Homework Assignment 18

Due: Friday, April 1, 2022 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (20 points)

Recall that $O(3) := \{A \in M_{3\times3}(\mathbb{R}) : A^T A = I\}$ is the set of orthogonal 3×3 matrices, and define $SO(3) := \{A \in O(3) : \det A = 1\}$. Let $A \in O(3)$ and $B \in SO(3)$; regard A and B also as linear transformations $A, B : \mathbb{R}^3_{x,y,z} \to \mathbb{R}^3_{x,y,z}$. Finally, recall that $S^2 = \{p \in \mathbb{R}^3 : |p| = 1\}$.

(a) Show that $A(S^2) = S^2$ (hence, $O(3)$ is often called "the group of symmetries of $\mathbb{R}^3$").

(b) Let $\omega := xdy \wedge dz + ydz \wedge dx + zdx \wedge dy \in \Omega^2(S^2)$. Show that $A^*\omega = (\det A)\omega$.

(c) Deduce that $B^*\omega = \omega$ and hence that B is orientation preserving.

Often ω is called "the volume form of S^2", as it is non-zero and invariant (not changing) under all orientation preserving symmetries (rotations) of S^2.

Q2 (20 points)

(a) Show that the following relations hold on $S^2 \subset \mathbb{R}^3_{x,y,z}$:

$$xdz \wedge dx = ydy \wedge dz, \quad ydx \wedge dy = zdz \wedge dx, \quad zdy \wedge dz = xdx \wedge dy.$$

Hint. Start with $xdx + ydy + zdz = 0$, and wedge it with dx, with dy, and with dz.

(b) With ω the same as in the previous question, show that on S^2 away from the north and the south poles (where $x = y = 0$),

$$\omega = \left(\frac{xdy - ydx}{x^2 + y^2}\right) \wedge dz.$$

(c) Deduce that when a spherical loaf of bread is put into a bread cutting machine, all slices come out with the same amount of crust.

Q3 (10 points)

On Monday March 28 I will say in class "the orientation on ∂M induced by a given orientation of M is well-defined" (meaning, it is independent of the choices made within the definition of the induction process). Turn this into a precise statement and prove it.

Q4 (10 points)

Prove that a manifold M is orientable iff an only if it has an atlas (a collection of coordinate patches that covers all of M) for which all transition functions have differentials with positive determinants.

Q5 (10 points)

Show that an $(n-1)$-dimensional manifold M in $\mathbb{R}^n$ is orientable if and only if one may find a consistent non-zero normal field ν to M in $\mathbb{R}^n$. Precisely, ν should be a smooth function on M which maps every $p \in M$ to a non-zero vector in $T_p\mathbb{R}^n$ such that for every p the vector $\nu(p)$ is perpendicular to T_pM.

(This exercise explains the relationship between "being orientable" and "having two sides". Don't write about this, but make sure that you understand this relationship).