finally, we will give a counterexample in $\mathbb{R}^2$ if $A$ and $B$ are closed but neither is compact. Let us define $A$ to be:

$$A = \{(n, \frac{1}{n}) : n \in \mathbb{N}\},$$

and let us define $B$ to be:

$$B = \{(n, -\frac{1}{n}) : n \in \mathbb{N}\}.$$

First, $A$ and $B$ are clearly disjoint because all points $(x_1, x_2) \in A$ have a positive $x_2$-coordinate and all points $(x_1, x_2) \in B$ have a negative $x_2$-coordinate. Moreover, we can show that $A$ and $B$ are not compact. The family $\{U_n\}_{n\in\mathbb{N}}$ of open sets defined by $U_n = (n-1, n+1) \times \mathbb{R}$ is an open cover for both $A$ and $B$ because the points $(n, \frac{1}{n})$ and $(n, -\frac{1}{n})$ are in $U_n$ for all $n \in \mathbb{N}$. Moreover, each $U_n$ only covers one point in $A$ and $B$, namely $(n, \frac{1}{n})$ and $(n, -\frac{1}{n})$, respectively. As a result, since $A$ and $B$ both have infinitely many points, they cannot be covered by a finite subcover using the $U_n$s, so $A$ and $B$ are both not compact.

Next, we will show that $A$ is closed. Let $(x_1, x_2)$ be any point in the complement $A^c$. Then, consider the four following cases:

**Case 1**: $x_1 < 1$. Then, consider the open rectangle $R = (x_1 - 1, 1) \times (x_2 - 1, x_2 + 1)$. Since $x_1 - 1 < x_1 < 1$ and $x_2 - 1 < x_2 < x_2 + 1$, we have $(x_1, x_2) \in R$. Additionally, since all points in $R$ have an $x_1$-coordinate less than $1$, they cannot be in $A$, so $R \subseteq A^c$.

**Case 2**: $x_1 > 1$ and $x_1 \notin \mathbb{N}$. Then, $x_1$ is in some open interval $(n, n+1)$, where $n \in \mathbb{N}$. Now, consider the open rectangle $R = (n, n+1) \times (x_2 - 1, x_2 + 1)$. Since $x_1 \in (n, n+1)$ and $x_2 - 1 < x_2 < x_2 + 1$, we have $(x_1, x_2) \in R$. Additionally, since all points in $R$ have an $x_1$-coordinate which is not an integer, they cannot be in $A$, so $R \subseteq A^c$.

**Case 3**: $x_1 \in \mathbb{N}$ and $x_2 > \frac{1}{x_1}$. Then, consider the open rectangle $R = (x_1 - 1, x_1 + 1) \times (\frac{1}{x_1}, x_2 + 1)$. Since $x_1 - 1 < x_1 < x_1 + 1$ and $\frac{1}{x_1} < x_2 < x_2 + 1$, we have $(x_1, x_2) \in R$. Additionally, the only integer in the interval $(x_1 - 1, x_1 + 1)$ is $x_1$, so the only point that could potentially be in $R \cap A$ is $(x_1, \frac{1}{x_1})$. Since $\frac{1}{x_1} \notin (\frac{1}{x_1}, x_2 + 1)$, we conclude that $R \subseteq A^c$.

**Case 4**: $x_1 \in \mathbb{N}$ and $x_2 < \frac{1}{x_1}$. Then, consider the open rectangle $R = (x_1 - 1, x_1 + 1) \times (x_2 - 1, \frac{1}{x_1})$. Since $x_1 - 1 < x_1 < x_1 + 1$ and $x_2 - 1 < x_2 < \frac{1}{x_1}$, we have $(x_1, x_2) \in R$. Additionally, similarly to Case 3, the only point that could potentially be in $R \cap A$ is $(x_1, \frac{1}{x_1})$, and we know that $(x_1, \frac{1}{x_1}) \notin R$ because $\frac{1}{x_1} \notin (x_2 - 1, \frac{1}{x_1})$. Thus, $R \subseteq A^c$.

Note that these cases are exhaustive – the case "$x_1 \in \mathbb{N}$ and $x_2 = \frac{1}{x_1}$" cannot occur because $(x_1, x_2) \notin A$. As a result, for all $(x_1, x_2) \in A^c$, we found an open rectangle $R$ around $(x_1, x_2)$ contained in $A^c$, so $A^c$ is open. Thus, $A$ is closed, as desired. In the same way, we can also prove that $B$ is closed.

Finally, we will prove for all $d > 0$ that there exist $x \in A$ and $y \in B$ such that $|y - x| < d$. Let us pick a natural number $n$ larger than $\frac{2}{d}$, and let us pick $x = (n, \frac{1}{n}) \in A$ and $y = (n, -\frac{1}{n}) \in B$. Then, we obtain:

$$|x - y| = \sqrt{(n-n)^2 + \left(\frac{1}{n} - \left(-\frac{1}{n}\right)\right)^2} = \frac{2}{n} < \frac{2}{\frac{2}{d}} = d.$$

Therefore, our choices of $A$ and $B$ are disjoint, closed, and not compact, and for all $d > 0$, there exist $x \in A$ and $y \in B$ such that $|y - x| < d$. This proves that $A$ and $B$ are valid counterexamples in $\mathbb{R}^2$, as required. $\square$

3. (Note: This solution was submitted to Crowdmark but not marked.)
If $U$ is open $C \subseteq U$ is compact, we will show that there is a compact set $D \subseteq U$ whose interior contains $C$.
First, the results proven in parts a) and b) of Question 2 will be useful to us, so their solutions are reproduced below for reference:

2. a) Given some closed set $A$ and some point $x \notin A$, we will prove that there is a number $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.
First, since $A$ is closed, its complement, $A^c$, is open. Then, since $x \in A^c$, there exists an open rectangle $R = \prod_{i=1}^{n}(a_i, b_i)$ around $x$ contained in $A^c$. In other words, if we denote the $i^{\text{th}}$ coordinate of $x$ by $x_i$, then $a_i < x_i < b_i$ for all $1 \leq i \leq n$. As a result, we can define the positive number $d$ by:

$$d = \min_{1 \leq i \leq n}(\min(x_i - a_i, b_i - x_i)).$$

Now, we claim that $|y - x| \geq d$ for all $y \in A$ for our choice of $d$. Since $y \in A$, and since $R \subseteq A^c$, we have $y \notin R$. As a result, there exists some $1 \leq j \leq n$ such that $y_j \notin (a_j, b_j)$. If $y_j \leq a_j$, we obtain $x_j - y_j \geq x_j - a_j \geq d$, and if $y_j \geq b_j$, we obtain $y_j - x_j \geq b_j - x_j \geq d$. Either way, we get $\left|x_j - y_j\right| \geq d$, so:

$$|x - y| = \sqrt{\sum_{i=1}^{n}|x_i - y_i|^2} \geq \sqrt{\left|x_j - y_j\right|^2} \geq d.$$

Therefore, our choice of $d > 0$ satisfies $|y - x| \geq d$ for all $y \in A$, as required. □
b) Given two disjoint sets $A, B$ such that $A$ is closed and $B$ is compact, we will prove that there exists $d > 0$ such that $|y - x| \geq d$ for all $x \in A$ and $y \in B$.
First, for all $y \in B$, we know from part a) that there exists a positive number $d_y$ such that $|x - y| \geq d_y$ for all $x \in A$. Then, we can define $B_y$ to be the open ball of radius $\frac{d_y}{2}$ around $y$. Since each $y \in B_y$ for all $y \in B$, these open balls form an open cover of $B$. Thus, since $B$ is compact, there exists a finite subcover $B_{y_1}, \ldots, B_{y_k}$ of $B$ with open balls around $y_1, \ldots, y_k$, respectively. Now, we can define the positive real number:

$$d = \min_{1 \leq i \leq k} \frac{d_{y_i}}{2}.$$

Then, we claim that $|y - x| \geq d$ for all $x \in A$ and all $y \in B$ for our choice of $d$. Since $y \in B$, $y$ is contained in some open ball $B_{y_i}$ in our finite subcover of $B$. Then, by definition of $d_{y_i}$, we have $|x - y_i| \geq d_{y_i}$. Moreover, since $y \in B_{y_i}$, and since $B_{y_i}$ is of radius $\frac{d_{y_i}}{2}$, we have $|y - y_i| < \frac{d_{y_i}}{2}$. As a result, since the triangle inequality gives us $|y - x| + |x - y_i| \geq |y - y_i|$, we obtain:

$$|y - x| \geq |y - y_i| - |x - y_i| \geq d_{y_i} - \frac{d_{y_i}}{2} = \frac{d_{y_i}}{2} \geq d.$$

Therefore, our choice of $d > 0$ satisfies $|y - x| \geq d$ for all $x \in A$ and all $y \in B$, as required. □

Now, we are ready to solve Question 3. Suppose we are given an open set $U$ and a compact set $C \subseteq U$. Then, since $U$ is open, its complement, $U^c$, is closed. Moreover, since $C \subseteq U$, we obtain that $C$ and $U^c$ are disjoint. Therefore, by Question 2 part b), there exists $d > 0$ such that $|x - y| \geq d$ for all $x \in C$ and $y \in U^c$. Now, for all $x \in C$, consider the open ball $B_x$ and closed ball $D_x$ around $x$ with radius $\frac{d}{2}$; in other words:

$$B_x = \{x' \in \mathbb{R}^n : \left|x' - x\right| < \frac{d}{2}\}, \qquad D_x = \{x' \in \mathbb{R}^n : \left|x' - x\right| \leq \frac{d}{2}\}.$$

(Note that $B_x \subseteq D_x$ because $\left|x' - x\right| < \frac{d}{2} \Rightarrow \left|x' - x\right| \leq \frac{d}{2}$.) Now, for all $x \in C$, we have $x \in B_x$ because $|x - x| = 0 < \frac{d}{2}$. As a result, the family $\{B_x\}_{x \in C}$ forms an open cover of $C$. Since $C$ is compact, we can extract a finite subcover $\{B_x\}_{x \in C'}$, where $C'$ is a finite subset of $C$. Then, let us define $D$ by $D = \bigcup_{x \in C'} D_x$. We will show that $D$ satisfies the desired properties.

First, we will show that $D \subseteq U$. For all $x' \in D$, there exists $x \in C'$ such that $x' \in D_x$, so $\left|x' - x\right| \leq \frac{d}{2}$. Then, as discussed above, all points in $U^c$ are a distance of at least $d$ away from $x$ because $x \in C$. Since $\left|x' - x\right| \leq \frac{d}{2}$, this implies that $x' \notin U^c$ for all $x' \in D$. As a result, $D \subseteq U$.

Next, we will show that $D$ is compact. For all $x \in C'$, the closed ball $D_x$ is bounded because every point in $D_x$ must be a distance of at most $\frac{d}{2}$ away from $x$. Then, since $D$ is a union of finitely many such balls, $D$ is also bounded. Additionally, for all $y \notin D_x$, we have $|y - x| > \frac{d}{2}$. Then, we can define the positive real number $r = |y - x| - \frac{d}{2}$, and we can consider the open ball $U$ around $y$ of radius $r$. For all $z \in U$, we have $|z - y| < r = |y - x| - \frac{d}{2}$, so by the triangle inequality, we obtain:

$$|z - x| \geq |y - x| - |y - z| > |y - x| - \left(|y - x| - \frac{d}{2}\right) = \frac{d}{2}.$$

As a result, $z \notin D_x$ for all $z \in U$. In other words, $U$ is an open neighbourhood around $y$ inside $D_x^c$. Since such a neighbourhood $U$ exists for all $y \notin D_x$, we conclude that $D_x^c$ is open, so each $D_x$ is closed, so $D$ is also closed as a finite union of closed sets. Thus, since $D$ is both closed and bounded, it is compact by Spivak's Corollary 1-7.

Finally, we will show that the interior of $D$ contains $C$. For all $x' \in C$, since $\{B_x\}_{x \in C'}$ is an open cover of $C$, there exists $x \in C'$ for which $x' \in B_x$. Moreover, $B_x \subseteq D_x \subseteq D$, so $B_x$ is an open neighbourhood of $x'$ contained in $D$. As a result, $x'$ is in the interior of $D$. Since this is true for all $x \in C$, we conclude that the interior of $D$ contains $C$. Overall, we found a compact set $D \subseteq U$ whose interior contains $C$, as required. □

4. (Note: This solution was submitted to Crowdmark but not marked.)
We will prove that any linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is continuous.
First, in Assignment 1 Question 2, we proved that there exists $M \in \mathbb{R}$ such that $\big|T(h)\big| \leq M|h|$ for all $h \in \mathbb{R}^n$. We can assume without loss of generality that $M > 0$; otherwise, if $M \leq 0$, we could select any $M' > 0$, and $M'$ would satisfy $\big|T(h)\big| \leq M|h| \leq M'|h|$ for all $h \in \mathbb{R}^n$.
Now, let any $a \in \mathbb{R}^n$ be given. Also, let any $\epsilon > 0$ be given. Then, let us define $\delta > 0$ by $\delta = \frac{\epsilon}{M}$; we can divide by $M$ because $M > 0$. Next, let $x$ be any point in $\mathbb{R}^n$ satisfying $|x - a| < \delta$. Then, since $T$ is linear, we obtain:

$$\big|T(x) - T(a)\big| = \big|T(x - a)\big| \leq M|x - a| < M\delta = M \cdot \frac{\epsilon}{M} = \epsilon.$$

Therefore, for all $\epsilon > 0$, we found $\delta > 0$ such that all $x \in \mathbb{R}^n$ which satisfy $|x - a| < \delta$ also satisfy $\big|T(x) - T(a)\big| < \epsilon$, so $T$ is continuous at $a$. Since this is true for all $a \in \mathbb{R}^n$, we conclude that $T$ is a continuous function, as required. $\square$

5. We are given $A = \{(x,y) \in \mathbb{R}^2 : x > 0 \text{ and } 0 < y < x^2\}$. We are also given the indicator function $f : \mathbb{R}^2 \to \mathbb{R}$ of $A$, defined by $f(x,y) = 1$ if $(x,y) \in A$, and $f(x,y) = 0$ otherwise. We will show that $f$ is not continuous at $(0,0)$, yet its restriction to every straight line through $(0,0)$ is continuous at $(0,0)$.
First, to show that $f$ is not continuous at $(0,0)$, let us pick $\epsilon = \frac{1}{2}$. Then, let any $\delta > 0$ be given. Clearly, $(0,0) \notin A$ because $(0,0)$ does not satisfy $x > 0$, so $f(0,0) = 0$. Next, consider the following two cases:
**Case 1**: $\delta \geq 2$. Then, let us define $z = (1, \frac{1}{2})$. We obtain $z \in A$ because $1 > 0$ and because $0 < \frac{1}{2} < 1 = 1^2$. As a result, $f(z) = 1$. We also have $\left|z - (0,0)\right| = \sqrt{1^2 + (\frac{1}{2})^2} < 2 \leq \delta$. Thus, our chosen $z \in \mathbb{R}^n$ satisfies $\left|z - (0,0)\right| < \delta$, and it also satisfies $\left|f(z) - f(0,0)\right| = |1 - 0| = 1 > \epsilon$.
**Case 2**: $0 < \delta < 2$. Then, let us define $z = (\frac{\delta}{2}, \frac{\delta^2}{8})$. We obtain $z \in A$ because $\frac{\delta}{2} > 0$ and because $0 < \frac{\delta^2}{8} < \frac{\delta^2}{4} = (\frac{\delta}{2})^2$. As a result, $f(z) = 1$. We also have:

$$\begin{aligned}
\left|z - (0,0)\right| &= \sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta^2}{8}\right)^2} \\
&\leq \sqrt{\left(\frac{\delta}{2}\right)^2 + \left(\frac{\delta \cdot 2}{8}\right)^2} && \text{(Since } 0 < \delta < 2\text{)} \\
&= \sqrt{\frac{5}{16}\delta^2} \\
&< \delta.
\end{aligned}$$

Thus, our chosen $z \in \mathbb{R}^n$ satisfies $\left|z - (0,0)\right| < \delta$, and it also satisfies $\left|f(z) - f(0,0)\right| = 1 > \epsilon$.
Overall, we found $\epsilon > 0$ such that for all $\delta > 0$, there exists $z \in \mathbb{R}^n$ such that $\left|z - (0,0)\right| < \delta$ and $\left|f(z) - f(0,0)\right| > \epsilon$, so $f$ is not continuous at $(0,0)$, as required.
Next, let $\ell$ be any straight line through $(0,0)$. Then, we will show that the restriction of $f$ to $\ell$ is continuous. First, if $\ell$ is vertical, or if $\ell$ has a slope that is not positive, $\ell$ will not pass through the first quadrant, so it will not intersect $A$ because $A$ is contained in the first quadrant. As a result, the restriction of $f$ to $\ell$ is the zero function, which is obviously continuous. (Formally: Given any $\epsilon > 0$, we can pick $\delta = 1 > 0$, then all $z \in \ell$ satisfying $\left|z - (0,0)\right| < \delta$ would also satisfy $\left|f(z) - f(0,0)\right| = |0 - 0| < \epsilon$.) From now on, we will focus on the nontrivial case where $\ell$ has some positive slope $m$.
Let any $\epsilon > 0$ be given. Then, let us define $\delta = m > 0$. Next, let $(x, mx)$ be any point in $\ell$ satisfying $\left|(x, mx) - (0,0)\right| < \delta$. If $x \leq 0$, then we cannot have $x > 0$, so $(x, mx) \notin A$. Otherwise, if $x > 0$, we obtain:

$$\delta > \left|(x, mx) - (0,0)\right| = \sqrt{(x-0)^2 + (mx-0)^2} \geq x.$$

As a result, $m = \delta > x$, so we obtain $mx > x^2$, giving us $(x, mx) \notin A$. In either case, we get $(x, mx) \notin A$, which means $f(x, mx) = 0$. As a result, $\left|f(x, mx) - f(0,0)\right| = |0 - 0| = 0 < \epsilon$. Therefore, given any $\epsilon > 0$, we found $\delta > 0$ such that all $(x,y) \in \ell$ satisfying $\left|(x,y) - (0,0)\right| < \delta$ also satisfy $\left|f(x,y) - f(0,0)\right| < \epsilon$, so the restriction of $f$ to any line $\ell$ through $(0,0)$ is continuous, as required. $\square$

6. (Note: This solution was submitted to Crowdmark but not marked.)
Given a set $A \subseteq \mathbb{R}^n$ which is not closed, we will show that there exists a continuous function $f : A \to \mathbb{R}$ which is unbounded.
Since $A$ is not closed, its complement, $A^c$, is not open. As a result, there exists $a \in A^c$ such that every open set around $a$ contains some point in $A$. Then, let us define the function $f : A \to \mathbb{R}^n$ by $f(x) = \frac{1}{|x-a|}$. Since $a \in A^c$, we know that $x \neq a$ for all $x \in A$, so $|x - a| > 0$, which means that $f(x)$ is well-defined.
Now, we will show that $f$ is unbounded. For all $M > 0$, consider the open ball $B_M$ around $a$ of radius $\frac{1}{M}$. Since every open set around $a$ contains some point in $A$, there exists some $x \in B_M \cap A$. Since $x \in B_M$, we know that $|x - a| < \frac{1}{M}$, so $f(x) = \frac{1}{|x-a|} > M$. As a result, for all $M > 0$, there exists some $x \in A$ such that $f(x) > M$, so $f$ is unbounded.
Finally, we will show that $f$ is continuous. Let $x_0$ be any point in $A$. Then, let any $\epsilon > 0$ be given. Now, let us define the positive number:

$$\delta = \min\left(\frac{|x_0 - a|}{2}, \frac{\epsilon|x_0 - a|^2}{4}\right).$$

Then, for all $x \in A$ such that $|x_0 - x| < \delta$, we obtain:

$$\begin{aligned}
\big|f(x) - f(x_0)\big| &= \left|\frac{1}{|x-a|} - \frac{1}{|x_0 - a|}\right| \\
&= \frac{\big||x_0 - a| - |x - a|\big|}{|x - a||x_0 - a|} \\
&\leq \frac{|x_0 - x|}{|x - a||x_0 - a|} && \text{(Applying triangle inequality)} \\
&\leq \frac{|x_0 - x|}{\big||x_0 - a| - |x_0 - x|\big| \cdot |x_0 - a|} && \text{(Applying triangle inequality)} \\
&\leq \frac{|x_0 - x|}{\left(|x_0 - a| - \frac{1}{2}|x_0 - a|\right) \cdot |x_0 - a|} && \left(\text{Since } |x_0 - x| < \delta \leq \frac{|x_0 - a|}{2}\right) \\
&= \frac{|x_0 - x|}{\frac{1}{2}|x_0 - a|^2} \\
&\leq \frac{\frac{\epsilon}{4}|x_0 - a|^2}{\frac{1}{2}|x_0 - a|^2} && \left(\text{Since } |x_0 - x| < \delta \leq \frac{\epsilon|x_0 - a|^2}{4}\right) \\
&= \frac{\epsilon}{2} \\
&< \epsilon.
\end{aligned}$$

Therefore, for all $\epsilon > 0$, we found $\delta > 0$ such that all $x \in A$ which satisfy $|x - x_0| < \delta$ also satisfy $\big|f(x) - f(x_0)\big| < \epsilon$. Since this is true for all $x_0 \in A$, we conclude that $f$ is continuous. Thus, our choice of $f$ is both unbounded and continuous, as required. ☐

7. We will prove that a set $C$ is compact if and only if every open cover $\mathcal{U}$ of $C$ that is closed under unions of pairs has a set $T$ such that $C \subseteq T$.
First, for the "$\Rightarrow$" direction, suppose that $C$ is compact, and let $\mathcal{U}$ be any open cover of $C$ that is closed under unions of pairs. Then, since $C$ is compact, there exists a finite subcover of $C$ contained in $\mathcal{U}$. As a result, we can define $k$ to be the smallest finite number of open sets in $\mathcal{U}$ required to cover $C$. Now, assume for contradiction that $k > 1$. Then, there exists a finite subcover $\mathcal{U}_1 \subseteq \mathcal{U}$ of $C$ with $k$ open sets $U_1, \ldots, U_k \in \mathcal{U}$. Since $\mathcal{U}$ is closed under unions of pairs, we have $U_1 \cup U_2 \in \mathcal{U}$. We also have:

$$(U_1 \cup U_2) \cup U_3 \cup \cdots \cup U_k = U_1 \cup U_2 \cup U_3 \cup \cdots \cup U_k,$$

so $(U_1 \cup U_2), U_3, \ldots, U_k$ cover $C$ because $U_1, \ldots, U_k$ also cover $C$. As a result, $(U_1 \cup U_2), \ldots, U_k$ is a subcover of $C$ contained in $\mathcal{U}$ with $k-1$ elements, contradicting the minimality of $k$. Therefore, by contradiction, $k = 1$, so there exists a subcover of $C$ contained in $\mathcal{U}$ with $1$ open set $T$. In other words, the set $T \in \mathcal{U}$ satisfies $C \subseteq T$, as required for the "$\Rightarrow$" direction.
Next, for the "$\Leftarrow$" direction, suppose that every open cover $\mathcal{U}$ of $C$ that is closed under unions of pairs has a set $T$ such that $C \subseteq T$. Then, let $\mathcal{U}_1$ be any general open cover of $C$. We can define $\mathcal{U}_2$ to be the set of all unions of finitely many sets in $\mathcal{U}_1$. Now, for all $(U_1 \cup \cdots \cup U_k), (V_1 \cup \cdots \cup V_j) \in \mathcal{U}_2$, where $U_1, \ldots, U_k, V_1, \ldots, V_j \in \mathcal{U}_1$, we find that the union:

$$(U_1 \cup \cdots \cup U_k) \cup (V_1 \cup \cdots \cup V_j)$$

is also a union of finitely many sets in $\mathcal{U}_1$, so it is also in $\mathcal{U}_2$. In other words, $\mathcal{U}_2$ is closed under unions of pairs. Then, by assumption, there exists a set $T \in \mathcal{U}_2$ such that $C \subseteq T$. Next, since $T \in \mathcal{U}_2$, we can write $T$ in the form $U_1 \cup \cdots \cup U_k$, where $U_1, \ldots, U_k \in \mathcal{U}_1$. Since $C \subseteq T = U_1 \cup \cdots \cup U_k$, it follows that $U_1, \ldots, U_k$ is a finite subcover of $C$ contained in $\mathcal{U}_1$. Therefore, since every open cover of $C$ has a finite subcover of $C$, we conclude that $C$ is compact, as required for the "$\Leftarrow$" direction. Since we have proven both directions, we are done. $\square$

## Notes on intuition

Now, let's develop some intuition on how to approach these problems and find solutions for them. (Note: This section was not submitted on Crowdmark.)

1. For the closed ball $A_1$, one could draw a geometric picture of a closed ball. Then, it becomes intuitive that the boundary of $A_1$ consists of the surface of $A_1$, which contains points with a magnitude (i.e, norm) of $1$. To motivate the formal proof, we are essentially picking some line passing through the boundary point $x$ – such as a line in the $x_1$-direction. This line would have to poke outside the closed ball, even if it is tangent to the ball, and we prove this algebraically. Next, we can also see that all points closer to the origin (i.e., with a smaller magnitude) will be in the interior of $A_1$, and all points farther away (with a larger magnitude) will be in the exterior. We formalize these ideas using standard proof techniques involving the triangle inequality. The intuition for $A_2$ is similar, except $A_2$ is a hollow sphere, so all points closer to the origin will now be outside $A_2$ and in its exterior.
   The solution for $A_3$ is similar to that of Assignment 1 Question 5. Since $\mathbb{Q}$ and $\mathbb{R} - \mathbb{Q}$ are both dense, we can show that every point in $\mathbb{R}^n$ is surrounded both by points with rational coordinates (points inside $A_3$) and by points with irrational coordinates (points outside $A_3$). It follows that, surprisingly, every point in $\mathbb{R}^n$ is in the boundary of $A_3$!

2. a) Here, we are only told $x$ is inside the complement $A^c$ of $A$, where $A^c$ is open. This is enough to construct an open ball around $x$ contained in $A^c$. In other words, this ball cannot contain points in $A$, so its radius naturally gives us a distance between $x$ and $A$.
   b) According to the textbook's hint, for all $y \in B$, we should find an open set $B_y$ containing $y$ such that $B_y \cap B$ has some positive distance away from $A$. Indeed, in part a), we already found that $y$ has some positive distance $d_y$ away from $B$. Then, we could try defining $B_y$ to be some open ball around $y$. This ball's radius must be chosen carefully – if we choose $d_y$ to be the radius, then some points inside the open ball may be too close to $B$ because $B$ is also a distance of $d_y$ from $y$! To amend this, we choose a radius of $\frac{d_y}{2}$ to leave some leeway of size $\frac{d_y}{2}$ between $B_y$ and $y$. Once we find these open balls around each $y$, the purpose of the textbook's hint becomes clear: These open balls cover $B$, so we could take the minimum of the $\frac{d_y}{2}$s to find the distance between $A$ and $B$. One problem remains: if there are infinitely $y$s, we cannot take the minimum of infinitely many numbers in general. Here is where we use the compactness of $B$. By taking a finite subcover using $B_y$s, we now only have to take the minimum of finitely many $\frac{d_y}{2}$s, which is possible.
   c) In a nutshell, our counterexample for $A$ and $B$ in this solution set consists of infinitely many single points, all separated from each other, which approach each other as we move farther away from the origin. First, the reason why we use single points is because we know that any finite collection of single points is closed. By separating all points in $A$ with a distance of at least $1$, there will be locally finitely many single points in $A$ around any $x \in A^c$, which allows us to easily find an open rectangle around $x$ contained in $A^c$. (The same thought process applies for $B$.) Next, from part b), we learned that if $A$ or $B$ is compact (which is equivalent to being closed and bounded), then $A$ and $B$ cannot be used as a counterexample. Instead, we need $A$ and $B$ to be unbounded in our counterexample. This motivates us to design $A$ and $B$ whose points approach each other, but do not coincide, as we move farther away from the origin.

3. First, if we want to surround $C$ with $D \subseteq U$, and we want to surround $C$ enough so that $C$

is inside the interior of $D$, it is helpful to have some distance between $C$ and $U^c$ so that we can surround $C$. Fortunately, Question 2 gives us this distance. Next, to have $D$ be closed and bounded (and thus compact), we can try taking a finite union of closed balls around points in $C$. (As with Question 2, we use a radius of $\frac{d}{2}$ instead of $d$ to create some leeway between $D$ and $U^c$.) These closed balls must form a finite cover of $C$ so that we can take their finite union. To facilitate this, the proof contains some technical details about using open balls of radius $\frac{d}{2}$ to approximate the closed balls, then picking a finite subcover with these balls because $C$ is compact.

4. First, the textbook's hint tells us to use Problem 1-10, which let us bound $|Th|$ in terms of $|h|$. In other words, if $h$ is small (i.e., near zero), then $Th$ is small (i.e., near zero). Since $T$ is linear, we can easily generalize this result: If some $x = a + h$ is near $a$ (i.e., $h$ is small), then $Tx$ is near $Ta$ because $Tx = Ta + Th$, where $Th$ is small. This gives us the continuity of $T$ that we wanted.

5. We are given the region $A$, which is bounded by the positive $x$-axis and the parabola $y = x^2$. Then, as the parabola approaches $(0, 0)$, it is geometrically clear that points in $A$ become arbitrarily close to $(0, 0)$. As a result, the indicator function is discontinuous at $(0, 0)$. Next, let us consider the direction that such points approach $(0, 0)$. Since $y = x^2$ is parabolic, the direction of approach becomes increasingly horizontal. In other words, if these points in $A$ were to approach $(0, 0)$ along some positively-sloped line through $(0, 0)$, this line would rise above $A$ near $(0, 0)$ as $A$ becomes more horizontal. The only lines left to consider are "special" cases, such as a horizontal line through $(0, 0)$. Fortunately, these cases are easy because those lines do not intersect $A$.

6. As stated in the textbook's hint, we pick some point $x$ that is outside $A$ and also on the boundary of $A$, then we want to define $f(y) := \frac{1}{|y-x|}$ for all $y \in A$. Since $x$ is on the boundary of $A$, there are points $y$ in $A$ arbitrarily close to $x$, so $f(y) = \frac{1}{|y-x|}$ becomes unbounded. Additionally, to prove that $f$ is continuous, our solution is heavily motivated by the MAT157 proof that the function $g : \mathbb{R} - \{0\} \to \mathbb{R}$ defined by $g(x) = \frac{1}{x}$ is continuous.

7. For the "$\Rightarrow$" direction, the key idea is that "Closed under pairwise unions" is equivalent to "Closed under finite unions" because pairwise unions can be performed multiple times in a row to form finite unions. In this case, when we are given an open cover $\mathcal{U}$ of $C$ that is closed under pairwise unions, $\mathcal{U}$ is also closed under finite unions. Since $C$ is compact, we extract a finite subcover from $\mathcal{U}$, and since $\mathcal{U}$ is closed under finite unions, this finite subcover gives a $T \in \mathcal{U}$ which covers $C$.
For the "$\Leftarrow$" direction, to prove $C$ is compact, we consider a general open cover $\mathcal{U}_1$ of $C$. The problem is that we cannot do anything with an open cover unless it is closed under pairwise unions, so we hope to construct some open cover $\mathcal{U}_2$ of $C$ which is closed under pairwise unions. If we construct such a cover $\mathcal{U}_2$, we would be able to extract some $T \in \mathcal{U}_2$ which covers $C$. Ideally, this $T$ should give us a finite subcover of $C$, which motivates us to take $\mathcal{U}_2$ to contain all finite unions of sets in $\mathcal{U}_1$. After verifying that this $\mathcal{U}_2$ is indeed closed under pairwise unions, we are done.