Homework Assignment 1

Due: Friday, September 24, 2021 11:59 pm (Eastern Daylight Time)

Assignment description

Solve and submit your solutions of the following questions. They are all taken from Spivak's book, pages 4,5, and 10.

Submit your assignment

Help

After you have completed the assignment, please save, scan, or take photos of your work and upload your files to the questions below. Crowdmark accepts PDF, JPG, and PNG file formats.

Q1 (10 points)

Spivak's 1-7. A linear transformation $T: \mathbb{R}^n \to \mathbb{R}^n$ is **norm preserving** if $|T(x)| = |x|$ for all x, and **inner product preserving** if $\langle Tx, Ty \rangle = \langle x, y \rangle$ for all x, y.

(a) Prove that T is norm preserving iff it is inner product preserving.

(b) Prove that such a linear transformation is 1-1 and onto, and that T^{-1} is also norm and inner product preserving.

Q2 (10 points)

Spivak's 1-10. If $T: \mathbb{R}^m \to \mathbb{R}^n$ is a linear transformation, show that there is a number M such that $|T(h)| \leq M|h|$ for all $h \in \mathbb{R}^m$. *Hint:* Estimate $|T(h)|$ in terms of $|h|$ and the entries of the matrix of T.

Q3 (10 points)

Spivak's 1-12. Let $(\mathbb{R}^n)^*$ denote the dual space of the vector space $\mathbb{R}^n$. If $x \in \mathbb{R}^n$, define $\varphi_x \in (\mathbb{R}^n)^*$ by $\varphi_x(y) = \langle x, y \rangle$. Define $T: \mathbb{R}^n \to (\mathbb{R}^n)^*$ by $T(x) = \varphi_x$. Show that T is a 1-1 linear transformation and conclude that every $\varphi \in (\mathbb{R}^n)^*$ is φ_x for a unique $x \in \mathbb{R}^n$.

Q4 (10 points)

Spivak's 1-13. If $x, y \in \mathbb{R}^n$, then x and y are called **perpendicular** if $\langle x, y \rangle = 0$. If x and y are perpendicular, show that $|x+y|^2 = |x|^2 + |y|^2$.

Q5 (10 points)

Spivak's 1-18. If $A \subset [0,1]$ contains all the rational numbers in $(0,1)$ and is the union of open intervals (a_i, b_i), show that the boundary of A is $[0,1] \setminus A$.

Q6 (10 points)

Spivak's 1-19. If A is a closed set that contains every rational number in $[0,1]$, show that $[0,1] \subset A$.