This is a preview of what students will see when they are submitting the assignment. Interactive features are disabled.

Homework Assignment 13

Due: Friday, February 11, 2022 11:59 pm (Eastern Standard Time)

Assignment description

Solve and submit your solutions of the following problems. Note also that the late policy remains strict - you will lose 5% for each hour that you are late. In other words, please submit on time!

Note that the questions in this assignment have unequal weights!

Note also that some questions depend on next Monday's material, and possibly even next Wednesday's.

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Find a good way of identifying $\Lambda^1(\mathbb{R}^3)$ and $\Lambda^2(\mathbb{R}^3)$ with $\mathbb{R}^3$. Under these identifications, $\wedge\colon \Lambda^1(\mathbb{R}^3) \times \Lambda^1(\mathbb{R}^3) \to \Lambda^2(\mathbb{R}^3)$ becomes a map $P\colon \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$. If you chose your identifications right, P is the vector product of two vectors in $\mathbb{R}^3$. See to it that this is the indeed the case!

Comment I was asked to explain what "identifying" two spaces means, so here's an example. We all know that if V is an n-dimensional vector space then, after we choose a basis, it is the "same" as $\mathbb{R}^n$. So, after you've chosen a basis, you can "identify" V with $\mathbb{R}^n$ - this means that from this point on you can refer to the vectors in V by the by the vectors in $\mathbb{R}^n$ that they correspond to, and vice versa. In general, whenever we have a bijection between two sets we can choose to identify them by using that bijection.

Q2 (10 points)

If $L: V \to W$ is an invertible linear transformation between oriented vector spaces (vector spaces equipped with an orientation), we say that L is *orientation preserving* if it pushes the orientation of V forward to the orientation of W (or equivalently, if it pulls the orientation of W back to the orientation of V). Otherwise, L is called *orientation reversing*. Decide for each of the cases below, if L_i is orientation preserving or reversing. In this question $\mathbb{R}^n$ always comes equipped with its standard orientation.

(a) $L_1: \mathbb{R}^2 \to \mathbb{R}^2$ via $(x, y) \mapsto (-x, y)$.

(b) $L_2: \mathbb{R}^2 \to \mathbb{R}^2$ via $(x, y) \mapsto (y, x)$.

(c) $L_3: \mathbb{R}^2 \to \mathbb{R}^2$, the counterclockwise rotation by $2\pi/7$.

(d) $L_4: \mathbb{R}^2 \to \mathbb{R}^2$, the clockwise rotation by $2\pi/7$.

(e) $L_5: \mathbb{R}^2 \to \mathbb{R}^2$, the complex conjugation map $z \mapsto \bar{z}$, where $\mathbb{R}^2$ is identified with $\mathbb{C}$ in the standard way.

(f) $L_6: \mathbb{R}^3 \to \mathbb{R}^3$ via $(x, y, z) \mapsto (y, z, x)$.

(g) $L_7: \mathbb{R}^n \to \mathbb{R}^n$ via $v \mapsto -v$.

(h) $L_8: \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^m$ via $(u, v) \mapsto (v, u)$, where $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$.

Q3 (15 points)

Let V be an n-dimensional vector space, and suppose you are given an isomorphism (an invertible linear transformation) $\chi: \Lambda^n(V) \to \mathbb{R}$. Let k be an integer such that $0 \leq k \leq n$. Explain how to construct an isomorphism $\psi_k: \Lambda^{n-k}(V) \to \left(\Lambda^k(V)\right)^*$ *without* making any additional choices (such as a basis of any of the spaces involved).

Q4 (20 points)

Given an n-dimensional vector space V with a basis (v_i) and a dual basis (φ_j) and an integer k with $0 \leq k \leq n$ define an inner product on $\Lambda^k(V)$ by declaring that $\langle \omega_I, \omega_J \rangle = \delta_{IJ}$ (with notation as in class). Let ω_n denote $\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$.

(a) Show that there is a unique isomorphism $*: \Lambda^k(V) \to \Lambda^{n-k}(V)$ that satisfies $\lambda \wedge (*\eta) = \langle \lambda, \eta \rangle \omega_n$ for every $\lambda, \eta \in \Lambda^k(V)$. (Sorry for the funny name for a linear map, yet this is standard notation).

(b) When $n = 3$ and $k = 1$ compute $*\omega_1$, $*\omega_2$, and $*\omega_3$. When $n = 4$ and $k = 2$ compute $*\omega_{12}$, $*\omega_{13}$, $*\omega_{14}$, $*\omega_{23}$, $*\omega_{24}$, and $*\omega_{34}$. Be considerate to the TA who will mark this and put your results in a very nice table!

(c) Show that $* \circ *$, which is a composition $\Lambda^k(V) \to \Lambda^{n-k}(V) \to \Lambda^k(V)$, is equal to $(-1)^{k(n-k)}$ times the identity map of $\Lambda^k(V)$.