A Bit on Maxwell's Equations

There's also a handout at http://drorbn.net/2122-257/ap/Maxwell.pdf

Dror Bar-Natan: Classes: 2021-22: 2122-257 Analysis II:

A Bit on Maxwell's Equations

Prerequisites.

- Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.
- Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.
- The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.
- The simplesest least action principle: the extremes of $q \mapsto S(q) = \int_a^b \left(\frac{1}{2}m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

Table 18–1 Classical Physics

Maxwell's equations

I. $\nabla \cdot E = \frac{\rho}{\epsilon_0}$ (Flux of E through a closed surface) = (Charge inside)/ϵ_0

II. $\nabla \times E = -\frac{\partial B}{\partial t}$ (Line integral of E around a loop) = $-\frac{d}{dt}$(Flux of B through the loop)

III. $\nabla \cdot B = 0$ (Flux of B through a closed surface) = 0

IV. $c^2\nabla \times B = \frac{j}{\epsilon_0} + \frac{\partial E}{\partial t}$ c^2(Integral of B around a loop) = (Current through the loop)/ϵ_0 $+\frac{\partial}{\partial t}$(Flux of E through the loop)

[Conservation of charge
$\nabla \cdot j = -\frac{\partial \rho}{\partial t}$ (Flux of current through a closed surface) = $-\frac{\partial}{\partial t}$(Charge inside)]

Force law
$F = q(E + v \times B)$

Law of motion
$\frac{d}{dt}(p) = F$, where $p = \frac{mv}{\sqrt{1 - v^2/c^2}}$ (Newton's law, with Einstein's modification)

Gravitation
$F = -G\frac{m_1 m_2}{r^2} e_r$

The Feynman Lectures on Physics vol. II, page 18-2

The Action Principle. The *4-Vector Potential* is a compactly supported 1-form A on $\mathbb{R}^4$ which extremizes the *action*

$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} \|dA\|^2 \, dtdxdydz + J \wedge A}$$

where the 3-form J is the *charge-current*.

The Euler-Lagrange Equations in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

These are the Maxwell equations! Indeed, writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

$dJ = 0 \Longrightarrow$	$\operatorname{div} j = -\frac{\partial \rho}{\partial t}$	"conservation of charge"
$dF = 0 \Longrightarrow$	$\operatorname{div} B = 0$	"no magnetic monopoles"
	$\operatorname{curl} E = -\frac{\partial B}{\partial t}$	that's how generators work!
$d * F = J \Longrightarrow$	$\operatorname{div} E = -\rho$	"electrostatics"
	$\operatorname{curl} B = j - \frac{\partial E}{\partial t}$	that's how electromagnets work!

Exercise. Use the Lorentz metric to fix the sign errors.

Exercise. Use pullbacks along Lorentz transformations to figure out how E and B (and j and ρ) appear to moving observers.

Exercise. With $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ use $S = mc\int_{e_1}^{e_2}(ds + eA)$ to derive Feynman's "law of motion" and "force law".

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[**Conservation of charge**

$$\nabla \cdot \boldsymbol{j} = -\frac{\partial \rho}{\partial t}$$

(Flux of current through a closed surface) = $-\dfrac{\partial}{\partial t}$ (Charge inside)]

Force law

$$\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$$

Law of motion

$$\frac{d}{dt}(\boldsymbol{p}) = \boldsymbol{F}, \quad \text{where} \quad \boldsymbol{p} = \frac{m\boldsymbol{v}}{\sqrt{1 - v^2/c^2}}$$

(Newton's law, with Einstein's modification)

Gravitation

$$\boldsymbol{F} = -G\,\frac{m_1 m_2}{r^2}\,\boldsymbol{e}_r$$

Prerequisites.

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Prerequisite 1.

Poincaré's Lemma, which says that on $\mathbb{R}^n$, every closed form is exact. That is, if $d\omega = 0$, then there exists η with $d\eta = \omega$.

Prerequisite 2.

Integration by parts: $\int_{\mathbb{R}^n} \omega \wedge d\eta = -(-1)^{\deg \omega} \int_{\mathbb{R}^n} (d\omega) \wedge \eta$ for compactly supported forms.

Prerequisite 3.

The Hodge star operator $\star$ which satisfies $\omega \wedge \star\eta = \langle \omega, \eta \rangle dx_1 \cdots dx_n$ whenever ω and η are of the same degree.

Prerequisite 4.

The simplesest least action principle: the extremes of $q \mapsto S(q) = \int_a^b \left(\frac{1}{2}m\dot{q}^2(t) - V(q(t))\right) dt$ occur when $m\ddot{q} = -V'(q(t))$. That is, when $F = ma$.

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$$\boxed{S_J(A) := \int_{\mathbb{R}^4} \frac{1}{2} |dA|^2 \, dtdxdydz + J \wedge A}$$

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in this case are $d \star dA = J$, meaning that there's no hope for a solution unless $dJ = 0$, and that we might as well (think Poincaré's Lemma!) change variables to $F := dA$. We thus get

$$\boxed{dJ = 0 \qquad dF = 0 \qquad d \star F = J}$$

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Writing $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$, we find:

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$dJ = 0$

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with $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$.

$$\boxed{dJ = 0 \Longrightarrow \quad \operatorname{div} j = -\frac{\partial \rho}{\partial t} \qquad \text{“conservation of charge”}}$$

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with $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and $J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$.

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with $F = (E_x dxdt + E_y dydt + E_z dzdt) + (B_x dydz + B_y dzdx + B_z dxdy)$ and
$J = \rho dxdydz - j_x dydzdt - j_y dzdxdt - j_z dxdydt$.
With $\omega \wedge *\omega = |\omega|^2 dtdxdydz$ we have

$$*dxdt = -dydz, \qquad *dydt = -dzdx, \qquad *dzdt = -dxdy,$$

$$*dydz = -dxdt, \qquad *dzdx = -dydt, \qquad *dxdy = -dxdt,$$

so $*F = (-B_x dxdt - B_y dydt - B_z dzdt) + (-E_x dydz - E_y dzdx - E_z dxdy)$.

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Feynman again. But wait, in our last two equations the sign of E is wrong!

Exercise 1.

Use the Lorentz metric to fix the sign errors.

Exercise 2.

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Exercise 3.

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