Dror Bar-Natan: Classes: 2021-22: MAT 257 Analysis II:

The 3D Theorems

Hour 72 on April 8, 2022.

There's also a handout at http: //drorbn.net/2122-257/ap/ 3DTheorems.pdf

Gauss' and Stokes' theorems:

► F and G are vector fields.

▶ *n* is the positive unit normal to $\partial D / S$. > T is the positive unit tangent to ∂S .

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The 3D Theorems	
Gaal, Gauss' and Stokes' theorems:	$Acca(S) = \int dA = \int c^{*}(dA) = \int dA(c, \sigma_1, c, \sigma_2)$
$\int_{\Omega} dh v G dV = \int_{\partial \Omega} G \cdot x dA$	$\int_{0}^{1} d\theta(\partial tx, \partial tx) = \int_{0}^{1} \sqrt{ \partial tx ^{2} \partial tx ^{2} - \partial tx, \partial tx }$
$\int_{X} (cutF) \cdot mdA = \int_{M} F \cdot T ds$	Recall that in R ³ ,
where $D \subset \mathbb{R}^{3}$ is a domain (a compact oriented 3D manifold with boundary) and $S \subset \mathbb{R}^{3}$ is a surface (a compact oriented 2D manifold with boundary). F and G are vector fields.	$D^{0} \xrightarrow{f} D^{0} \xrightarrow{f} D^{0} \xrightarrow{f} D^{0} \xrightarrow{f} D^{0}$ $\downarrow^{r} \xrightarrow{r} \downarrow^{r} \xrightarrow{r} \downarrow^{r} \xrightarrow{r} \downarrow^{r}$ there $0 \xrightarrow{r} D^{0} \xrightarrow{r} D^{0} \xrightarrow{r} D^{0} \xrightarrow{r}$ there $r \xrightarrow{r} D^{0} \xrightarrow{r} D^{0} \xrightarrow{r} D^{0}$
 n is the positive unit normal to AD / S. 	$via \omega_{i}^{0} = f, \omega_{i}^{1} = F_{1}ds + F_{2}dy + F_{3}dz, \omega_{i}^{2} = G_{3}dy \wedge dz + G_{2}$
 T is the positive unit tangent to 26. 	$dx + G_y dx \wedge dy$, and $\omega_y^2 = g dx \wedge dy \wedge dy$.
 dV is the 3D volume form on D, dA the 2D volume form on dD / S ("Ansa"), and ds is the 1D volume form on dS ("are length"). 	
Volumes, Recall, "the volume form" dV (or dA or ds/dl) on M	Proof. Compute both sides on (a. v), a positive orthonormal
is the multiple of the orientation form which when fed with a po-	
idve orthonormal basis of T_pM (at any $p \in M$) outputs 1. For a 3D $D \subset \mathbb{R}^3$, this is $dV = dx \wedge dy \wedge dy$. If $S \subset \mathbb{R}^3$ is 2D and a is	$\omega_{\mathcal{C}}^2(a,v) = G \cdot (a \times v) = (G \cdot a)[a \times v] = (G \cdot a)dA(a,v)$
its positive unit normal, we have	Claim 3. On an oriented domain D in \mathbb{R}^3 , $\omega_t^3 = gdV$.
le e el	Proof. Of course.
$dA(\mathbf{x}, \mathbf{v}) = \begin{vmatrix} -\mathbf{x} & -\mathbf{v} \\ -\mathbf{v} & -\mathbf{v} \\ -\mathbf{x} & - \end{vmatrix} = (\mathbf{a} \times \mathbf{v}) \cdot \mathbf{a}$	The 3D Theorems. With all that,
$= (u_1dy \wedge dz + u_2dz \wedge dx + u_3dx \wedge dy)(u, v)$ so $dA = u_2dy \wedge dz + u_2dz \wedge dx + u_3dx \wedge dy.$ Example. On S^2 , $dA = zdy \wedge dz + vdz \wedge dx + zdx \wedge dy.$ as so we	$\int_{\Omega} du_{\Omega}^2 = \int_{\partial \Omega} u_{\Omega}^2 \Longrightarrow \int_{\Omega} du G dV = \int_{\partial \Omega} G \cdot u dt$
(a) (b) (an - an)	$\int_{\mathbb{T}} d\omega_{t}^{2} = \int_{\mathbb{T}^{d}} \omega_{t}^{2} \Longrightarrow \int_{\mathbb{T}^{d}} (\operatorname{curl} F) \cdot \pi dA = \int_{\mathbb{T}^{d}} F \cdot T$
Reminder $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 x_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_1 x_1 \\ a_1 x_2 \end{bmatrix}$ is the vector perpendice	
far to both a and v whose length is the area of the parallelogram generated by a and v, and if that is not 0, then (a, v, a × v) make a positive basis.	
Note. The area of a parallelogram is	 No The Mession and Deep regrad had notes process also been to that used note testing. Not Westworks have the additional space of the Albanetic been (process roles) and the Mession Process performance applied to The Albanetic been (process) role of Mession Process performance and the The Albanetic been (Process).
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$A = q \cdot q \cdot da \theta$ = $a b d \cdot b d \cdot \sqrt{1 - cos^2 \theta} = a \sqrt{a t^2 b t^2 - (a \cdot y)^2}$.	 Manual Analysis actualing the second at Manual spatians, with early high tables explores and instant second in the province table balls. Yes, receipting we find at the persons of the second at the second second second second second se
$= a(a) - b(-\nabla 1 - cos^{2} \theta) = a q(a)(b)^{2} - (a, v)^{2}$, if you remember that $cos \theta = \frac{b(a)}{b(a)}$.	 Name and the Association of the Associ
	 A proper threads for parts for any state with a part of the part of the state of the of the state of the state of the state of the balance of the state of the state of the state of the state of the state of the state of the state of the state of the state of the state of the state of the state of the state of the st
Aside: $dA(u, v) = (u \times v) \cdot v = u(u \times v) = u \sqrt{ u ^2/ v ^2} + (u, v)^2$ we can compute surface area as follows: if S is the image of an	
erientation proserving 2-cube c:	Final-Info.pdf

Some Small Print.

- The Final Exam will take place on Wednesday April 27, 7-10pm in person at EX200.
- Our TAs Sebastian and Shuyang will hold extra pre-exam office hours, in their usual zoom rooms. Sebastian on Wednesday April 20 and Friday April 22 at 11am-1pm at Sebastian's Zoom (password vchat), and Shuyang on Monday, Tuesday, and Wednesday April 25-27 at 11am-1pm at Shuyang's Zoom (password vchat).
- I will hold my regular office hours on Tuesday April 12, at 9:30-10:30am, I will skip my regular office hours on April 19 and 26, and I will hold extra office hours on Monday, Tuesday, and Wednesday April 25-27 1:30-4pm. All these in person at BA 6178 and online at http://drorbn.net/vchat.
- Material: Everything excluding the material on Maxwell equations, with very light further emphasis on the material not covered in the previous term tests. Very roughly, one third of the questions will come straight from class material, one third from HW, and the last third will be fresh questions.
- No outside material will be allowed other than stationery, minimal hydration and snacks, and stuffed animals.
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "8 of 8", in 3 hours.
- To prepare: Consider last year's final assessment and last year's "reject" questions, yet bear in mind that these are all "fresh" questions as last year the test was online. Much more important: make sure that you understand every single bit of class material!
- More, more to date, and practice questions at http://drorbn.net/2122-257/ap/Final-Info.pdf.

Volumes.

Recall, "the volume form" dV (or dA or $ds/d\ell$) on M is the multiple of the orientation form which when fed with a positive orthonormal basis of $T_{p}M$ (at any $p \in M$) outputs 1. For a 3D $D \subset \mathbb{R}^3$, this is $dV = dx \wedge dy \wedge dz$. If $S \subset \mathbb{R}^3$ is 2D and n is its positive unit normal, we have

$$dA(u,v) = \begin{vmatrix} -u & -\\ -v & -\\ -n & - \end{vmatrix} = (u \times v) \cdot n$$
$$= (n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy)(u,v)$$

so $dA = n_1 dy \wedge dz + n_2 dz \wedge dx + n_3 dx \wedge dy$. **Example.** On S^2 , $dA = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$, as seen before.

Reminder.

Goal.

where

 $\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} \times \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix}$ is the vector perpendicular to both u and v

 $\int_{D} \operatorname{div} G \, dV = \int_{\partial D} G \cdot n \, dA \qquad \int_{S} (\operatorname{curl} F) \cdot n \, dA = \int_{\partial S} F \cdot T \, ds$

 $\blacktriangleright \ D \subset \mathbb{R}^3$ is a domain (a compact oriented 3D manifold with boundary) and $S \subset \mathbb{R}^3$ is a surface (a compact oriented 2D manifold with boundary).

• dV is the 3D volume form on D, dA the 2D volume form on $\partial D / S$

("Area"), and ds is the 1D volume form on ∂S ("arc length").

whose length is the area of the parallelogram generated by u and v, and if that is not 0, then $(u, v, u \times v)$ make a positive basis.

Note. The area of a parallelogram is given by

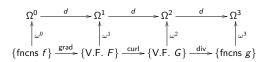
$$A = |u| \cdot |v| \cdot \sin \theta$$

$$=\pm |u| \cdot |v| \cdot \sqrt{1 - \cos^2 \theta} = \pm \sqrt{|u|^2 |v|^2 - \langle u, v \rangle^2},$$

if you remember that $\cos \theta = \frac{\langle u, v \rangle}{|u| \cdot |v|}$

Recall

That in \mathbb{R}^3 ,



via $\omega_f^0 = f$, $\omega_F^1 = F_1 dx + F_2 dy + F_3 dz$, $\omega_G^2 = G_1 dy \wedge dz + G_2 dz \wedge dx + G_3 dx \wedge dy$, and $\omega_g^3 = g dx \wedge dy \wedge dz$.

Aside.

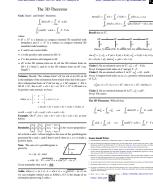
 $dA(u, v) = (u \times v) \cdot n = \pm |u \times v| = \pm \sqrt{|u|^2 |v|^2 - \langle u, v \rangle^2}$ so we can compute surface area as follows, if S is the image of an orientation preserving 2-cube c:

$$Area(S) = \int_{S} dA = \int_{I^{2}} c^{*}(dA) = \int_{I^{2}} dA(c_{*}e_{1}, c_{*}e_{2})$$
$$\int_{I^{2}} dA(\partial_{1}c, \partial_{2}c) = \int_{I^{2}} \sqrt{|\partial_{1}c|^{2}|\partial_{2}c|^{2} - \langle\partial_{1}c, \partial_{2}c\rangle^{2}}$$

Claim 1. On an oriented curve in \mathbb{R}^3 , $\omega_F^1 = (T \cdot F)ds$. *Proof.* Compute both sides on T and get $T \cdot F$. **Claim 2.** On an oriented surface S in \mathbb{R}^3 , $\omega_G^2 = (G \cdot n) dA$. *Proof.* Compute both sides on (u, v), a positive orthonormal basis of T_pS :

$$\omega_G^2(u,v) = G \cdot (u \times v) = (G \cdot n)|u \times v| = (G \cdot n)dA(u,v).$$

Claim 3. On an oriented domain D in \mathbb{R}^3 , $\omega_g^3 = gdV$. Proof. Of course.



$\frac{1}{2}|v|\sin\theta$

With all that,

$$\int_{D} d\omega_{G}^{2} = \int_{\partial D} \omega_{G}^{2} \implies \int_{D} \operatorname{div} G \, dV = \int_{\partial D} G \cdot n \, dA$$
$$\int_{S} d\omega_{F}^{1} = \int_{\partial S} \omega_{F}^{1} \implies \int_{S} (\operatorname{curl} F) \cdot n \, dA = \int_{\partial S} F \cdot T \, ds$$
and that's all, for now!

There's so much more, and I wish we had the time. Anyway,

Good luck with the final!

Have a wonderful summer!

Have a wonderful year, next year!