#### Dror Bar-Natan: Classes: 2021-22: MAT 257 Analysis II:



#### A Bit on Maxwell's Equations

There's also a handout at http://drorbn.net/2122-257/ap/Maxwell.pdf

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#### A Bit on Maxwell's Equations

#### Prerequisites.

- Poincaré's Lemma, which says that on R<sup>n</sup>, every closed form is exact.
   That is, if dω = 0, then there exists η with dη = ω.
- Integration by parts: ∫<sub>R\*</sub> ω ∧ dη =
   -(-1)<sup>degω</sup> ∫<sub>R\*</sub>(dω) ∧ η for compactly supported forms.
- The Hodge star operator \* which satisfies ω ∧ \*η = (ω, η)dx<sub>1</sub> · · · dx<sub>n</sub> whenever ω and n are of the same degree.
- •ver ω and η are of the same degree.
   The simplesest least action principle: the extremes of q ↦ S(q) = ∫<sub>a</sub><sup>b</sup> (½mq<sup>2</sup>(t) − V(q(t))) dt occur when mq = −V'(q(t)). That is, when F = ma.
- Primary  $F=a(E+p)\times B,$  for all such that  $F=a(E+p)\times B,$  for all such that  $F=a(E+p)\times B,$  where  $F=a(E+p)\times B,$  (Nowher) has will Densit's residuation of the primary  $F=a(E+p)\times B$

Table 18-1 Classical Physics

(Flor of # through a should surface) = 0

19. At 10 K - A + H Addressed Respectations - Cornel Security Section (1)

(Plus of E shough a closed surface) = (Charge inside)/s<sub>0</sub>

(Line integral of E around a loop)  $= -\frac{d}{2}$  (Flux of E through the loop)

(Plax of current through a closed surface) =  $-\frac{\partial}{\partial t}$  (Charge inside)

 $+\frac{\partial}{\partial t}$  (Planef Ethrough the loop)

The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The 4-Vector Potential is a compactly supported 1-form A on  $\mathbb{R}^4$  which extremizes the action

Maximily equation  $L \cdot \nabla \cdot E = L$ 

B X X X - - #

III. V : # ~ 0

$$S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} ||dA||^2 dt dx dy dz + J \wedge A$$

where the 3-form J is the charge-current

The Euler-Lagrange Equations in this case are d \* dA = J, meaning that there's no hope for a solution unless dJ = 0, and that we might as well (think Poincaré's Lemmal) change variables to F := dA. We thus get

$$dJ = 0$$
  $dF = 0$   $d \star F = J$ 

These are the Maxwell equations! Indeed, writing  $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and  $J = \rho dxdydz - j_z dydzdt - j_y dzdxdt - j_z dxdydt$ , we find:

$dJ = 0 \longrightarrow$	$\operatorname{div} j = -\frac{\partial \rho}{\partial t}$	"conservation of charge"
$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
	$\operatorname{curl} E = -\frac{\partial B}{\partial t}$	that's how generators work!
$d*F=J \Longrightarrow$	$\operatorname{div} E = -\rho$	"electrostatics"
	$\operatorname{curl} B = j - \frac{\partial E}{\partial t}$	that's how electromagnets work!

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how E and B (and j and  $\rho$ ) appear to moving observers. Exercise. With  $dx^2 = c^2dt^2 - dx^2 - dy^2 - dx^2$  use  $S = mc \int_{a_0}^{a_0} (ds + cA)$  to derive Feynman's "law of motion" and "force

#### Table 18-1 Classical Physics

Maxwell's equations

I. 
$$\nabla \cdot E = \frac{\rho}{\epsilon}$$
 (Flux of E through a closed surface) = (Charge inside)/ $\epsilon_0$ 

II. 
$$\nabla \times E = -\frac{\partial B}{\partial t}$$
 (Line integral of E around a loop) =  $-\frac{d}{dt}$  (Flux of B through the loop)

III. 
$$\nabla \cdot \mathbf{B} = 0$$
 (Flux of  $\mathbf{B}$  through a closed surface) = 0

IV. 
$$c^2 \nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$$
  $c^2$  (Integral of  $B$  around a loop) = (Current through the loop)/ $\epsilon_0$   $+ \frac{\partial}{\partial t}$  (Flux of  $E$  through the loop)

$$\nabla \cdot j = -\frac{\partial \rho}{\partial t}$$
 (Flux of current through a closed surface)  $= -\frac{\partial}{\partial t}$  (Charge inside)

Force law

$$F = q(E + v \times B)$$

Law of motion

$$\frac{d}{dt}(p) = F$$
, where  $p = \frac{mv}{\sqrt{1 - v^2/c^2}}$  (Newton's law, with Einstein's modification)

Gravitation

$$F = -G \frac{m_1 m_2}{r^2} e_r$$

The Feynman Lectures on Physics vol. II, page 18-2

# Prerequisites.

- Poincaré's Lemma, which says that on  $\mathbb{R}^n$ , every closed form is exact. That is, if  $d\omega = 0$ , then there exists  $\eta$  with  $d\eta = \omega$ .
- ▶ Integration by parts:  $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$  for compactly supported forms.
- ▶ The Hodge star operator  $\star$  which satisfies  $\omega \wedge \star \eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$  whenever  $\omega$  and  $\eta$  are of the same degree.
- The simplesest least action principle: the extremes of  $q\mapsto S(q)=\int_a^b\left(\frac{1}{2}m\dot{q}^2(t)-V(q(t))\right)dt$  occur when  $m\ddot{q}=-V'(q(t))$ . That is, when F=ma.

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### Prerequisite 2.

Integration by parts:  $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$  for compactly supported forms.

## Prerequisite 3.

The Hodge star operator  $\star$  which satisfies  $\omega \wedge \star \eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$  whenever  $\omega$  and  $\eta$  are of the same degree.

## Prerequisite 4.

The simplesest least action principle: the extremes of  $q\mapsto S(q)=\int_a^b\left(\frac{1}{2}m\dot{q}^2(t)-V(q(t))\right)dt$  occur when  $m\ddot{q}=-V'(q(t))$ . That is, when F=ma.

# The Action Principle.

The 4-Vector Potential is a compactly supported 1-form A on  $\mathbb{R}^4$  which extremizes the action

$$S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} |dA|^2 dt dx dy dz + J \wedge A$$

where the 3-form J is the *charge-current*.

# The Euler-Lagrange Equations

in this case are  $d \star dA = J$ , meaning that there's no hope for a solution unless dJ = 0, and that we might as well (think Poincaré's Lemma!) change variables to F := dA. We thus get

$$dJ = 0$$
  $dF = 0$   $d \star F = J$ 

# These are the Maxwell equations!

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Writing  $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$  and  $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$ , we find:

$$dJ=0 \Longrightarrow \quad \text{div}\, j=-\frac{\partial \rho}{\partial t} \qquad \text{"conservation of charge"}$$
 
$$dF=0 \Longrightarrow \quad \text{div}\, B=0 \qquad \text{"no magnetic monopoles"}$$
 
$$\text{curl}\, E=-\frac{\partial B}{\partial t} \qquad \text{that's how generators work!}$$
 
$$d*F=J \Longrightarrow \quad \text{div}\, E=-\rho \qquad \text{"electrostatics"}$$
 
$$\text{curl}\, B=j-\frac{\partial E}{\partial t} \qquad \text{that's how electromagnets work!}$$

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With  $\omega \wedge *\omega = |\omega|^2 dt dx dy dz$  we have

$$*dxdt = -dydz, \qquad *dydt = -dzdx, \qquad *dzdt = -dxdy,$$
  $*dydz = -dxdt, \qquad *dzdx = -dydt, \qquad *dxdy = -dxdt,$  so  $*F = (-B_x dxdt - B_y dydt - B_z dzdt) + (-E_x dydz - E_y dzdx - E_z dxdy).$ 

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$$\nabla \cdot \mathbf{B} = 0$$
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IV. 
$$c^2 \nabla \times \mathbf{B} = \frac{\mathbf{j}}{\epsilon_0} + \frac{\partial \mathbf{E}}{\partial t} - c^2 (\text{Integral of } \mathbf{B} \text{ around a loop}) = (\text{Current through the loop})/\epsilon_0 + \frac{\partial}{\partial t} (\text{Flux of } \mathbf{E} \text{ through the loop})$$

$$\nabla \cdot f = -\frac{\partial \rho}{\partial t}$$
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Force law

$$F = q(E + v \times B)$$

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Feynman again. But wait, in our last two equations the sign of E is wrong!

### Exercise 1.

Use the Lorentz metric to fix the sign errors.

#### Exercise 2.

Use pullbacks along Lorentz transformations to figure out how E and B (and j and  $\rho$ ) appear to moving observers.

### Exercise 3.

With  $ds^2=c^2dt^2-dx^2-dy^2-dz^2$  use  $S=mc\int_{e_1}^{e_2}(ds+eA)$  to derive Feynman's "law of motion" and "force law".