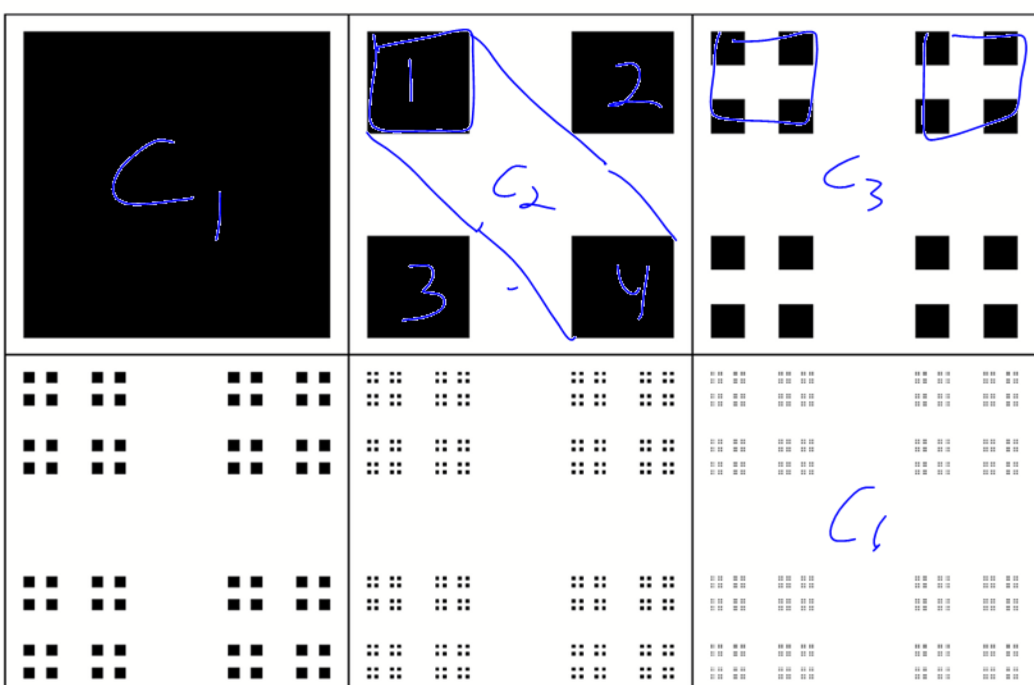
```
GraphicsGrid[Partition[Table[
  Graphics[Table[
    x = Sum[a[[i]] 3^-i, {i, n}]; y = Sum[b[[i]] 3^-i, {i, n}];
    Rectangle[{x, y}, {x, y} + 3^-n],
    {a, Tuples[{0, 2}, n]}, {b, Tuples[{0, 2}, n]}]],
  {n, 0, 5}], 3], Frame → All]
```



$C_n + C_n = [0,2]$

$S(\bigcap C_n)$

$= \bigcap S(C_n)$

Hour 46 MAT257 Analysis II on January 31, 2022: Alternating tensors (2).
Read Along: Spivak 78-85.

$C+\mathcal{L}=?$

<u>Standing Assumption</u> V w/ basis $(v_i)_{i=1}^n$ & dual basis $(\varphi_i)_{i=1}^n$

<u>Thm</u> $\{\varphi_I\}_{I\in \underline{n}^k}$ is a basis of T^kV

$$\boxed{V \xrightarrow{F} W \qquad \Lambda^k(V) \xleftarrow{F^*} \Lambda^k(W)}$$

<u>Claim</u> Alternating $\iff$ kills repetitions

<u>Def</u> $\Lambda^k(V)$ ~ sub-v.s. of $T^k(V)$ Warning: $\Lambda^k(V^*)$

$\underline{n}^k_\wedge = \binom{\underline{n}}{k} = \{(i_1,\dots,i_k)\in\underline{n}^k : i_1<\dots<i_k\}$ $\quad \left|\binom{\underline{n}}{k}\right| = \binom{n}{k}$

$S_k = \{\text{bijections } \sigma : \underline{k}\to\underline{k}\}$ A group! 1. Associative 2. Identity 3. Inverses

"permutations"   "The permutation of k elements"   4. Not commutative!

$\sigma\tau \neq \tau\sigma$

$S_3 \ni \sigma = [1\ 3\ 2] = [\sigma 1\ \sigma 2\ \sigma 3]$

$\tau = [2\ 1\ 3] = \dots$

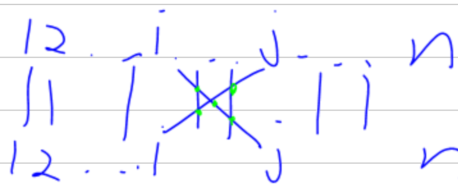
$\sigma\circ\tau = [3\ 1\ 2]$

$\tau\circ\sigma = [2\ 3\ 1]$

<u>Thm</u> $\exists!$ Sign: $S_k \to \{\pm\}$ s.t.

1. $\text{sign}(\sigma\tau) = \text{sign}(\sigma)\,\text{sign}(\tau)$
2. $\text{sign}(\tau_{ij}) = -1$

$$\tau_{ij}(l) = \begin{cases} j & l=i \\ i & l=j \\ l & \end{cases}$$



Notation $\text{sign}(\sigma) = (-1)^\sigma$

$\text{sign}(\sigma) = 1$ "σ is even" $(-1)^n$

$\text{sign}(\sigma) = -1$ "σ is odd"

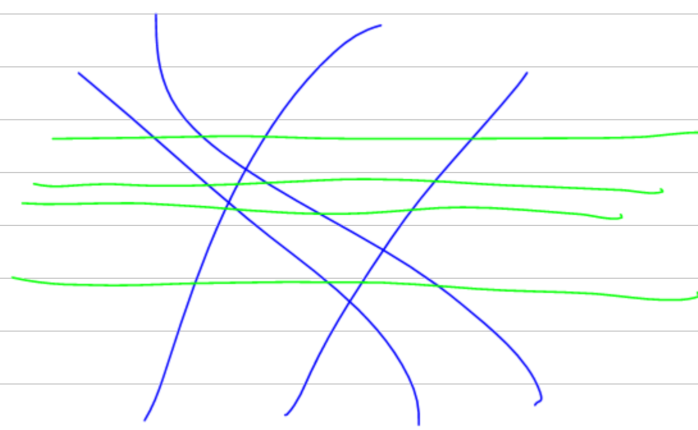
<u>PF</u> (240/247/347) $\square$

$$\text{sign}(\sigma) = (-1)^{\#\text{ xings in a string diag}} = \prod_{i<j} \text{sign}(\sigma j - \sigma i)$$

$$= \det(P_\sigma) = (-1)^{\#\text{ of transposition}}$$

$$P_\sigma = \begin{array}{c} \\ \\ \sigma 2 \\ \\ \sigma 1 \\ \\ \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & & \end{array} \\ \begin{pmatrix} 0 & 0 & & \\ 0 & 1 & & \\ 0 & 0 & \cdots & \\ 1 & 0 & & \\ 0 & 0 & & \end{pmatrix} \end{array}$$

<u>Claim</u> Every σ is a composition of T_{ij}'s.



<u>Claim</u> If $T \in \Lambda^k V$ & $\sigma \in S_k$

then $T \circ \sigma^* = (-1)^\sigma \cdot T$

where $\sigma^*(v_1 \ldots v_k) = (v_{\sigma 1} \ldots v_{\sigma k})$ $\quad T : V^k \to \mathbb{R}$

$$(T\circ\sigma^*)(v_1\ldots v_k) = T(v_{\sigma 1}\ldots v_{\sigma k})$$

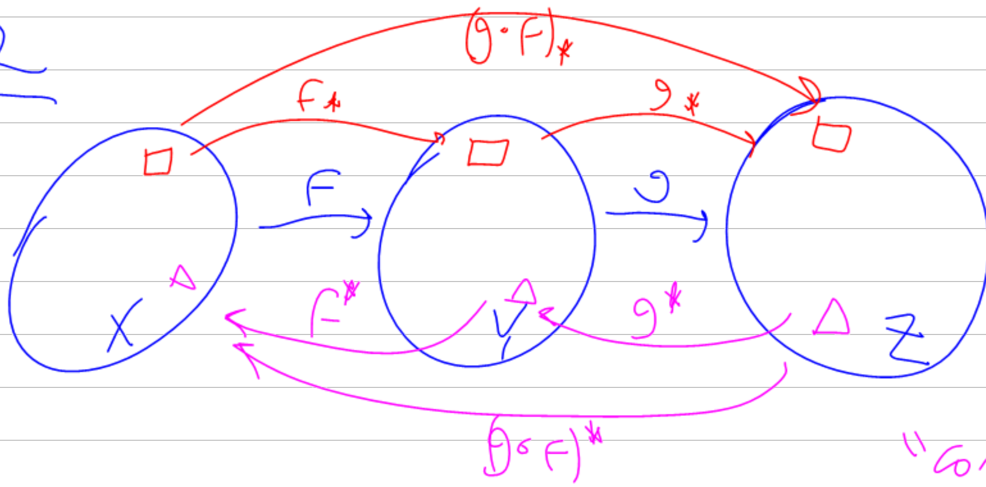
Aside 1

$$V^k = \left\{ \begin{array}{l}\text{fnctns} \\ \underline{k}\longrightarrow V\end{array} \right\}$$

$$\underbrace{\underline{k}\xrightarrow{\sigma}\underbrace{\underline{k}\longrightarrow V}}$$

"pull back of a seq of vects via a perm."

Aside 2



$(g\circ f)_* = g_*\circ f_*$ "covariant"

$(g\circ f)^* = f^*\circ g^*$ "contra-variant"

In particular $(\sigma\circ\tau)^* = \tau^*\circ\sigma^*$ as $V^k\to V^k$

Pf of claim write $\sigma = \underbrace{\tau_1}\circ\cdots\circ\underbrace{\tau_\ell}$ transpositions

$T\circ\sigma^* =$

$$= T\circ(\tau_1\cdots\tau_\ell)^*$$

$$= (T\circ\tau_\ell^*)\tau_{\ell-1}^*\cdots\tau_1^* = (-T)\circ\tau_{\ell-1}^*\cdots\tau_1^*$$

$$= \ldots = (-1)^\ell T = (-1)^\sigma T.$$

Basis for $\Lambda^k V = \{w_I : I\in\underline{n}^k_a\}$

Def If $I \in \underline{\Lambda}^k$ (especially if $I \in \underline{\Lambda}^k_{\sim}$)

$$W_I = \sum_{\sigma \in S_k} (-1)^\sigma \cdot \varphi_I \circ \sigma^* \quad \text{"anti-symmetrization"}$$

Claim W_I is alternating.

Pf $W_I \circ \overset{\text{transposition}}{\tau^*} =$

$$\left(\sum_\sigma (-1)^\sigma \varphi_I \circ \sigma^*\right) \circ \tau^*$$

$$= \sum_\sigma (-1)^\sigma \varphi_I \circ \sigma^* \circ \tau^*$$

$$= \sum_\sigma (-1)^\sigma \varphi_I \circ (\tau\sigma)^*$$

$$= -\sum_\sigma (-1)^{\tau\sigma} \varphi_I \circ (\tau\sigma)^*$$

$$= -\sum_\lambda (-1)^\lambda \varphi_I \circ \lambda^* = -W_I \qquad \square$$

$$\lambda = \tau\sigma$$

example $f : \mathbb{R}^2 \to \mathbb{R}$ "anti-symmetric" if $f(x,y) = -f(y,x)$

If $g : \mathbb{R}^2 \to \mathbb{R}$ is any set

$$f(x,y) = g(x,y) - g(y,x)$$

& $h(x,y) = g(x,y) + g(y,x)$

$$g = \tfrac{1}{2}(f + h)$$

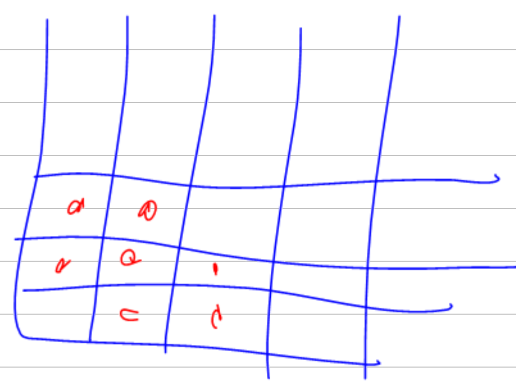
$$\begin{tikzcd} X \arrow[r, "g"] \arrow[rr, bend right, "g^* f = f \circ g"'] & Y \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

$$\begin{tikzcd} \underline{k} \arrow[r, "\sigma"] & \underline{k} \arrow[r, "v = (v_1 \ldots v_k)"] & V \end{tikzcd}$$

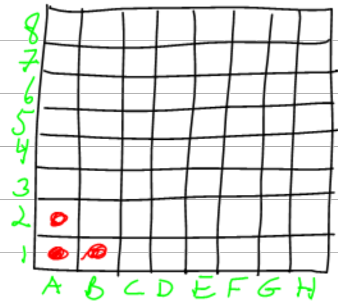
$\sigma^* V = V \circ \sigma = (V_{\sigma 1}, V_{\sigma 2} \dots V_{\sigma k})$

So σ^* maps seqs of vects to seqs of vects.

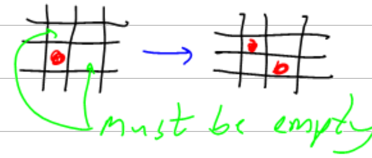
Trick: Find an "invariant".



Hour 47 MAT257 Analysis II on 2/2/2022: Alternating tensors (3).
Read Along: Spivak 78-85.
Riddle Along: On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied. Can you



A move:



$S_k = \{\text{bijections } \sigma : \underline{k} \to \underline{k}\}$

$T \in \Lambda^k \iff T \circ \sigma^* = (-1)^\sigma T$ $\quad \text{Sign}(\sigma) = (-1)^\sigma = (-1)^{\#\text{transpositions to make } \sigma}$

$W_I = \sum_{\sigma \in S_k} (-1)^\sigma \varphi_I \circ \sigma^*$ $\quad \sigma^* : V^k \to V^k$ "pullback" $\quad \sigma^*(v_1 \dots v_k) = (v_{\sigma 1} \dots v_{\sigma k})$ $\quad \underline{n}^k_a = \binom{\underline{n}}{k} = \left\{ \begin{matrix} (a_1 < \dots < a_k) \\ \in \underline{n}^k \end{matrix} \right\}$

$f(x) \to f(x) - f(-x)$

$\to f(x) + f(-x)$

<u>Thm</u> $\{W_I : I \in \underline{n}^k_a\}$ is a basis of $\Lambda^k V$ and so $\dim \Lambda^k V = \binom{n}{k}$

V is a v.s. $v_1 \dots v_n$ a basis $\varphi_1 \dots \varphi_n$ a dual basis.

<u>PF</u> 1. $W_I(v_J) = \delta_{IJ}$ $\quad I, J \in \underline{n}^k_a$

<u>subpf</u>

$$W_I(v_J) = \sum_{\sigma \in S_k} (-1)^\sigma \varphi_I(\sigma^* v_J)$$

$v_J = (v_{j_1}, \dots v_{j_k})$

$$= \sum_\sigma (-1)^\sigma (\varphi_{i_1} \otimes \varphi_{i_2} \otimes \dots \varphi_{i_k})(v_{j_{\sigma 1}} \dots v_{j_{\sigma k}})$$

$$= \sum_\sigma (-1)^\sigma \varphi_{i_1}(v_{j_{\sigma 1}}) \cdot \varphi_{i_2}(v_{j_{\sigma 2}}) \cdots \varphi_{i_k}(v_{j_{\sigma k}})$$

$$= \sum_\sigma (-1)^\sigma \begin{cases} 1 & i_1 = j_{\sigma 1}, \dots i_k = j_{\sigma k} \iff \begin{matrix} (i_1 \dots i_k) \text{ asc.} \\ \parallel \\ (j_{\sigma 1} \dots j_{\sigma k}) \end{matrix} \\ 0 & \text{otherwise} \end{cases}$$

$= (-1)^{id} \delta_{I \cdot J} = \delta_{IJ}.$

2. $\lambda_1, \lambda_2 \in \Lambda^k V$ $\lambda_1 = \lambda_2 \iff \forall I \in \Omega_\wedge^k$

$\lambda_1(V_I) = \lambda_2(V_I)$

$\Rightarrow$ obvious

$\Leftarrow$ NTS: $(\lambda_1 - \lambda_2)(u_1 \ldots u_k) = 0$

Enough $(\lambda_1 - \lambda_2)(V_{i_1}, \ldots V_{i_k}) = 0$ $\forall i_1 \ldots i_k$.

Enough $(\lambda_1 - \lambda_2)(V_{i_1} \ldots V_{i_k}) = 0$ $\forall (i_1 \ldots i_k) \in \Omega_\wedge^k$

Same as

$\lambda_1(V_I) = \lambda_2(V_I)$ $\forall I \in \Omega_\wedge^k$ $\square$.

3 <u>Spans</u> Given λ find a_I s.t.

$\lambda = \sum a_I W_I$

take $a_I = \lambda(V_I)$

NTS, $\lambda = \sum a_I W_I$

Enough to show

$\forall J \in \Omega_a^k$ $\lambda(V_J) = (\sum_I a_I W_I)(V_J)$

$\lambda(V_J) \stackrel{?}{=} \sum_I a_I \delta_{IJ} = a_J$!

4 Lin indep:

$$\sum b_I w_I = 0$$

Eval on v_J

$$\sum b_I w_I(v_J) = 0$$

$$\Downarrow$$

$$b_J = 0$$

$\square$

Example $V = \mathbb{R}^3$

$v_1 = e_1$, $v_2 = e_2$, $v_3 = e_3$

$\varphi_1 = x$, $\varphi_2 = y$, $\varphi_3 = z$

$$\mathbb{R}^3 = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right\}$$

basis for

$\Lambda^0 \mathbb{R}^3$	$\Lambda^1 \mathbb{R}^3$	Λ^2	Λ^3	Λ^4	Λ^5
$w_{()}$	w_1, w_2, w_3	w_{12}, w_{23}, w_{13}	w_{123}	$\emptyset$	$\emptyset$
1	3	3	1	0	0 $\binom{3}{k}$

Aside

$$\Omega_n^k = \left\{ (i_1 \ldots i_k) : \begin{array}{c} 1 \le i_\ell \le n \\ i_1 < i_2 \ldots < i_k \end{array} \right\} \quad |\Omega_n^k| = \binom{n}{k}$$

$$\Omega_{nd}^k = \left\{ (i_1 \ldots i_k) : \begin{array}{c} 1 \le i_\ell \le n \\ i_1 \le i_2 \le \ldots \le i_k \end{array} \right\} \quad |\Omega_{nd}^k| = ?$$

non-descending

$n = 5 \quad k = 7$

$$\{(1 2 2 3 5 5 5)\} \longleftrightarrow \{(\underline{1} \boxed{* 2 * * 3 * 4 5 * * *})\}$$

a seq's of digits & stars, w/ exactly k stars, w/ total length $n+k$, beginning w/ 1.

numb of elements = # of ways of placing k stars within strings of length $n+k-1$ $\Leftarrow$

$$= \binom{n+k-1}{k}$$

$$\mathrm{Alt}: T^k V \to \Lambda^k V$$

$$\mathrm{Alt}(T)(u_1 \dots u_k) = \frac{1}{k!}\sum_{\sigma} (-1)^{\sigma}\, T(u_{\sigma 1} \dots u_{\sigma k})$$

$$k!\,\mathrm{Alt}(\varphi_I) = \omega_I$$

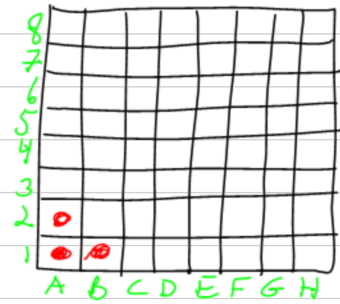
$$\Lambda^k \times \Lambda^l \to \Lambda^{k+l}$$ "wedge product"

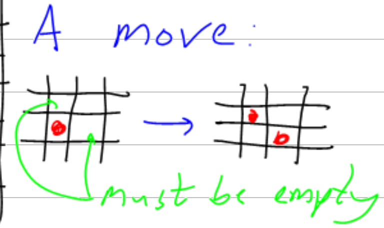
$$\omega_I = \varphi_{i_1} \wedge \varphi_{i_2} \wedge \dots \wedge \varphi_{i_k}$$

1/8				
1/4	1/8			
1/2	1/4	1/8		
1	1/2	1/4	1/8	

Start w/ \$2

Hour 48 MAT257 Analysis II on Feb 4, 2022: Alternating tensors (4).
Starting on Monday we're back to in-person!!
HW12 due and HW13 on web by 11:59pm!
Read Along: Spivak 78-85.
Riddle Along: On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied. Can you clear the original 3 pawns?





Reminder $\{W_I = \sum_{\sigma\in S_k} (-1)^\sigma \varphi_I \circ \sigma\}_{I \in \underline{n}^k_{\nearrow}}$ "elementary alt. ten." make a basis of $\Lambda^k V$, so $\dim = \binom{n}{k}$

$n=5$ 1 5 10 10 5 1
$k=0$ 1 2 3 4 5

$\lambda, \eta \longmapsto \lambda \cdot \eta$ for k-tensors.

Thm $\exists!$ $(\lambda, \eta) \longmapsto \lambda \wedge \eta$ ["wedge product"]
$\lambda \in \Lambda^k$, $\eta \in \Lambda^l V$, $\lambda\wedge\eta \in \Lambda^{k+l} V$ s.t.

0. Bilinear: $(\alpha\lambda_1 + \beta\lambda_2)\wedge\eta = \alpha\lambda_1\wedge\eta + \beta\lambda_2\wedge\eta$
$\lambda\wedge(\alpha\eta_1+\beta\eta_2) = \_\ \_\ \_\ \_$

1. Associative $(\lambda\wedge\eta)\wedge\phi = \lambda\wedge(\eta\wedge\phi)$

2. Super-commutative (graded-commutative)
$\lambda\wedge\eta = (-1)^{kl}\eta\wedge\lambda$ $\lambda\in\Lambda^k$, $\eta\in\Lambda^l$

3. $W_I = \varphi_{i_1}\wedge\varphi_{i_2}\wedge\dots\wedge\varphi_{i_k}$ if $I \in \underline{n}^k_{\nearrow}$

Pf uniqueness [if w & η are given we can compute $w\wedge\eta$ using only 0-3.]

$$\varphi_i \wedge \varphi_j = \begin{cases} w_{(ij)} & \text{if } i<j \quad (ij) \in \underline{n}^2_\wedge \\ -\varphi_j \wedge \varphi_i = -w_{(ji)} & j<i \\ 0 & i=j \end{cases}$$

$$\varphi_i \wedge \varphi_i = -\varphi_i \wedge \varphi_i \Rightarrow \varphi_i \wedge \varphi_i = 0$$

Suppose $I \in \underline{n}^k_\wedge$ $\quad J \in \underline{n}^l_\wedge$

$$w_I \wedge w_J \overset{3}{=} (\varphi_{i_1} \wedge \varphi_{i_2} \wedge \ldots \wedge \varphi_{i_k}) \wedge (\varphi_{j_1} \wedge \ldots \wedge \varphi_{j_l})$$

$$\overset{1}{=} \underset{\varphi_3}{\varphi_{i_1}} \wedge \ldots \wedge \varphi_{i_k} \wedge \varphi_{j_1} \wedge \ldots \wedge \underset{\varphi_3}{\varphi_{j_l}}$$

$$= \begin{cases} 0 & \text{if } I \cap J \neq \emptyset \\ \pm w_{\underbrace{I \cup J}_{\text{take union of the sigs \& sort}}} & \text{if } I \cap J = \emptyset \end{cases}$$

E.g. $w_{(23)} \wedge w_{(14)} = (-1)^2 w_{(1234)} = w_{(1234)}$

$$w_{(13)} \wedge w_{(24)} = -w_{(1234)}$$

$\omega \wedge \eta =$ $\qquad \omega = \sum a_I \omega_I$

$\left(\sum a_I \omega_I\right) \wedge \left(\sum b_J \omega_J\right)$ $\qquad \eta = \sum b_J \omega_J$

$$\overset{?}{=} \sum_{I,J} a_I b_J \underbrace{\omega_I \wedge \omega_J}_{\text{known.}}$$

$$\{\omega_I\} \longrightarrow \{\omega_I'\}$$

$$\omega = \sum a_I \omega_I = \sum a_I' \omega_I'$$

$$\eta = \sum b_J \qquad b_J'$$

<u>Existence</u> need basis-indep formulas for $\omega \wedge \eta$: $\lambda \in \Lambda^k$, $\eta \in \Lambda^l$

$$(\lambda \wedge \eta)(u_1 \ldots u_{k+l})$$

$$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma \overbrace{\lambda(u_{\sigma 1} \ldots u_{\sigma k})} \cdot \overbrace{\eta(u_{\sigma(k+1)} \ldots u_{\sigma(k+l)})}$$

$$= \cancel{\frac{k!\,l!}{k!\,l!}} \sum_{\substack{\sigma \in S_{k+l} \\ \sigma 1 < \sigma 2 < \dots \sigma k \\ \sigma(k+1) < \sigma(k+2) < \dots < \sigma(k+l)}} (-1)^5 \lambda(u_{\sigma 1} \dots u_{\sigma k})\, \eta(u_{\sigma(k+1)} \dots u_{\sigma(k+l)})$$

Example $\quad k=3,\ \lambda \in \Lambda^3 \quad l=2 \quad \eta \in \Lambda^2$

$$(\lambda \wedge \eta)(u_1 \dots u_5) =$$

$$u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \qquad \uparrow \sigma$$

$$\sum_{\sigma} (-1)^5 \lambda(u_{\sigma 1}, u_{\sigma 2}, u_{\sigma 3}) \cdot \eta(u_{\sigma 4}, u_{\sigma 5})$$

$$\sigma = [2\ 4\ 5\ 1\ 3]$$