Homework Assignment 1

Due: Friday, September 24, 2021 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following questions. They are all taken from Spivak's book, pages 4,5, and 10.

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 1-7. A linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ is **norm preserving** if $|T(x)| = |x|$ for all x, and **inner product preserving** if $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all x, y.

(a) Prove that T is norm preserving iff it is inner product preserving.

(b) Prove that such a linear transformation is 1-1 and onto, and that T^{-1} is also norm and inner product preserving.

Q2 (10 points)

Spivak's 1-10. If $T : \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, show that there is a number M such that $|T(h)| \leq M|h|$ for all $h \in \mathbb{R}^m$. *Hint:* Estimate $|T(h)|$ in terms of $|h|$ and the entries of the matrix of T.

Q3 (10 points)

Spivak's 1-12. Let $(\mathbb{R}^n)^*$ denote the dual space of the vector space $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, define $\varphi_x \in (\mathbb{R}^n)^*$ by $\varphi_x(y) = \langle x, y \rangle$. Define $T : \mathbb{R}^n \to (\mathbb{R}^n)^*$ by $T(x) = \varphi_x$. Show that T is a 1-1 linear transformation and conclude that every $\varphi \in (\mathbb{R}^n)^*$ is φ_x for a unique $x \in \mathbb{R}^n$.

Q4 (10 points)

Spivak's 1-13. If $x, y \in \mathbb{R}^n$, then x and y are called **perpendicular** if $\langle x, y \rangle = 0$. If x and y are perpendicular, show that $|x + y|^2 = |x|^2 + |y|^2$.

Q5 (10 points)

Spivak's 1-18. If $A \subset [0, 1]$ contains all the rational numbers in $(0, 1)$ and is the union of open intervals (a_i, b_i), show that the boundary of A is $[0, 1] \setminus A$.

Q6 (10 points)

Spivak's 1-19. If A is a closed set that contains every rational number in $[0, 1]$, show that $[0, 1] \subset A$.

Homework Assignment 2

Due: Friday, October 1, 2021 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following questions. Note that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 1-16. Find the interior, exterior, and boundary of the sets

$$A_1 = \{x \in \mathbb{R}^n : |x| \leq 1\},$$

$$A_2 = \{x \in \mathbb{R}^n : |x| = 1\},$$

$$A_3 = \{x \in \mathbb{R}^n : \forall i \, x_i \in \mathbb{Q}\}.$$

Q2 (10 points)

Spivak's 1-21. (a) If A is closed and $x \notin A$, prove that there is a number $d > 0$ such that $|y - x| \geq d$ for all $y \in A$.

(b) If A is closed, B is compact and $A \cap B = \emptyset$, prove that there is $d > 0$ such that $|y - x| \geq d$ for all $x \in A$ and $y \in B$ (hint available in textbook).

(c) Give a counterexample in $\mathbb{R}^2$ if A and B are closed but neither is compact.

Q3 (0 points)

Spivak's 1-22. If U is open and $C \subset U$ is compact, show that there is a compact set $D \subset U$ whose interior contains C.

Q4 (0 points)

Spivak's 1-25. Prove that a linear transformation $T\colon \mathbb{R}^n \to \mathbb{R}^m$ is continuous (hint available in textbook).

Q5 (10 points)

Spivak's 1-26, rephrased. Let $A = \{(x,y) \in \mathbb{R}^2 \colon x > 0 \text{ and } 0 < y < x^2\}$. Let $f = 1_A \colon \mathbb{R}^2 \to \mathbb{R}$ be the *indicator* function of A, defined by $f(x,y) = 1$ if $(x,y) \in A$, and $f(x,y) = 0$ otherwise. Show that f is not continuous at $(0,0)$, yet its restriction to every straight line through $(0,0)$ is continuous at $(0,0)$.

Q6 (0 points)

Spivak's 1-28. If $A \subset \mathbb{R}^n$ is not closed, show that there is a continuous function $f\colon A \to \mathbb{R}$ which is unbounded (hint available in textbook).

Q7 (10 points)

For submission. Prove that a set C is compact if and only if every open cover $\mathcal{U}$ of C that is closed under unions of pairs (namely, $(A \in \mathcal{U}) \text{ and } (B \in \mathcal{U}) \Longrightarrow (A \cup B \in \mathcal{U})$) has a set T such that $C \subset T$.

Do not submit, yet ponder. Okay, so we have a statement that can serve as an alternative definition of compactness. Perhaps it is better? Go over all the theorems about compactness shown in class and re-derive them with the definition of compactness replaced with the alternative one. Have they become easier or harder? Less or more "natural"?

This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 3

Due: Friday, October 8, 2021 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following problems. They are mostly taken from Spivak's book, pages 17,18, and 23. Note that the late policy is very strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 2-1. Prove that if $f\colon \mathbb{R}^n \to \mathbb{R}^m$ is diffable at $a \in \mathbb{R}^n$, then it is continuous at a. (Hint in text).

Q2 (10 points)

Spivak's 2-4. Let g be a continuous real-valued function on the unit circle $S^1 = \{x \in \mathbb{R}^2 \colon |x| = 1\}$ such that $g(0,1) = g(1,0) = 0$ and $g(-x) = -g(x)$. Define $f\colon \mathbb{R}^2 \to \mathbb{R}$ by $f(0) = 0$ and by $f(x) = |x| g(x/|x|)$ otherwise.

(a) Show that the restriction of f to any line through the origin is diffable. In other words, if $x \in \mathbb{R}^2$ and $h\colon \mathbb{R} \to \mathbb{R}$ is defined by $h(t) = f(tx)$, show that h is diffable.

(b) Show that f is not diffable at $(0,0)$ unless $g = 0$. (Hint in text).

Q3 (10 points)

Spivak's 2-7. Let $f: \mathbb{R}^n \to \mathbb{R}$ be a function such that $|f(x)| \leq |x|^2$. Show that f is diffable at 0.

Q4 (10 points)

Spivak's 2-10abg. Use the results from class and from Spivak's book up to page 23 to find f' in the following cases:

(a) $f(x, y, z) = x^y$.

(b) $f(x, y, z) = (x^y, z)$.

(g) $f(x, y, z) = (x + y)^z$.

Q5 (10 points)

Spivak's 2-12. A function $f: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^p$ is **bilinear** if it is linear as a function of the first $\mathbb{R}^n$ input when the second is held constant, and of the second $\mathbb{R}^m$ input when the first is held constant (details in text).

(a) Prove that if f is bilinear, then

$$\lim_{(h,k)\to 0} \frac{|f(h,k)|}{|(h,k)|} = 0.$$

(b) Prove that $Df(a,b)(x,y) = f(a,y) + f(x,b)$.

(c) Show that the formula for the differential of a product that we've seen in class (also in Spivak's Theorem 2-3) is a special case of (b).

Q6 (10 points)

Spivak's 2-14 and 2-15 cut short. The space of 2×2 matrices with entries in $\mathbb{R}$ is equivalent to $\mathbb{R}^4$ via $(a, b, c, d) \leftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Is the determinant function on 2×2 matrices, regarded as a function $f: \mathbb{R}^4 \to \mathbb{R}$, diffable? What is its differential?

This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 4

Due: Friday, October 15, 2021 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following problems. They are taken from Spivak's book, pages 23, 33, and 34. Note that the late policy remains strict - you will lose 10% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 2-18. Find the partial derivatives of the following functions (where $g: \mathbb{R} \to \mathbb{R}$ is continuous):

(a) $f(x, y) = \int_a^{x+y} g$.

(b) $f(x, y) = \int_y^{x} g$.

(c) $f(x, y) = \int_a^{xy} g$.

(d) $f(x, y) = \int_a^{\int_b^y g} g$.

Q2 (10 points)

Spivak's 2-19. If $f(x, y) = x^{x^{x^{x^y}}} + (\log x)(\arctan(\arctan(\arctan(\sin(\cos xy) - \log(x+y)))))$ find $D_2 f(1, y)$. (Hint in textbook).

Q3 (10 points)

Spivak's 2-21. Let $g_1, g_2: \mathbb{R}^2 \to \mathbb{R}$ be continuous. Define $f: \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt.$$

(a) Show that $D_2 f(x,y) = g_2(x,y)$.

(b) How would you change the definition of f so that $D_1 f(x,y) = g_1(x,y)$?

(c) Find a function $f_c: \mathbb{R}^2 \to \mathbb{R}$ such that $D_1 f_c(x,y) = x$ and $D_2 f_c(x,y) = y$.

(c) Find a function $f_d: \mathbb{R}^2 \to \mathbb{R}$ such that $D_1 f_d(x,y) = y$ and $D_2 f_d(x,y) = x$.

Q4 (10 points)

Spivak's 2-29. Let $f: \mathbb{R}^n \to \mathbb{R}$. For $x \in \mathbb{R}^n$, the limit

$$\lim_{t \to 0} \frac{f(a+tx) - f(a)}{t},$$

if it exists, is denoted $D_x f(a)$, and called the **directional derivative** of f at a, in the direction x.

(a) Show that $D_{e_i} f(a) = D_i f(a)$ (recall that e_i is the ith standard basis vector of $\mathbb{R}^n$).

(b) Show that $D_{tx} f(a) = t D_x f(a)$.

(c) If f is diffable at a, show that $D_x f(a) = Df(a)(x)$ and therefore $D_{x+y} f(a) = D_x f(a) + D_y f(a)$.

Q5 (10 points)

Spivak's 2-34. A function $f: \mathbb{R}^n \to \mathbb{R}$ is **homogeneous** of degree m if $f(tx) = t^m f(x)$ for all x. If f is also diffable, show that

$$\sum_{i=1}^n x_i D_i f(x) = m f(x).$$

(Hint in textbook).

Q6 (10 points)

Spivak's 2-35. If $f: \mathbb{R}^n \to \mathbb{R}$ is diffable and $f(0) = 0$, prove that there exist $g_i: \mathbb{R}^n \to \mathbb{R}$ such that

$$f(x) = \sum_{i=1}^n x_i g_i(x).$$

(Hint in textbook).

Homework Assignment 5

Due: Friday, October 22, 2021 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following problems. They are taken from Spivak's book, pages 39 and 40. Note that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 2-36. Let $A \subset \mathbb{R}^n$ be an open set and $f: A \to \mathbb{R}^n$ a continuously diffable 1-1 function such that $f'(x)$ is invertible for all $x \in A$. Show that $f(A)$ is an open set and that $f^{-1}: f(A) \to A$ is diffable. Show also that $f(B)$ is open for any open set $B \subset A$.

Q2 (10 points)

Spivak's 2-37. (a) Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a continuously diffable function. Show that f is *not* 1-1 (hint in textbook).

It is not required to submit part (b) of this question (though if you submit it, you will not lose any points, of course).

(b) Generalize this result to the case of a continuously diffable $f: \mathbb{R}^n \to \mathbb{R}^m$, where $n > m$.

Q3 (10 points)

Spivak's 2-38. (a) If $f: \mathbb{R} \to \mathbb{R}$ satisfies $f'(a) \neq 0$ for all $a \in \mathbb{R}$, show that f is 1-1 on $\mathbb{R}$.

(b) Define $f: \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (e^x \cos y, e^x \sin y)$. Show that $f'(x, y)$ is always invertible yet f is not 1-1.

This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 6

Due: Friday, October 29, 2021 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following problems. They are taken from Munkres' book, pages 78 and 79. Note that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Note that C^1 means "continuously differentiable".

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Let $f\colon \mathbb{R}^3 \to \mathbb{R}^2$ be a C^1 function; we write f in the form $f(x, y_1, y_2)$. Assume that $f(3, -1, 2) = 0$ and

$$f'(3,-1,2) = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \end{pmatrix}.$$

(a) Show that there is a function $g\colon B \to \mathbb{R}^2$ defined on an open set B in $\mathbb{R}$ such that $3 \in B$ and such that $g(3) = (-1, 2)$ and

$$f(x, g_1(x), g_2(x)) = 0$$

for $x \in B$.

(b) Find $g'(3)$.

(c) Discuss the problem of solving the equation $f(x, y_1, y_2) = 0$ for an arbitrary pair of the unknowns in terms of the third, near the point $(3, -1, 2)$.

Q2 (10 points)

Let $f: \mathbb{R}^2 \to \mathbb{R}$ be C^1, with $f(2,-1) = -1$. Set

$$G(x,y,u) = f(x,y) + u^2,$$

$$H(x,y,u) = ux + 3y^3 + u^3.$$

The equations $G(x,y,u) = 0$ and $H(x,y,u) = 0$ have the solution $(x,y,u) = (2,-1,1)$.

(a) What conditions on f' ensure that there are C^1 functions $x = g(y)$ and $u = h(y)$ defined on an open set in $\mathbb{R}$ that satisfy both equations, and such that $g(-1) = 2$ and $h(-1) = 1$?

(b) Under the conditions of (a) and assuming that $f'(2,-1) = (1 \quad -3)$, find $g'(-1)$ and $h'(-1)$.

Q3 (10 points)

Let $f, g: \mathbb{R}^3 \to \mathbb{R}$ be C^1 functions. "In general", one expects that each of the equations $f(x,y,z) = 0$ and $g(x,y,z) = 0$ represents a "nice" surface in $\mathbb{R}^3$ and that their intersections is a smooth curve. Show that if (x_0, y_0, z_0) satisfies both of these equations, and if $\partial(f,g)/\partial(x,y,z)$ has rank 2 at $p_0 = (x_0, y_0, z_0)$, then near p_0 one can solve these equations for two of x, y, z in terms of the third, thus representing the solution set locally as a parametrized curve.

Note. There is only one reasonable way to interpret the notation $\partial(f,g)/\partial(x,y,z)$.

Q4 (10 points)

Let $f: \mathbb{R}^{k+n} \to \mathbb{R}^n$ be a C^1 function; suppose that $f(a) = 0$ and that $f'(a)$ has rank n. Show that if c is a point of $\mathbb{R}^n$ sufficiently close to 0, then the equation $f(x) = c$ has a solution.

Homework Assignment 7

Due: Friday, November 19, 2021 11:59 pm (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. They are taken from Spivak's book, pages 49 and 52. Note that the late policy is strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

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Q1 (10 points)

Spivak's 3-3. Let A be a rectangle in $\mathbb{R}^n$ and let $f, g\colon A \to \mathbb{R}$ be integrable.

(a) For any partition P of A and any subrectangle S, show that

$$m_S(f) + m_S(g) \le m_S(f+g) \quad \text{and} \quad M_S(f+g) \le M_S(f) + M_S(g)$$

and therefore

$$L(f,P) + L(g,P) \le L(f+g,P) \quad \text{and} \quad U(f+g,P) \le U(f,P) + U(g,P).$$

(b) Show that $f+g$ is integrable and $\int_A (f+g) = \int_A f + \int_A g$.

(c) For any constant c, show that cf is integrable and $\int_A cf = c\int_A f$.

Q2 (10 points)

Spivak's 3-4. Let $f\colon A \to \mathbb{R}$ and let P be a partition of A. Show that f is integrable if and only if for each subrectangle S the function $F|_S$, the restriction of f to S, is integrable, and that in this case, $\int_A f = \sum_{S\in P} \int_S f|_S$.

Q3 (10 points)

Spivak's 3-5. Let $f, g: A \to \mathbb{R}$ be integrable and suppose $f \leq g$. Show that $\int_A f \leq \int_A g$.

Q4 (10 points)

Spivak's 3-6. If $f: A \to \mathbb{R}$ is integrable, show that $|f|$ is integrable and $\left|\int_A f\right| \leq \int_A |f|$.

Q5 (10 points)

Spivak's 3-9. (a) Show that an unbounded set cannot have content 0.

(b) Give an example of a closed set of measure 0 which does not have content 0.

Q6 (10 points)

Spivak's 3-11. Let $A = \bigcup_{i=1}^{\infty} (a_i, b_i)$ be a countable union of open intervals, and assume that $([0,1] \cap \mathbb{Q}) \subset A$. Show that if $\sum_{i=1}^{\infty} (b_i - a_i) < 1$ then the boundary of A is not of measure 0.

Q7 (10 points)

Spivak's 3-12. Let $f: [a, b] \to \mathbb{R}$ be an increasing function. Show that the set of discontinuities of f is of measure 0. (Hint in text).

Homework Assignment 8

Due: Friday, November 26, 2021 11:59 pm (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. They are taken from Spivak's book, pages 56 and 61-62. The late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Sorry for being nearly 10 hours late in assigning this assignment.

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Recall from reading the text that a set C is called "Jordan-measurable" if it is bounded and its boundary has measure 0, or equivalently, if it is bounded and its characteristic function is integrable in some rectangle that contains C.

Spivak's 3-22. If A is a Jordan-measurable set and $\epsilon > 0$, show that there is a compact Jordan-measurable set $C \subset A$ such that the volume of $A \setminus C$ is less than ϵ.

Q2 (10 points)

Recall from page 26 of the text that if f is a real-valued function, then $D_i f$ denotes its ith partial derivative, and $D_{i,j} f$ denotes the jth partial derivative of its ith partial derivative:

$$D_{i,j} f = D_j(D_i f)$$

(assuming all these quantities exist).

Spivak's 3-28. Use Fubini's theorem to give an easy proof that $D_{1,2} f = D_{2,1} f$, if these are continuous (hint in text).

Q3 (10 points)

Spivak's 3-32. Let $f\colon [a,b] \times [c,d] \to \mathbb{R}$ be continuous and suppose $D_2 f$ is continuous. Define $F(y) = \int_a^b f(x,y)dx$. Prove *Leibnitz' rule:* $F'(y) = \int_a^b D_2 f(x,y)dx$. (Hint in text).

Q4 (10 points)

Spivak's 3-34 (modified). Let $g_1, g_2\colon \mathbb{R}^2 \to \mathbb{R}$ be continuously differentiable and suppose $D_1 g_2 = D_2 g_1$. Let

$$f(x,y) = \int_0^x g_1(t,0)dt + \int_0^y g_2(x,t)dt.$$

Show that $D_1 f = g_1$ and $D_2 f = g_2$. (Hint: Use the previous question).

Homework Assignment 9

Due: Friday, December 3, 2021 11:59 pm (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

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Q1 (30 points)

Spivak's 3-38, modified. A function $f\colon \mathbb{R} \to \mathbb{R}$ vanishes for $x < 1$ and all on intervals of the form $(n - 1/3, n + 1/3)$, where n is a positive integer, yet for every such n, $\int_{n+1/3}^{n+2/3} f = (-1)^n/n$.

(a) Prove that such a function f exists and draw an approximate plot thereof.

(b) Show that $\int_{\mathbb{R}} f$ does not exist.

(c) Find two partitions of unity Φ and Ψ of $\mathbb{R}$, such that the sums $\sum_{\phi \in \Phi} \int \phi f$ and $\sum_{\psi \in \Psi} \int \psi f$ both absolutely converge, yet to different values.

Q2 (20 points)

Let A be an arbitrary subset of $\mathbb{R}^n$, and let $f\colon A \to \mathbb{R}$ be given. We say that "f is smooth at $a \in A$" if there is an open set U containing a and a smooth (C^∞) function $g\colon U \to \mathbb{R}$ such that f and g agree on $A \cap U$.

Prove that if f is smooth at every $a \in A$ then it can be extended to a smooth function on some open set $V \supset A$.

Warning. The function g from the definition of "smooth" may depend on a!

Hint. PO1.

This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 10

Due: Thursday, December 9, 2021 11:59 pm (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. They are all taken from Munkres' *Analysis on Manifolds*, page 151. Note the unusual due day - Thursday rather than Friday! Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

If $V = \{(x, y, z) : x^2 + y^2 + z^2 < a^2 \text{ and } z > 0\}$, use the spherical coordinate transformation $g(r, \phi, \theta) = (r\cos\phi\cos\theta, r\cos\phi\sin\theta, r\sin\phi)$ to express $\int_V z$ as an integral over an appropriate set in $\mathbb{R}^3_{r,\phi,\theta}$.

Q2 (10 points)

Let $f(x, y) = 1/(x^2 + y^2)$. Determine if f is integrable over $U_1 = \{(x, y) : 0 < x^2 + y^2 < 1\}$ and over $U_2 = \{(x, y) : x^2 + y^2 > 1\}$.

Q3 (10 points)

Let $B = \{(x, y) : x > 0,\ y > 0,\ 1 < xy < 2,\ x < y < 4x\}$. Compute $\int_B x^2y^3$. Hint: Set $x = u/v$ and $y = uv$.

Q4 (10 points)

Let T be the tetrahedron in $\mathbb{R}^3$ having vertices $(0,0,0)$, $(1,2,3)$, $(0,1,2)$, and $(-1,1,1)$. Compute $\int_T f$, where $f(x,y,z) = x + 2y - z$. Hint: use a linear transformation to change variables.

Q5 (10 points)

Let $0 < a < b$, and let T be a solid torus, the result of spinning around the z axis the disk of radius a around the point $(b,0,0)$ in the xz-plane. What is the volume of T? Hint: T is the image under $g(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$ of $A = \{(r,\theta,z)\colon (r-b)^2 + z^2 \le a^2 \text{ and } 0 \le \theta < 2\pi\}$.

Homework Assignment 11

Due: Friday, January 28, 2022 11:59 pm (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Let V be a vector space of dimension n. Define a map $\iota\colon V \to (V^*)^*$ from V to the dual space of the dual space of V as follows. For $v \in V$ and $\phi \in V^*$, set

$$\iota(v)(\phi) = \phi(v).$$

(a) If $(v_1, \dots, v_n) \in V^n$ is a basis of V and $(\phi_1, \dots, \phi_n) \in (V^*)^n$ is its dual basis, show that $(\iota(v_1), \dots, \iota(v_n))$ is a basis of $(V^*)^*$, which is in fact dual to the basis $(\phi_1, \dots, \phi_n)$ of V^*.

(b) Deduce that ι is an isomorphism of vector spaces.

Q2 (10 points)

Let V be the vector spaces of polynomials p of degree ≤ 2 with coefficients in $\mathbb{R}$. Let $\phi_{-1}, \phi_0, \phi_1$ be the elements of V^* defined as follows:

$$\phi_x(p) = p(x) \qquad \text{for } x = -1, 0, 1.$$

Show that $\gamma = (\phi_{-1}, \phi_0, \phi_1)$ is a basis of V^* and find a basis β of V whose dual is γ (namely, such that $\beta^* = \gamma$).

Q3 (10 points)

Let V be a finite dimensional vector space and let $(v_1, \dots, v_n)$ be a basis of V. Let k be a natural number. Define a map $B\colon \mathcal{T}^k(V) \times \mathcal{T}^k(V) \to \mathbb{R}$ as follows:

$$B(T_1, T_2) := \sum_{i_1, \dots, i_k = 1}^{n} T_1(v_{i_1}, \dots, v_{i_k}) T_2(v_{i_1}, \dots, v_{i_k}).$$

(a) Show that $B \in \mathcal{T}^2(\mathcal{T}^k(V))$ (what does this even mean??).

(b) Show that B is an inner product. Namely, show that it is symmetric $B(T_1, T_2) = B(T_2, T_1)$, that for any T we have $B(T, T) \geq 0$, and that $B(T, T) = 0$ iff $T = 0$.

This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 12

Due: Friday, February 4, 2022 11:59 pm (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Note that the questions in this assignment have unequal weights!

Note also that some questions depend on next Monday's material, and possibly even next Wednesday's.

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Munkres' 27-1. Which of the following are alternating tensors in $\mathbb{R}^4$?

$$f(x,y) = x_1y_2 - x_2y_1 + x_1y_1$$

$$g(x,y) = x_1y_3 - x_3y_2$$

$$h(x,y) = (x_1)^3(y_2)^3 - (x_2)^3(y_1)^3$$

Q2 (10 points)

The determinant, as a function of a list of column vectors, is alternating. Write it in terms of the elementary alternating functions ω_I (assuming the standard basis of $\mathbb{R}^n$).

Q3 (15 points)

Munkres' 28-6 (modified). Let $L\colon \mathbb{R}^m \to \mathbb{R}^n$ be a linear transformation presented by a matrix $A \in M_{n\times m}(\mathbb{R})$ relative to the standard bases of the spaces involved. If ω_I (where $I \in \underline{n}_a^k$) is an elementary alternating k-tensor on $\mathbb{R}^n$, then $L^*\omega_I$ is a linear combination $\sum_{J\in\underline{m}_a^k} c_J\omega_J$ of the elementary alternating k-tensors ω_J on $\mathbb{R}^m$. Write formulas for the coefficients c_J in terms of the matrix A.

Q4 (20 points)

Along the lines of our development of a theory of "tensors" and a theory of "alternating tensors", develop a theory of "symmetric tensors" $S^k(V)$ (a symmetric tensor is a tensor whose values are unchanged if its arguments are permuted). Your theory should include definitions for specific tensors σ_I for $I \in \underline{n}_s^k$ (what should $\underline{n}_s^k$ be)?), a proof that the σ_I indeed belong to $S^k(V)$ and that they form a basis of that space, and a computation of the dimension of $S^k(V)$.

This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 13

Due: Friday, February 11, 2022 11:59 pm (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Note that the questions in this assignment have unequal weights!

Note also that some questions depend on next Monday's material, and possibly even next Wednesday's.

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Find a good way of identifying $\Lambda^1(\mathbb{R}^3)$ and $\Lambda^2(\mathbb{R}^3)$ with $\mathbb{R}^3$. Under these identifications, $\wedge\colon \Lambda^1(\mathbb{R}^3) \times \Lambda^1(\mathbb{R}^3) \to \Lambda^2(\mathbb{R}^3)$ becomes a map $P\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. If you chose your identifications right, P is the vector product of two vectors in $\mathbb{R}^3$. See to it that this is the indeed the case!

Comment I was asked to explain what "identifying" two spaces means, so here's an example. We all know that if V is an n-dimensional vector space then, after we choose a basis, it is the "same" as $\mathbb{R}^n$. So, after you've chosen a basis, you can "identify" V with $\mathbb{R}^n$ - this means that from this point on you can refer to the vectors in V by the by the vectors in $\mathbb{R}^n$ that they correspond to, and vice versa. In general, whenever we have a bijection between two sets we can choose to identify them by using that bijection.

Q2 (10 points)

If $L: V \to W$ is an invertible linear transformation between oriented vector spaces (vector spaces equipped with an orientation), we say that L is *orientation preserving* if it pushes the orientation of V forward to the orientation of W (or equivalently, if it pulls the orientation of W back to the orientation of V). Otherwise, L is called *orientation reversing*. Decide for each of the cases below, if L_i is orientation preserving or reversing. In this question $\mathbb{R}^n$ always comes equipped with its standard orientation.

(a) $L_1: \mathbb{R}^2 \to \mathbb{R}^2$ via $(x, y) \mapsto (-x, y)$.

(b) $L_2: \mathbb{R}^2 \to \mathbb{R}^2$ via $(x, y) \mapsto (y, x)$.

(c) $L_3: \mathbb{R}^2 \to \mathbb{R}^2$, the counterclockwise rotation by $2\pi/7$.

(d) $L_4: \mathbb{R}^2 \to \mathbb{R}^2$, the clockwise rotation by $2\pi/7$.

(e) $L_5: \mathbb{R}^2 \to \mathbb{R}^2$, the complex conjugation map $z \mapsto \bar{z}$, where $\mathbb{R}^2$ is identified with $\mathbb{C}$ in the standard way.

(f) $L_6: \mathbb{R}^3 \to \mathbb{R}^3$ via $(x, y, z) \mapsto (y, z, x)$.

(g) $L_7: \mathbb{R}^n \to \mathbb{R}^n$ via $v \mapsto -v$.

(h) $L_8: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ via $(u, v) \mapsto (v, u)$, where $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$.

Q3 (15 points)

Let V be an n-dimensional vector space, and suppose you are given an isomorphism (an invertible linear transformation) $\chi: \Lambda^n(V) \to \mathbb{R}$. Let k be an integer such that $0 \le k \le n$. Explain how to construct an isomorphism $\psi_k: \Lambda^{n-k}(V) \to \left(\Lambda^k(V)\right)^*$ *without* making any additional choices (such as a basis of any of the spaces involved).

Q4 (20 points)

Given an n-dimensional vector space V with a basis (v_i) and a dual basis (φ_j) and an integer k with $0 \le k \le n$ define an inner product on $\Lambda^k(V)$ by declaring that $\langle \omega_I, \omega_J \rangle = \delta_{IJ}$ (with notation as in class). Let ω_n denote $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$.

(a) Show that there is a unique isomorphism $*: \Lambda^k(V) \to \Lambda^{n-k}(V)$ that satisfies $\lambda \wedge (*\eta) = \langle \lambda, \eta \rangle \omega_n$ for every $\lambda, \eta \in \Lambda^k(V)$. (Sorry for the funny name for a linear map, yet this is standard notation).

(b) When $n = 3$ and $k = 1$ compute $*\omega_1$, $*\omega_2$, and $*\omega_3$. When $n = 4$ and $k = 2$ compute $*\omega_{12}$, $*\omega_{13}$, $*\omega_{14}$, $*\omega_{23}$, $*\omega_{24}$, and $*\omega_{34}$. Be considerate to the TA who will mark this and put your results in a very nice table!

(c) Show that $* \circ *$, which is a composition $\Lambda^k(V) \to \Lambda^{n-k}(V) \to \Lambda^k(V)$, is equal to $(-1)^{k(n-k)}$ times the identity map of $\Lambda^k(V)$.

This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 14

Due: Friday, February 18, 2022 11:59 pm (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 4-13b. If $f, g: \mathbb{R}^n \to \mathbb{R}$ are differentiable, show that $d(fg) = fdg + gdf$.

(Note. This question will make sense after Wednesday's class).

Q2 (10 points)

Spivak's 4-15. Let $f: \mathbb{R} \to \mathbb{R}$ be differentiable and define $\gamma: \mathbb{R} \to \mathbb{R}^2$ by $\gamma(t) = (t, f(t))$. Show that the end point of the tangent vector of γ at t lies on the tangent line of f at $(t, f(t))$. (Assuming the MAT157 definition of "tangent line").

Q3 (10 points)

Spivak's 4-16. Let $\gamma: [0,1] \to \mathbb{R}^2$ be a differentiable curve such that $|\gamma(t)| = 1$ for all t. Show that the tangent vector to $\gamma(t)$ at t is perpendicular to $\gamma(t)_{\gamma(t)}$.

Q4 (10 points)

Spivak's 4-18 (modified). If $f: \mathbb{R}^n \to \mathbb{R}$ is differentiable, define a vector field $\operatorname{grad} f$ by

$$(\operatorname{grad} f)(p) = D_1 f(p)(e_1)_p + \ldots + D_n f(p)(e_n)_p.$$

If v_p is some other tangent vector at p, prove that $D_{v_p} f = \langle (\operatorname{grad} f)(p), v_p \rangle$ and therefore $(\operatorname{grad} f)(p)$ is the direction in which f is changing the fastest at p.

Homework Assignment 15

Due: Friday, March 4, 2022 11:59 pm (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 4-19ab. If f is a function on $\mathbb{R}^3$ and F is a vector field on $\mathbb{R}^3$ with component functions F_1, F_2, F_3, define

$$\omega_F^1 := F_1 dx + F_2 dy + F_3 dz,$$

$$\omega_F^2 := F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy.$$

(a) Prove that

$$df = \omega^1_{\text{grad} f},$$

$$d(\omega_F^1) = \omega^2_{\text{curl} F},$$

$$d(\omega_F^2) = (\operatorname{div} F) dx \wedge dy \wedge dz.$$

(b) Use (a) to prove that

$$\operatorname{curl} \operatorname{grad} f = 0,$$

$$\operatorname{div} \operatorname{curl} F = 0.$$

(Note that the operators div, curl, and grad are defined in Spivak's text on pages 88 and 96).

Q2 (10 points)

Spivak's 4-20 (modified). Recall that a differential form ω is called "closed" if $d\omega = 0$, and "exact" if there exist another differential form λ such that $\omega = d\lambda$. Suppose the open sets $U, V \subset \mathbb{R}^n$ are diffeomorphic (meaning that there is a differentiable $g: U \to V$ that has a differentiable inverse $g^{-1}: V \to U$). Suppose it is known that every closed form on U is exact. Show that the same is true on V.

Q3 (10 points)

Spivak's 4-21 (modified). Let $\theta: \mathbb{R}^2_{x,y} \to \mathbb{R}$ be the function that assigns to every point (x, y) its "angle" θ when it is written using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$. Note that in order to make θ well-defined we need to exclude some ray from the x, y plane; for example, we can exclude the ray $\{(x, 0) : x \leq 0\}$. Prove that where θ is defined, we have

$$d\theta = \frac{-y}{x^2 + y^2} dx + \frac{x}{x^2 + y^2} dy.$$

Q4 (10 points)

Munkres' 30-2 Consider the forms

$$\omega = xydx + 3dy - yzdz,$$

$$\eta = xdx - yz^2 dy + 2xdz,$$

in $\mathbb{R}^3$. Verify by direct computations that $d(d\omega) = 0$ and that $d(\omega \wedge \eta) = (d\omega) \wedge \eta - \omega \wedge (d\eta)$.

Q5 (10 points)

Munkres' 30-6 (modified). Often in mathematical notation, we put a hat on top of elements in a sequence that we wish to omit. For example, $(1 \ldots \hat{4} \ldots 7)$ means (123567). With this in mind, let ω be the form

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} \frac{x_i}{|x|^p} dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n,$$

in $\Omega^{n-1}(\mathbb{R}^n \setminus \{0\})$, where p is some positive real number.

(a) Compute $d\omega$.

(b) For what value of p is $d\omega = 0$?

Homework Assignment 16

Due: Friday, March 18, 2022 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

(a) Using $\partial^2 = 0$, show that if a chain b is the boundary of another chain c, namely if $b = \partial c$, then the boundary of b is zero, namely $\partial b = 0$.

(b) Use Stokes' theorem to show that the 1-cube $b(t) = (\cos 2\pi t, \sin 2\pi t)$ in $\mathbb{R}^2 \setminus \{0\}$ has $\partial b = 0$, yet it is not the boundary of any 2-chain $c \in C_2(\mathbb{R}^2 \setminus \{0\})$.

Q2 (10 points)

Spivak's 4-29 (modified, hint in text). Show that if $\omega = f dx \in \Omega^1([0,1])$ where f is smooth and $f(0) = f(1)$, then there is a unique real number λ so that $\omega - \lambda dx$ is dg for some smooth $g\colon [0,1] \to \mathbb{R}$ for which $g(0) = g(1)$.

Q3 (10 points)

Spivak's 4-30 (modified, hint in text) Prove that if $\omega \in \Omega^1(\mathbb{R}^2 \setminus \{0\})$ is closed, then there is a unique real number λ such that $\omega - \lambda\eta$ is exact, where $\eta := \frac{-y dx + x dy}{x^2+y^2}$.

Homework Assignment 17

Due: Friday, March 25, 2022 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 5-6 (modified).

(a) If $f: \mathbb{R}^n \to \mathbb{R}^m$ is a function, the graph of f is defined to be $\Gamma_f = \{(x,y) : y = f(x)\} \subset \mathbb{R}^{n+m}$. Prove that Γ_f is a smooth n-manifold if f is smooth.

(b) This isn't an "if and only if"! Find an example of a non-smooth f for which nevertheless Γ_f is a smooth manifold.

Q2 (10 points)

Spivak's 5-8(a) (modified). Prove any 12-dimensional manifold M in $\mathbb{R}^{22}$ is of measure 0 in $\mathbb{R}^{22}$.

Q3 (10 points)

Show that $U(2) := \{A \in M_{2\times 2}(\mathbb{C}) : \bar{A}^T A = I\} \subset M_{2\times 2}(\mathbb{C}) = \mathbb{C}^4 = \mathbb{R}^8$ is a manifold. What is its dimension $\dim U(2)$?

If you have a problem visualizing $U(2)$, that's okay. So do I.

Note. $\bar{A}^T$ is the conjugate-transpose of A. Namely, you conjugate every entry of A and then take the transpose.

Hint. It is best to present $U(2)$ as the zero set of some function g and then show that the appropriate condition on the differential of g is satisfied.

Q4 (10 points)

Recall that the notion of "differentiable", and therefore also of "smooth", extends to functions $\phi: A \to \mathbb{R}^m$ defined on not-necessarily-open subsets $A \subset \mathbb{R}^n$. We simply say that ϕ is differentiable at a point $p \in A$ if ϕ has an extension $\bar{\phi}: U \to \mathbb{R}^m$ to some neighborhood U of p such that $\phi|_{U\cap A} = \bar{\phi}|_{U\cap A}$ and such that $\bar{\phi}$ is differentiable at p.

Show that if U and V are open subsets of $\mathbb{R}^k_+ = \{x \in \mathbb{R}^k : x_k \geq 0\}$ (meaning, there exists U' and V' open in $\mathbb{R}^k$ such that $U = U' \cap \mathbb{R}^k$ and $V = V' \cap \mathbb{R}^k$), and if $\phi: U \to V$ is a diffeomorphism, then $\phi(U \cap \mathbb{R}^{k-1}) = V \cap \mathbb{R}^{k-1}$, where $\mathbb{R}^{k-1}$ is considered as a subset of $\mathbb{R}^k$ in the obvious manner.

(It follows that the notion of "the boundary of a manifold" is independent of the perspective).

Q5 (10 points)

(a) Show that if $M^k \subset \mathbb{R}^m$ and $N^l \subset \mathbb{R}^n$ are manifolds without boundary, then $M \times N$ is a $(k+l)$-dimensional manifold without boundary in $\mathbb{R}^{m+n}$.

(b) Show that if $M^k \subset \mathbb{R}^m$ and $N^l \subset \mathbb{R}^n$ are manifolds with boundary, then $M \times N$ can be presented as the disjoint union of a $(k+l)$-manifold with boundary and a $(k+l-2)$-manifold without boundary.

Hint. Start by understanding the case of $M = N = [0,1]$.

Homework Assignment 18

Due: Friday, April 1, 2022 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (20 points)

Recall that $O(3) := \{A \in M_{3\times 3}(\mathbb{R}) : A^T A = I\}$ is the set of orthogonal 3×3 matrices, and define $SO(3) := \{A \in O(3) : \det A = 1\}$. Let $A \in O(3)$ and $B \in SO(3)$; regard A and B also as linear transformations $A, B : \mathbb{R}^3_{x,y,z} \to \mathbb{R}^3_{x,y,z}$. Finally, recall that $S^2 = \{p \in \mathbb{R}^3 : |p| = 1\}$.

(a) Show that $A(S^2) = S^2$ (hence, $O(3)$ is often called "the group of symmetries of $\mathbb{R}^3$").

(b) Let $\omega := xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(S^2)$. Show that $A^*\omega = (\det A)\omega$.

(c) Deduce that $B^*\omega = \omega$ and hence that B is orientation preserving.

Often ω is called "the volume form of S^2", as it is non-zero and invariant (not changing) under all orientation preserving symmetries (rotations) of S^2.

Q2 (20 points)

(a) Show that the following relations hold on $S^2 \subset \mathbb{R}^3_{x,y,z}$:

$$xdz \wedge dx = ydy \wedge dz, \quad ydx \wedge dy = zdz \wedge dx, \quad zdy \wedge dz = xdx \wedge dy.$$

Hint. Start with $xdx + ydy + zdz = 0$, and wedge it with dx, with dy, and with dz.

(b) With ω the same as in the previous question, show that on S^2 away from the north and the south poles (where $x = y = 0$),

$$\omega = \left(\frac{xdy - ydx}{x^2 + y^2}\right) \wedge dz.$$

(c) Deduce that when a spherical loaf of bread is put into a bread cutting machine, all slices come out with the same amount of crust.

Q3 (10 points)

On Monday March 28 I will say in class "the orientation on ∂M induced by a given orientation of M is well-defined" (meaning, it is independent of the choices made within the definition of the induction process). Turn this into a precise statement and prove it.

Q4 (10 points)

Prove that a manifold M is orientable iff an only if it has an atlas (a collection of coordinate patches that covers all of M) for which all transition functions have differentials with positive determinants.

Q5 (10 points)

Show that an $(n-1)$-dimensional manifold M in $\mathbb{R}^n$ is orientable if and only if one may find a consistent non-zero normal field ν to M in $\mathbb{R}^n$. Precisely, ν should be a smooth function on M which maps every $p \in M$ to a non-zero vector in $T_p\mathbb{R}^n$ such that for every p the vector $\nu(p)$ is perpendicular to T_pM.

(This exercise explains the relationship between "being orientable" and "having two sides". Don't write about this, but make sure that you understand this relationship).

This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 19

Due: Friday, April 8, 2022 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 5-19&20 (combined and modified). (a) Show that Stokes's Theorem fails if the manifold is not compact (hint in text), yet that it holds again if the form has a compact support.

(b) Show that the integral of an exact form on a compact oriented manifold with no boundary vanishes, and give a counterexample where the manifold is not compact.

Q2 (10 points)

Munkres' 37.4 (modified). Let $M^2 = \{(x,y,z)\colon 4x^2+y^2+4z^2=4 \;\&\; y\geq 0\}\subset\mathbb{R}^3_{xyz}$. The map $\alpha(u,v)=(u,2(1-u^2-v^2)^{1/2},v)$ defined when $u^2+v^2<1$ is a coordinate patch that covers $M\setminus\partial M$. Orient M so that α is orientation preserving, and give ∂M the induced orientation. Let $\omega = y\,dx+3x\,dz$.

(a) At any point $(x,0,z)$ of ∂M, write a tangent vector that defines the given orientation of ∂M.

(b) Evaluate $\int_{\partial M}\omega$ directly.

(c) Evaluate $\int_M d\omega$ directly.

(d) Repeat until steps (b) and (c) give the same answer.

Q3 (10 points)

Munkres' 37-5 (modified). For $r > 0$ let $D_r^3 = \{x \in \mathbb{R}^3 \colon |x| \le r\}$ be the 3-disk of radius r oriented with the orientation induced from the standard orientation of $\mathbb{R}^3$, and let $S_r^2 = \partial D_r^3$ be its boundary with its induced orientation. Assume that $\omega \in \Omega^2(\mathbb{R}^3 \setminus \{0\})$ satisfies

$$\int_{S_r^2} \omega = a + (b/r)$$

for some constants a and b and for all $r > 0$.

(a) Given $0 < c < d$, compute $\int_{D_d^3 \setminus (\text{int } D_c^3)} d\omega$.

(b) If ω is closed, what can you say about a and b?

(c) If ω is exact, what can you say about a and b?

Q4 (10 points)

Munkres' 37.6 (modified). Given a compact oriented $(k + l + 1)$-dimensional manifold without boundary and given $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$, prove that the following "integration by parts" formula holds,

$$\int_M \omega \wedge d\eta = s \int_M d\omega \wedge \eta,$$

for some sign s. What is s?

Problem 1. A continuous function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$|(x_1 - x_2) - (f(x_1) - f(x_2))| \le \frac{1}{7}|x_1 - x_2|$$

for every x_1 and x_2 in $\mathbb{R}^n$. Prove that for every $y \in \mathbb{R}^n$ there is an $x \in \mathbb{R}^n$ such that $f(x) = y$.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Tip. You may want to start by writing "draft solutions" on the last pages of this notebook and only then write the perfected versions in the space left here for solutions.

Problem 2. Show that if a function f is defined near a point $a \in \mathbb{R}^n$ and has continuous partial derivatives near a, then it is differentiable at a.

Tip. In math exams, "show" means "prove".

Problem 3. A set $A \subset [0, 1]$ is a (possibly infinite) union of open intervals and it contains all the rational numbers in $[0, 1]$. Show that the boundary of A is $[0, 1] \setminus A$.

Problem 4. Let $f\colon [0, 1] \times [0, 1] \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} 1 & x > y \\ 0 & x \le y \end{cases}.$$

Prove that f is integrable on $[0, 1] \times [0, 1]$ directly by using partitions (namely, without using theorems about continuity and integrability).

Term Test 2

Due: Tuesday, January 18, 2022 7:30 pm (Eastern Standard Time)

Assignment description

Solve all 5 problems on this test, and do Task 6.

Each problem is worth 20 points.

You have two hours to write this test, and another 25 minutes for Task 6 and for uploading.

Allowed material. Open book(s), open notes, but you can only use the internet (during the exam) to read the exam, to submit the exam, and to connect with the instructor/TAs to ask clarification questions. No contact allowed with other students or with any external advisors, online or in person.

Neatness counts! Language counts! The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (20 points)

Directly from the definition, show that the set $\{(t,t)\colon t\in\mathbb{Q}\cap[0,1]\}\subset\mathbb{R}^2$ is of content 0 in $\mathbb{R}^2$.

Tip. Don't start working! Read the whole test first. You may wish to start with the questions that are easiest for you.

Q2 (20 points)

Show that every open set in $\mathbb{R}^n$ is the union of countably many compact sets.

Tip. In math exams, "show" means "prove".

Q3 (20 points)

Compute $\int_{x^2+y^2\leq R^2}\frac{dx\,dy}{1+x^2+y^2}$.

Tip. Explain every step of your computation!

Note added 5:34pm. R is a positive real number.

Q4 (20 points)

A bounded non-negative function f is continuous on $\mathbb{R}^n$ except for a set of measure 0, and it is known that there is a constant M such that $\int_R f \leq M$ for every rectangle R in $\mathbb{R}^n$. Show that f is integrable (NT) on $\mathbb{R}^n$

Tip. Remember that the definition of integrability (NT) starts with a PO1. You can't prove anything about integrability (NT) unless you start the same way.

Q5 (20 points)

A continuously differentiable map $g\colon \mathbb{R}^n \to \mathbb{R}^n$ is called "volume-preserving" if for every Jordan-measurable set B, the set $g^{-1}(B)$ is also Jordan measurable and its volume is equal to the volume of B. Show that the function $h\colon \mathbb{R}^2 \to \mathbb{R}^2$ given by $h(x,y) = (y, x+y)$ is volume-preserving.

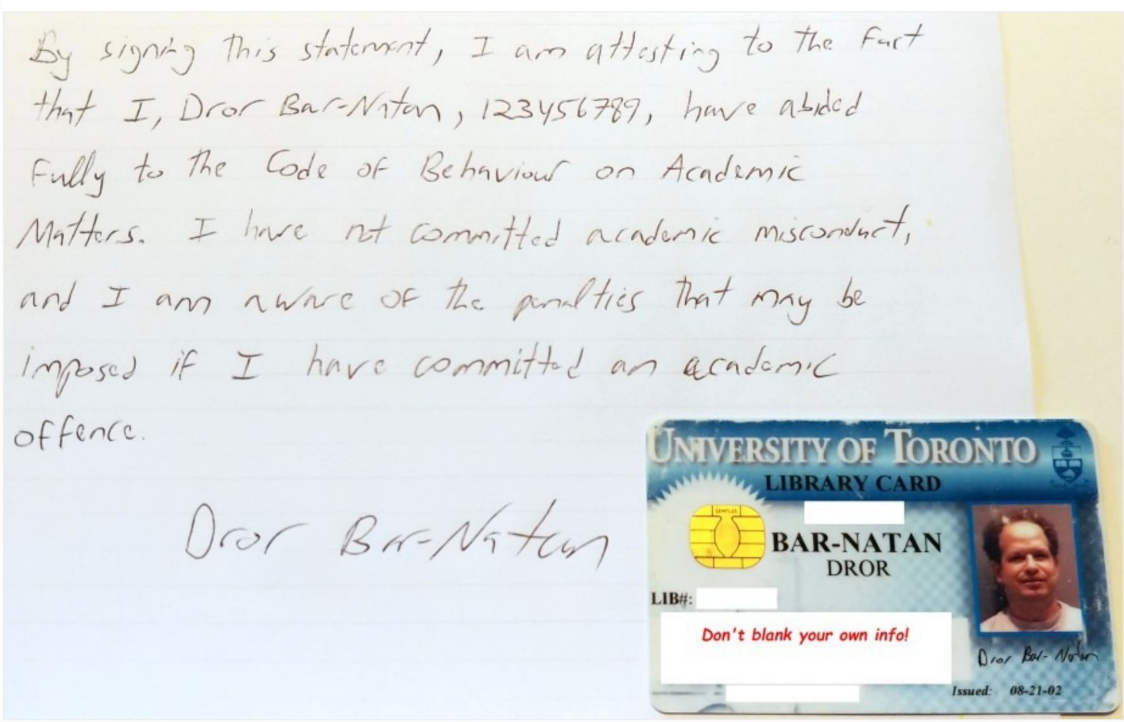
Tip. Once you have finished writing a test, if you have time left, it is always a good idea to go back and re-read and improve everything you have written, and perhaps even completely rewrite any parts that came out messy.

Task 6 (0 points)

Please copy in your own handwriting, fill in the missing details, and sign the statement below, and then submit it along with a photo of your student ID card to complete this test. (See a sample submission below)

By signing this statement, I am attesting to the fact that I, [name], [student number], have abided fully to the Code of Behaviour on Academic Matters. I have not committed academic misconduct, and I am aware of the penalties that may be imposed if I have committed an academic offence.

Signature:



2021-22 MAT257 Term Test 2 Rejects

The following questions were a part of a question pool for the 2021-22 MAT257 Term Test 2, but at the end, they were not included.

1. Prove that the set of irrational numbers is not of measure-0.
2. Prove that the collection of all finite sequences of rational numbers is countable.
3. Given a set A, an "accumulation point" for A is a point x such that every open neighbourhood of x contains infinitely many elements of A. Show that if A is bounded and has finitely many accumulation points, then A is of content 0.
4. Prove that every closed set is the intersection of countably many open sets.
5. Give an example of two functions that differ only on a bounded set of measure 0, yet such that one is integrable and the other is not.
6. Use Fubini's Theorem to compute the volume of the set $\{x \in \mathbb{R}^5 \colon 0 \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq x_5 \leq 1\}$.
7. Show that there is a smooth function on $\mathbb{R}^3$ whose support is precisely the cube $[-1, 1]^3$.
8. Find an example of a continuous function on $\mathbb{R}$ for which there is a constant M such that $\int_I f \leq M$ for every internal $I \subset \mathbb{R}$, but yet such that f is not integrable (NT).
9. We've shown in class that $\int_{\mathbb{R}^n} e^{-|x|^2/2} dx = (2\pi)^{n/2}$. Let λ be a positive real number. Compute $\int_{\mathbb{R}^n} e^{-\lambda|x|^2/2} dx$.

Problem 1. In this question, we say that a function $g\colon \mathbb{R}^n \to \mathbb{R}^n$ preserves one coordinate if there is some $k \in \underline{n}$ such that $g_k(x_1, \ldots, x_n) = x_k$, where g_k is the kth component function of g. Prove that if $n \geq 2$ and $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and $f'(0)$ is invertible, then on a neighborhood of 0 we can write $f = g_1 \circ g_2$ where g_1 and g_2 are continuously differentiable functions $\mathbb{R}^n \to \mathbb{R}^n$ and each preserves one coordinate.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Tip. You may want to start by writing "draft solutions" on the last pages of this notebook and only then write the perfected versions in the space left here for solutions.

Ouch. This problem is buggy. During the test I added the necessary assumption $\exists i \, \frac{\partial f_i(0)}{\partial x_i} \neq 0$. Some further comments:

- The functions g_1 and g_2 need only be defined on a neighborhood of 0, and not on all of $\mathbb{R}^n$ as was written. I did not make this comment during the test because it seemed minor and clear enough, and no points will be deducted if you completely ignored this issue.
- The naming g_1 and g_2 conflicts with the notation for the components of a function, g_k. Strictly sppeaking, this is not a bug. It's just a point of inelegance. Sorry.
- With the new assumption the condition that $f'(0)$ is invertible is no longer needed.

Problem 2. Let $\phi\colon \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{u,v}$ be given by $\phi(x,y) = (e^x \cos y, e^x \sin y)$. Compute $\phi^*(du \wedge dv)$ and $\phi_* \xi$, where ξ is the tangent vector to $\mathbb{R}^2_{x,y}$ given by $\xi = \left(\begin{pmatrix} 0 \\ \pi/2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right)$.

Problem 3. Let V be a vector space, let $\phi\colon V \to V \times V$ be given by $\phi(v) = (v, v)$ and let $\psi\colon V \times V \to V \times V$ be given by $\psi(v, w) = (w, v)$. Let $B\colon V \times V \to \mathbb{R}$ be a bilinear function. Prove that $\phi^* B = 0$ iff $B + \psi^* B = 0$.

Problem 4. Explain in detail how the vector field operator curl arises as an instance of the exterior derivative operator $d\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, for some k and n.

Reminder. $\operatorname{curl} \begin{pmatrix} F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial F_3}{\partial x_2} - \frac{\partial F_2}{\partial x_3} \\ \frac{\partial F_1}{\partial x_3} - \frac{\partial F_3}{\partial x_1} \\ \frac{\partial F_2}{\partial x_1} - \frac{\partial F_1}{\partial x_2} \end{pmatrix}$.

Problem 5. If $L\colon V \to W$ is an invertible linear transformation between oriented vector spaces (vector spaces equipped with an orientation), we say that L is *orientation preserving* if it pushes the orientation of V forward to the orientation of W (or equivalently, if it pulls the orientation of W back to the orientation of V). Otherwise, L is called *orientation reversing*. Decide for each of the cases below, if L_i is orientation preserving or reversing. In this question $\mathbb{R}^n$ always comes equipped with its standard orientation $(e_1, e_2, \ldots, e_n)$.

1. $L_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ via $(x,y) \mapsto (-x, y)$.
2. $L_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ via $(x,y) \mapsto (y, x)$.
3. $L_3\colon \mathbb{R}^2 \to \mathbb{R}^2$, the counterclockwise rotation by $2\pi/7$.
4. $L_4\colon \mathbb{R}^2 \to \mathbb{R}^2$, the clockwise rotation by $2\pi/7$.
5. $L_5\colon \mathbb{R}^2 \to \mathbb{R}^2$, the complex conjugation map $z \mapsto \bar{z}$, where $\mathbb{R}^2$ is identified with $\mathbb{C}$ via $(x,y) \leftrightarrow x + iy$.
6. $L_6\colon \mathbb{R}^3 \to \mathbb{R}^3$ via $(x,y,z) \mapsto (y, z, x)$.
7. $L_7\colon \mathbb{R}^n \to \mathbb{R}^n$ via $v \mapsto -v$.

8. $L_8\colon \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ via $(u, v) \mapsto (v, u)$, where $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$.

Tip. The answers for L_7 and for L_8 may depend on n and m.

2021/22 MAT257 Term Test 3 Information and Rejected Questions

- The test will take place on Tuesday March 8, 5-7PM, at EX320. It will be a "closed book" exam: no books and no notes of any kind will be allowed, no cell-phones, no calculators, no devices of any kind that can display text. So only stationary will be allowed, as well as minimal hydration and snacks, and stuffed animals for joy and comfort. Don't forget to bring your UofT ID!
- Our TA Jessica Liu will hold extra pre-test office hoursin her usual zoom room. (password vchat), on Monday at 4-5:30PM and on Tuesday at 1-2:30PM.
- I will hold my regular office hours, plus an additional hour and a half, on Tuesday at 9:30-12 at Bahen 6178 and simultaneously at http://drorbn.net/vchat.
- Material: Everything up to and including Friday's material, chains and boundaries of chains, with greater emphasis on the material that was not included in Term Test 2 (meaning, starting with the proof of the COV formula, and then k-tensors and all that followed). The questions will be a mix of direct class material, questions from homework, and "fresh" questions. This is more similar to TT1 than to TT2 which was "all fresh".
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "5 of 5".
- To prepare: Do last years' 2021-257-TT3 and the TT3 "rejects" available below. But more important: make sure that you understand every single bit of class material so far!

The following questions were a part of a question pool for the 2020-21 MAT257 Term Test 3, but at the end, they were not included.

1. Prove that the Change of Variables (COV) theorem holds even without the assumption on the invertibility of g'.
2. It is common to identify $\mathbb{R}^3$ with the space of column vectors of length 3, and to identify $(\mathbb{R}^3)^*$ with the space of row vectors of length 3. With this in mind, find the dual basis to the basis $v_1 = \begin{pmatrix}1\\3\\0\end{pmatrix}$, $v_2 = \begin{pmatrix}2\\4\\0\end{pmatrix}$, $v_3 = \begin{pmatrix}0\\0\\5\end{pmatrix}$ of $\mathbb{R}^3$.
3. Let V be a vector space, let $\phi\colon V \to V \times V$ be given by $\phi(v) = (v, v)$ and let $\psi\colon V \times V \to V \times V$ be given by $\psi(v, w) = (w, v)$. Let $B\colon V \times V \to \mathbb{R}$ be a bilinear function. Prove that $\phi^*B = 0$ iff $B + \psi^*B = 0$.
4. Prove that in S_k, for $k > 1$, there is an equal number of odd and even permutations.
5. Let $\sigma \in S_n$ be the permutation given by $\sigma i = i + 1$ for $i < n$ and $\sigma n = 1$. What is $\text{sign}(\sigma)$?
6. Let $\phi\colon \mathbb{R}^2_{x,y} \to \mathbb{R}^2_{u,v}$ be given by $\phi(x, y) = (x^2 - y^2, 2xy)$. Compute $\phi^*(du \wedge dv)$ and $\phi_*\xi$, where ξ is the tangent vector to $\mathbb{R}^2_{x,y}$ given by $\xi = ((0, 1), (1, 0))$.
7. Let $\xi = (p, v)$ be a tangent vector to $\mathbb{R}^n$. Prove that there exists a path $\gamma\colon \mathbb{R} \to \mathbb{R}^n$ such that for every differentiable function $f\colon \mathbb{R}^n \to \mathbb{R}$ we have that $D_\xi f = (f \circ \gamma)'(0)$.
8. Let $\omega = \dfrac{xdy - ydx}{x^2 + y^2} \in \Omega^1\left(\mathbb{R}^2_{x,y} \setminus \{0\}\right)$, and let $f\colon Q = (0, \infty)_r \times [0, 2\pi]_\theta \to \mathbb{R}^2$ be given by $f(r, \theta) = (r\cos\theta, r\sin\theta)$.
   (a) Compute $f^*(\omega)$.
   (b) Show that ω is closed.

(c) Show that $f^*(\omega)$ is exact on Q.

(d) Show that ω is not exact on $\mathbb{R}^2_{x,y} \setminus \{0\}$.

9. Explain in detail how the vector-field operator grad arises as an instance of the exterior derivative operator $d\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, for some k and n.

10. Explain in detail how the vector-field operator div arises as an instance of the exterior derivative operator $d\colon \Omega^k(\mathbb{R}^n) \to \Omega^{k+1}(\mathbb{R}^n)$, for some k and n.

11. (Not in this years' material!) Let $f\colon [0,1] \to \mathbb{R}$ be a differentiable function, and let $c\colon [0,1]^2_{u,v} \to \mathbb{R}^2_{x,y}$ be the 2-cube given by $c(u,v) = (u, f(u)v)$. Use Stokes' theorem and the form $\omega = -y dx$ to show that $\int_c dx \wedge dy = \int_0^1 f(x) dx$. Can you interpret this result geometrically?

12. It is common to identify $\mathbb{R}^n$ with the space of column vectors of length n and to identify $(\mathbb{R}^n)^*$ with the space of row vectors of length n. Suppose $\phi \in (\mathbb{R}^m)^*$ is a row vector, and suppose $L\colon \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation presented relative to the standard bases of $\mathbb{R}^n$ and $\mathbb{R}^m$ by the matrix $A \in M_{m\times n}(\mathbb{R})$. Compute the row vector $L^*\phi$ (the pullback of ϕ via L).

Please watch this page for changes — I may add to it later. Last modified: Wednesday 2nd March, 2022, 13:37