A Bit on Maxwell's Equations

Prerequisites.

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q \mapsto S(q) = \int_a^b \left(\frac{1}{2} m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

Table 18–1 Classical Physics

Maxwell's equations

I. $\nabla \cdot \boldsymbol{E} = \frac{\rho}{\epsilon_0}$ — (Flux of $\boldsymbol{E}$ through a closed surface) = (Charge inside)/ϵ_0

II. $\nabla \times \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}$ — (Line integral of $\boldsymbol{E}$ around a loop) = $-\frac{d}{dt}$(Flux of $\boldsymbol{B}$ through the loop)

III. $\nabla \cdot \boldsymbol{B} = 0$ — (Flux of $\boldsymbol{B}$ through a closed surface) = 0

IV. $c^2 \nabla \times \boldsymbol{B} = \frac{\boldsymbol{j}}{\epsilon_0} + \frac{\partial \boldsymbol{E}}{\partial t}$ — c^2(Integral of $\boldsymbol{B}$ around a loop) = (Current through the loop)/ϵ_0 $+ \frac{\partial}{\partial t}$(Flux of $\boldsymbol{E}$ through the loop)

[Conservation of charge
$\nabla \cdot \boldsymbol{j} = -\frac{\partial \rho}{\partial t}$ — (Flux of current through a closed surface) = $-\frac{\partial}{\partial t}$(Charge inside)]

Force law

$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$

Law of motion

$\frac{d}{dt}(\boldsymbol{p}) = \boldsymbol{F}$, where $\boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification)

Gravitation

$\boldsymbol{F} = -G\frac{m_1 m_2}{r^2}\boldsymbol{e}_r$

The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The *4-Vector Potential* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} \|dA\|^2 \, dtdxdydz + J \wedge A$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$dJ = 0 \qquad dF = 0 \qquad d \star F = J$$

These are the Maxwell equations! Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$dJ = 0 \Longrightarrow$	$\operatorname{div} j = -\frac{\partial \rho}{\partial t}$	"conservation of charge"
$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
	$\operatorname{curl} E = -\frac{\partial B}{\partial t}$	that's how generators work!
$d * F = J \Longrightarrow$	$\operatorname{div} E = -\rho$	"electrostatics"
	$\operatorname{curl} B = j - \frac{\partial E}{\partial t}$	that's how electromagnets work!

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise. With $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e_2}(ds + eA)$ to derive Feynman's "law of motion" and "force law".