



# The 3D Theorems

Better with video  
on! (if you can)

Hour 72 on April 8, 2022.

There's also a handout at <http://drorbn.net/2122-257/ap/3DTheorems.pdf>

Dror Bar-Natan: Classes: 2021-22: MAT 257 Analysis II:

<http://drorbn.net/2122-257/ap/3DTheorems.pdf>



## The 3D Theorems

**Goal.** Gauss' and Stokes' theorems:

$$\int_D \operatorname{div} G \, dV = \int_{\partial D} G \cdot n \, dA$$

$$\int_S (\operatorname{curl} F) \cdot n \, dA = \int_{\partial S} F \cdot T \, ds$$

where

- $D \subset \mathbb{R}^3$  is a domain (a compact oriented 3D manifold with boundary) and  $S \subset \mathbb{R}^3$  is a surface (a compact oriented 2D manifold with boundary).
- $F$  and  $G$  are vector fields.
- $n$  is the positive unit normal to  $\partial D / S$ .
- $T$  is the positive unit tangent to  $\partial S$ .
- $dV$  is the 3D volume form on  $D$ ,  $dA$  the 2D volume form on  $\partial D / S$  ("Area"), and  $ds$  is the 1D volume form on  $\partial S$  ("arc length").

**Volumes.** Recall, "the volume form"  $dV$  (or  $dx \wedge dy \wedge dz$ ) on  $M$  is the multiple of the orientation forms which when fed with a positive orthonormal basis of  $T_p M$  (at any  $p \in M$ ) outputs 1. For a 3D  $D \subset \mathbb{R}^3$ , this is  $dV = dx \wedge dy \wedge dz$ . If  $S \subset \mathbb{R}^3$  is 2D and  $n$  is its positive unit normal, we have

$$dA(u, v) = \begin{vmatrix} -u & -v \\ v & -u \end{vmatrix} = (u \times v) \cdot n$$

$$= (n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy)(u, v)$$

so  $dA = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy$ .

**Example.** On  $S^2$ ,  $dA = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$ , as seen before.

**Reminder.**  $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix}$  is the vector perpendicular to both  $u$  and  $v$  whose length is the area of the parallelogram generated by  $u$  and  $v$ , and if that is not 0, then  $(u, v, u \times v)$  make a positive basis.

**Note.** The area of a parallelogram is given by

$$A = |u| \cdot |v| \cdot |\sin \theta|$$

$$= \pm |u| \cdot |v| \cdot \sqrt{1 - \cos^2 \theta} = \pm \sqrt{|u|^2 |v|^2 - (u, v)^2}$$

if you remember that  $\cos \theta = \frac{(u, v)}{|u| |v|}$

**Aside.**  $dA(u, v) = (u \times v) \cdot n = \pm |u \times v| = \pm \sqrt{|u|^2 |v|^2 - (u, v)^2}$  so we can compute surface area as follows, if  $S$  is the image of an orientation preserving 2-cube  $c$ :

$$\operatorname{Area}(S) = \int_c dA = \int_c c^*(dA) = \int_P dA(c, c_1, c_2, c_3)$$

$$= \int_P \sqrt{|u_1 c_1^1 + u_2 c_2^1|^2 - (u_1 c_1^1 + u_2 c_2^1)^2} = (u_1 c_1^1 + u_2 c_2^1)$$

Recall that in  $\mathbb{R}^3$ ,

$$\begin{matrix} \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 \\ \int_P & & \int_P & & \int_P & & \int_P \\ \{\text{funcs } f\} & \xrightarrow{d} & \{V.F. F\} & \xrightarrow{d} & \{V.F. G\} & \xrightarrow{d} & \{\text{funcs } g\} \end{matrix}$$

via  $\omega_1^0 = 0, \omega_2^0 = F_1 dx + F_2 dy + F_3 dz, \omega_3^0 = G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy$ , and  $\omega_1^1 = g dx \wedge dy \wedge dz$ .

**Claim 1.** On an oriented curve in  $\mathbb{R}^3, \omega_2^1 = (T \cdot F) ds$ .

*Proof.* Compute both sides on  $T$  and get  $T \cdot F$ .

**Claim 2.** On an oriented surface  $S$  in  $\mathbb{R}^3, \omega_3^1 = (G \cdot n) dA$ .

*Proof.* Compute both sides on  $(u, v)$ , a positive orthonormal basis of  $T_p S$ :

$$\omega_3^1(u, v) = G \cdot (u \times v) = (G \cdot n)(u \times v) = (G \cdot n)dA(u, v).$$

**Claim 3.** On an oriented domain  $D$  in  $\mathbb{R}^3, \omega_3^2 = g dV$ .

*Proof.* Of course.

**The 3D Theorem.** With all that,

$$\int_D d\omega_2^1 = \int_D \omega_3^2 \implies \int_D \operatorname{div} G \, dV = \int_D G \cdot n \, dA$$

$$\int_S d\omega_1^0 = \int_{\partial S} \omega_2^0 \implies \int_S (\operatorname{curl} F) \cdot n \, dA = \int_{\partial S} F \cdot T \, ds$$

### Some Small Print.

- The Final Exam will take place on Wednesday April 27, 7:00pm in room 4333B.
- On the Final Exam and during all other exam periods, you are allowed to use a calculator, but you are not allowed to use a computer, a cell phone, or any other electronic device. You are also not allowed to use any notes, books, or any other material.
- I will hold my regular office hours on Tuesday April 12, at 9:00-10:00 AM, and I will also hold my regular office hours on April 19 and 26, which will hold my office hours on Thursdays. Thursday, my office hours will be on Wednesday April 25, 2:00-3:00 PM. All other days you can email me at [drorbn@math.toronto.edu](mailto:drorbn@math.toronto.edu).
- Material: If you're looking for the material on Maxwell's equations, with very light buter emphasis on the material we covered in the previous course, you can find it in the book "Classical Electrodynamics" by Jackson, 3rd edition, Wiley, 1998. The book is available in the library.
- The book will be "Classical Electrodynamics" by Jackson, 3rd edition, Wiley, 1998.
- The program: Consider the program and the program's "input" operators, you have to realize that there are all "input" operators in the program. You have to realize that there are all "input" operators in the program. You have to realize that there are all "input" operators in the program.
- More: more to do, and more to do, and more to do. <http://drorbn.net/2122-257/ap/>

Final-Info.pdf



# Some Small Print.

- ▶ The Final Exam will take place on Wednesday April 27, 7-10pm in person at EX200.
- ▶ Our TAs Sebastian and Shuyang will hold extra pre-exam office hours, in their usual zoom rooms. Sebastian on Wednesday April 20 and Friday April 22 at 11am-1pm at [Sebastian's Zoom](#) (password vchat), and Shuyang on Monday, Tuesday, and Wednesday April 25-27 at [Shuyang's Zoom](#) (password vchat).
- ▶ I will hold my regular office hours on Tuesday April 12, at 9:30-10:30am, I will skip my regular office hours on April 19 and 26, and I will hold extra office hours on Monday, Tuesday, and Wednesday April 25-27 1:30-4pm. All these in person at BA 6178 and online at <http://drorbn.net/vchat>.
- ▶ Material: *Everything* excluding the material on Maxwell equations, with very light further emphasis on the material not covered in the previous term tests. Very roughly, one third of the questions will come straight from class material, one third from HW, and the last third will be fresh questions.
- ▶ No outside material will be allowed other than stationery, minimal hydration and snacks, and stuffed animals.
- ▶ The format will be "Solve 7 of 7", or maybe "6 of 6" or "8 of 8", in 3 hours.
- ▶ To prepare: Consider last year's final assessment and last year's "reject" questions, yet bear in mind that these are all "fresh" questions as last year the test was online. Much more important: make sure that you understand every single bit of class material!
- ▶ More, more to date, and practice questions at <http://drorbn.net/2122-257/ap/Final-Info.pdf>.

## Goal.

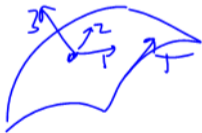
Gauss' and Stokes' theorems:

$$\int_M dw = \int_{\partial M} w$$

$$\int_D \operatorname{div} G \, \underline{dV} = \int_{\partial D} G \cdot \underline{n} \, dA \quad \int_S (\operatorname{curl} F) \cdot n \, dA = \int_{\partial S} F \cdot \underline{T} \, ds$$

where

- ▶  $D \subset \mathbb{R}^3$  is a domain (a compact oriented 3D manifold with boundary) and  $S \subset \mathbb{R}^3$  is a surface (a compact oriented 2D manifold with boundary).
- ▶  $F$  and  $G$  are vector fields.
- ▶  $n$  is the positive unit normal to  $\partial D / S$ .
- ▶  $T$  is the positive unit tangent to  $\partial S$ .
- ▶  $dV$  is the 3D volume form on  $D$ ,  $dA$  the 2D volume form on  $\partial D / S$  ("Area"), and  $ds$  is the 1D volume form on  $\partial S$  ("arc length").



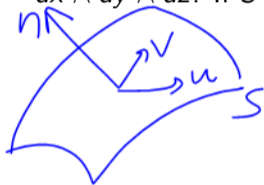
## Volumes.



$$(dy \wedge dz)/(u, v) = u_2 v_3 - u_3 v_2$$

Recall, "the volume form"  $dV$  (or  $dA$  or  $ds/d\ell$ ) on  $M$  is the multiple of the orientation form which when fed with a positive orthonormal basis of  $T_p M$  (at any  $p \in M$ ) outputs 1. For a 3D  $D \subset \mathbb{R}^3$ , this is  $dV = dx \wedge dy \wedge dz$ . If  $S \subset \mathbb{R}^3$  is 2D and  $n$  is its positive unit normal, we have

$$dA(u, v) = \begin{vmatrix} - & u & - \\ - & v & - \\ - & n & - \end{vmatrix} = (u \times v) \cdot n$$



$$= (n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy)(u, v)$$

so  $dA = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy$ .

**Example.** On  $S^2$ ,  $dA = x dy \wedge dz + y dz \wedge dx + z dx \wedge dy$ , as seen before.

$$S = S^2 \quad n = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

## Reminder.

$$\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2 v_3 - u_3 v_2 \\ u_3 v_1 - u_1 v_3 \\ u_1 v_2 - u_2 v_1 \end{pmatrix} \begin{matrix} n_1 \\ n_2 \\ n_3 \end{matrix}$$

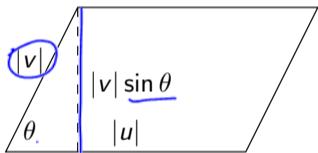
is the vector perpendicular to both  $u$  and  $v$  whose length is the area of the parallelogram generated by  $u$  and  $v$ , and if that is not 0, then  $(u, v, u \times v)$  make a positive basis.

$$\parallel |u \times v|$$

**Note.** The area of a parallelogram is given by

$$\begin{aligned} A &= |u| \cdot |v| \cdot \sin \theta \\ &= \pm |u| \cdot |v| \cdot \sqrt{1 - \cos^2 \theta} = \pm \sqrt{|u|^2 |v|^2 - \langle u, v \rangle^2} \end{aligned}$$

if you remember that  $\cos \theta = \frac{\langle u, v \rangle}{|u| \cdot |v|}$ .



Aside.



$$Area = \int_S dA = \int_S n_1 dy dz + \dots$$

$dA(u, v) = (u \times v) \cdot n = \pm |u \times v| = \pm \sqrt{|u|^2 |v|^2 - \langle u, v \rangle^2}$  so we can compute surface area as follows, if  $S$  is the image of an orientation preserving 2-cube  $c$ :

$$Area(S) = \int_S dA = \int_{I^2} c^*(dA) = \int_{I^2} dA(c_* e_1, c_* e_2)$$

$$\int_{I^2} dA(\partial_1 c, \partial_2 c) = \int_{I^2} \sqrt{|\partial_1 c|^2 |\partial_2 c|^2 - \langle \partial_1 c, \partial_2 c \rangle^2}$$



# Recall

$$\int_M d\omega = \int_{\partial M} \omega \quad \int_S \omega_G^2$$

That in  $\mathbb{R}^3$ ,

$$\begin{array}{ccccccc} \Omega^0 & \xrightarrow{d} & \Omega^1 & \xrightarrow{d} & \Omega^2 & \xrightarrow{d} & \Omega^3 \\ \uparrow \omega^0 & & \uparrow \omega^1 & & \uparrow \omega^2 & & \uparrow \omega^3 \\ \{\text{fncns } f\} & \xrightarrow{\text{grad}} & \{\text{V.F. } F\} & \xrightarrow{\text{curl}} & \{\text{V.F. } G\} & \xrightarrow{\text{div}} & \{\text{fncns } g\} \end{array}$$

via  $\omega_f^0 = dx$ ,  $\omega_F^1 = F_1 dx + F_2 dy + F_3 dz$ ,  $\omega_G^2 = G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy$ ,  
and  $\omega_g^3 = g dx \wedge dy \wedge dz$ .



$$\omega_F^1(T) = (F_1 dx + F_2 dy + F_3 dz)(T) = F \cdot T$$

$$(T \cdot F) ds(T) = T \cdot F$$

**Claim 1.** On an oriented curve in  $\mathbb{R}^3$ ,  $\omega_F^1 = (T \cdot F) ds$ .

*Proof.* Compute both sides on  $T$  and get  $T \cdot F$ .

**Claim 2.** On an oriented surface  $S$  in  $\mathbb{R}^3$ ,  $\omega_G^2 = (G \cdot n) dA$ .

*Proof.* Compute both sides on  $(u, v)$ , a positive orthonormal basis of  $T_p S$ :

$$\begin{aligned} & F_1 T_1 + F_2 T_2 \\ & \quad + F_3 T_3 \\ & = F \cdot T \end{aligned}$$

$$\omega_G^2(u, v) = G \cdot (u \times v) = (G \cdot n) |u \times v| = (G \cdot n) dA(u, v).$$

**Claim 3.** On an oriented domain  $D$  in  $\mathbb{R}^3$ ,  $\omega_g^3 = g dV$ .

*Proof.* Of course.



$$u \times v = n \cdot |u \times v|$$



## The 3D Theorems.

$$\int_M d\omega = \int_{\partial M} \omega \qquad d\omega_G^2 = \omega_{\text{div} G}^3 = \text{div}(G) \cdot dV$$

With all that,

$$\int_D d\omega_G^2 = \int_{\partial D} \omega_G^2 \implies \int_D \text{div} G \, dV = \int_{\partial D} G \cdot n \, dA$$
$$\int_S d\omega_F^1 = \int_{\partial S} \omega_F^1 \implies \int_S (\text{curl} F) \cdot n \, dA = \int_{\partial S} F \cdot T \, ds$$

and that's all, for now!

$$d\omega_F^1 = \omega_{\text{curl} F}^2 = (\text{curl} F) \cdot n \cdot dA$$

There's so much more, and I wish we had the time. Anyway,

Good luck with the final!

Have a wonderful summer!

Have a wonderful year, next year!