The 3D Theorems

Better with video on! (if you can)

Hour 72 on April 8, 2022.

There's also a handout at http://drorbn.net/2122-257/ap/3DTheorems.pdf

Dror Bar-Natan: Classes: 2021-22: MAT 257 Analysis II: http://drorbn.net/2122-257/ap/3DTheorems.pdf

The 3D Theorems

Goal. Gauss' and Stokes' theorems:

$$\int_D \operatorname{div} G\, dV = \int_{\partial D} G \cdot n\, dA$$

$$\int_S (\operatorname{curl} F) \cdot n\, dA = \int_{\partial S} F \cdot T\, ds$$

where

- $D \subset \mathbb{R}^3$ is a domain (a compact oriented 3D manifold with boundary) and $S \subset \mathbb{R}^3$ is a surface (a compact oriented 2D manifold with boundary).
- F and G are vector fields.
- n is the positive unit normal to ∂D / S.
- T is the positive unit tangent to ∂S.
- dV is the 3D volume form on D, dA the 2D volume form on ∂D / S ("Area"), and ds is the 1D volume form on ∂S ("arc length").

Volumes. Recall, "the volume form" dV (or dA or $ds/d\ell$) on M is the multiple of the orientation form which when fed with a positive orthonormal basis of T_pM (at any $p \in M$) outputs 1. For a 3D $D \subset \mathbb{R}^3$, this is $dV = dx \wedge dy \wedge dz$. If $S \subset \mathbb{R}^3$ is 2D and n is its positive unit normal, we have

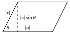
$$dA(u,v) = \begin{vmatrix} - & u & - \\ - & v & - \\ - & n & - \end{vmatrix} = (u \times v) \cdot n$$
$$= (n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy)(u,v)$$

so $dA = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy$.

Example. On S^2, $dA = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$, as seen before.

Reminder. $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$ is the vector perpendicular to both u and v whose length is the area of the parallelogram generated by u and v, and if that is not 0, then $(u, v, u \times v)$ make a positive basis.

Note. The area of a parallelogram is given by



$$A = |u| \cdot |v| \cdot \sin\theta$$
$$= \pm|u| \cdot |v| \cdot \sqrt{1 - \cos^2\theta} = \pm\sqrt{|u|^2|v|^2 - \langle u, v\rangle},$$

if you remember that $\cos\theta = \frac{\langle u,v\rangle}{|u|\cdot|v|}$.

Aside. $dA(u,v) = (u \times v) \cdot n = \pm|u \times v| = \pm\sqrt{|u|^2|v|^2 - \langle u,v\rangle}$ so we can compute surface area as follows, if S is the image of an orientation preserving 2-cube c:

$$\text{Area}(S) = \int_S dA = \int_{I^2} c^*(dA) = \int_{I^2} dA(c_*e_1, c_*e_2)$$
$$\int_{I^2} dA(\partial_1 c, \partial_2 c) = \int_{I^2} \sqrt{|\partial_1 c|^2 |\partial_2 c|^2 - \langle \partial_1 c, \partial_2 c\rangle}$$

Recall that in $\mathbb{R}^3$,

$$\begin{array}{ccccccc} \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 \\ \uparrow{\scriptstyle\omega^0} & & \uparrow{\scriptstyle\omega^1} & & \uparrow{\scriptstyle\omega^2} & & \uparrow{\scriptstyle\omega^3} \\ \{\text{fncns } f\} & \xrightarrow{\text{grad}} & \{\text{V.F. } F\} & \xrightarrow{\text{curl}} & \{\text{V.F. } G\} & \xrightarrow{\text{div}} & \{\text{fncns } g\} \end{array}$$

via $\omega^0_f = 0$, $\omega^1_F = F_1 dx + F_2 dy + F_3 dz$, $\omega^2_G = G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy$, and $\omega^3_g = g dx \wedge dy \wedge dz$.

Claim 1. On an oriented curve in $\mathbb{R}^3$, $\omega^1_F = (T \cdot F) ds$.

Proof. Compute both sides on T and get $T \cdot F$.

Claim 2. On an oriented surface S in $\mathbb{R}^3$, $\omega^2_G = (G \cdot n) dA$.

Proof. Compute both sides on (u, v), a positive orthonormal basis of T_pS:

$$\omega^2_G(u,v) = G \cdot (u \times v) = (G \cdot n)|u \times v| = (G \cdot n) dA(u,v).$$

Claim 3. On an oriented domain D in $\mathbb{R}^3$, $\omega^3_g = g dV$.

Proof. Of course.

The 3D Theorems. With all that,

$$\int_D d\omega^2_G = \int_{\partial D} \omega^2_G \implies \int_D \operatorname{div} G\, dV = \int_{\partial D} G \cdot n\, dA$$

$$\int_S d\omega^1_F = \int_{\partial S} \omega^1_F \implies \int_S (\operatorname{curl} F) \cdot n\, dA = \int_{\partial S} F \cdot T\, ds$$

Some Small Print.

- The Final Exam will take place on Wednesday April 27, 7-10pm in person at EX200.
- Our TAs Sebastian and Shuyang will hold extra pre-exam office hours, in their usual zoom rooms. Sebastian on Wednesday April 20 and Friday April 22 at 11am-1pm at Sebastian's Zoom (password vchat), and Shuyang on Monday, Tuesday, and Wednesday April 25-27 at Shuyang's Zoom (password vchat).
- I will hold my regular office hours on Tuesday April 12, at 9:30-10:30am, I will skip my regular office hours on April 19 and 26, and I will hold extra office hours on Monday, Tuesday, and Wednesday April 25-27 1:30-4pm. All these in person at BA 6178 and online at http://drorbn.net/vchat.
- Material: *Everything* excluding the material on Maxwell equations, with very light further emphasis on the material not covered in the previous term tests. Very roughly, one third of the questions will come straight from class material, one third from HW, and the last third will be fresh questions.
- No outside material will be allowed other than stationery, minimal hydration and snacks, and stuffed animals.
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "8 of 8", in 3 hours.
- To prepare: Consider last year's final assessment and last year's "reject" questions, yet bear in mind that these are all "fresh" questions as last year the test was online. Much more important: make sure that you understand every single bit of class material!
- More, more to date, and practice questions at http://drorbn.net/2122-257/ap/Final-Info.pdf:

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Goal.

$$\int_{M} d\omega = \int_{\partial M} \omega$$

Gauss' and Stokes' theorems:

$$\int_D \operatorname{div} G \, dV = \int_{\partial D} G \cdot n \, dA \qquad \int_S (\operatorname{curl} F) \cdot n \, dA = \int_{\partial S} F \cdot T \, ds$$

where

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Volumes.

$(dy \wedge dz)(u, v) = u_2 v_3 - u_3 v_2$

Recall, "the volume form" dV (or dA or $ds/d\ell$) on M is the multiple of the orientation form which when fed with a positive orthonormal basis of T_pM (at any $p \in M$) outputs 1. For a 3D $D \subset \mathbb{R}^3$, this is $dV = dx \wedge dy \wedge dz$. If $S \subset \mathbb{R}^3$ is 2D and n is its positive unit normal, we have

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so $dA = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy$.

Example. On S^2, $dA = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$, as seen before.

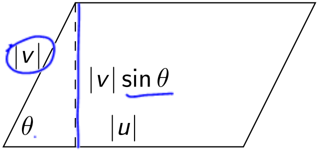
$S = S^2 \quad n = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Reminder.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{array}{c} \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \end{array} \begin{array}{l} n_1 \\ n_2 \\ n_3 \end{array}$$

is the vector perpendicular to both u and v whose length is the area of the parallelogram generated by u and v, and if that is not 0, then $(u, v, u \times v)$ make a positive basis.

$// \ |u \times v|$



Note. The area of a parallelogram is given by

$$A = \underline{|u| \cdot |v| \cdot \sin\theta}$$

$$= \pm |u| \cdot |v| \cdot \underline{\sqrt{1 - \cos^2\theta}} = \pm \sqrt{|u|^2 |v|^2 - \langle u, v \rangle^2},$$

if you remember that $\cos\theta = \frac{\langle u,v \rangle}{|u| \cdot |v|}$.

Aside.

$$\text{Area} = \int_S dA = \int_S n_1\, dy\, dz + \dots$$

$dA(u, v) = (u \times v) \cdot n = \pm|u \times v| = \pm\sqrt{|u|^2|v|^2 - \langle u, v\rangle^2}$ so we can compute surface area as follows, if S is the image of an orientation preserving 2-cube c:

$$\text{Area}(S) = \int_S dA = \int_{I^2} c^*(dA) = \int_{I^2} dA(c_* e_1, c_* e_2)$$

$$\int_{I^2} dA(\partial_1 c, \partial_2 c) = \int_{I^2} \sqrt{|\partial_1 c|^2|\partial_2 c|^2 - \langle \partial_1 c, \partial_2 c\rangle^2}$$

C

Recall

$$\int_M d\omega = \int_{\partial M} \omega \qquad \int_S \omega_G^2$$

That in $\mathbb{R}^3$,

$$\begin{tikzcd}
\Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r, "d"] & \Omega^2 \arrow[r, "d"] & \Omega^3 \\
\{\text{fncns } f\} \arrow[u, "\omega^0"'] \arrow[r, "\text{grad}"] & \{\text{V.F. } F\} \arrow[u, "\omega^1"'] \arrow[r, "\text{curl}"] & \{\text{V.F. } G\} \arrow[u, "\omega^2"'] \arrow[r, "\text{div}"] & \{\text{fncns } g\} \arrow[u, "\omega^3"']
\end{tikzcd}$$

via $\omega_f^0 = \overset{f}{\cancel{0}}$, $\omega_F^1 = F_1 dx + F_2 dy + F_3 dz$, $\omega_G^2 = G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy$, and $\omega_g^3 = g dx \wedge dy \wedge dz$.

$(\omega_F^1)(T) = (F_1\,dx + F_2\,dy + F_3\,dz)(T)$

$(T\cdot F)\,ds\,(T) = T\cdot F$

$F_1T_1 + F_2T_2 + F_3T_3 = F\cdot T$

Claim 1. On an oriented curve in $\mathbb{R}^3$, $\omega^1_F = (T \cdot F)ds$.

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Claim 3. On an oriented domain D in $\mathbb{R}^3$, $\omega^3_g = gdV$.

Proof. Of course.

$u \times v = n \cdot |u \times v|$

The 3D Theorems.

$$\int_M dw = \int_{\partial M} w \qquad dw_G^2 = w_{\text{div}\,G}^3$$
$$= \text{div}(G) \cdot dV$$

With all that,

$$\int_D d\omega_G^2 = \int_{\partial D} \omega_G^2 \quad \Longrightarrow \quad \int_D \operatorname{div} G \, dV = \int_{\partial D} G \cdot n \, dA$$

$$\int_S d\omega_F^1 = \int_{\partial S} \omega_F^1 \quad \Longrightarrow \quad \int_S (\operatorname{curl} F) \cdot n \, dA = \int_{\partial S} F \cdot T \, ds$$

and that's all, for now!

$$dw_F^1 = w_{\text{curl}\,F}^2 = (\text{curl}\,F) \cdot n \cdot dA$$

There's so much more, and I wish we had the time. Anyway,

Good luck with the final!

Have a wonderful summer!

Have a wonderful year, next year!