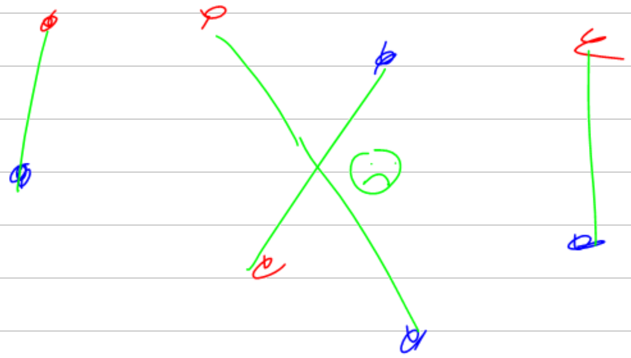
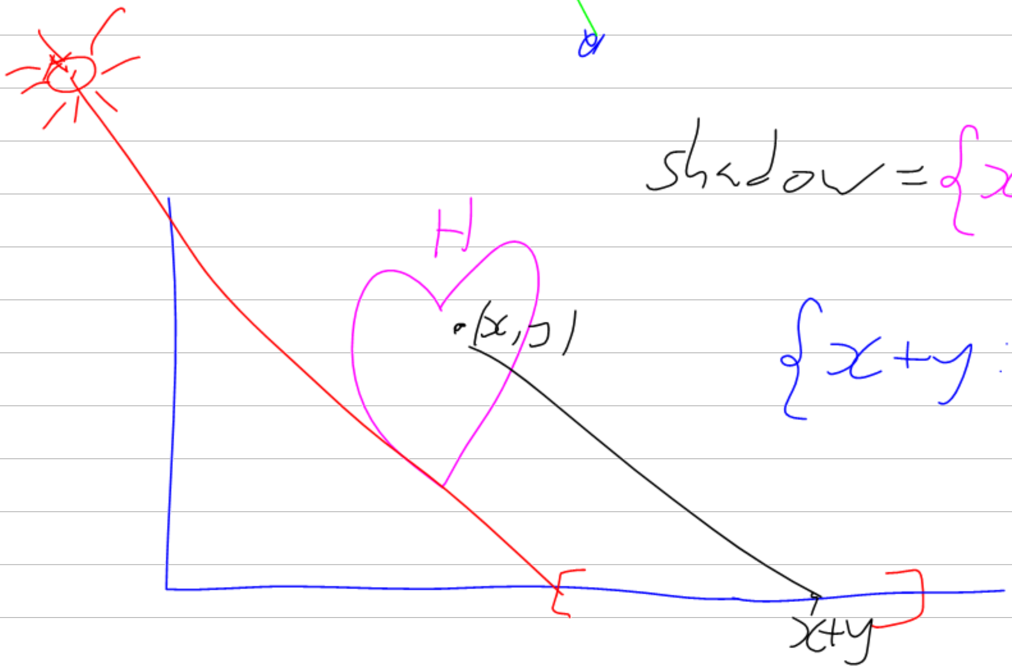
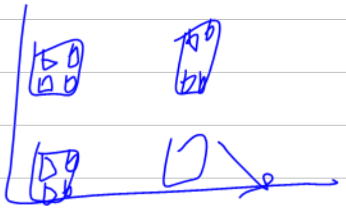


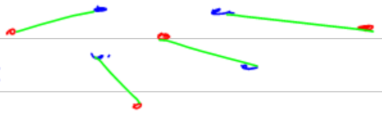


shadow $= \{x+y : (x,y) \in H\}$

$\{x+y : (x,y) \in C \times C\}$



Hour 43 MAT257 Analysis II on January 24, 2022: k-tensors (2).
Read Along: Spivak 75-78.
Riddle Along: n red dots and n blue dots are placed in the plane with no 3 on the same line. Prove that it is possible to pair them up using n straight line segments so that no two of the segments will intersect.



Def $\mathcal{T}^k V$ = "k-tensors on V" $= \left\{ \begin{array}{c} \text{k-linear maps} \\ V^k \to \mathbb{R} \end{array} \right\}$. A v.s.!

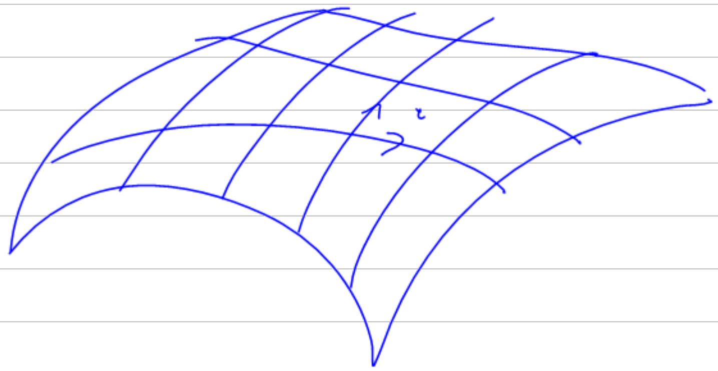
(Elsewhere $\mathcal{T}^k(V^*)$)

$\mathcal{T}^0 V = \mathbb{R}$ $\quad \mathcal{T}^1 V = V^*$ $\quad \langle , \rangle \in \mathcal{T}^2 V$ $\quad \det|_{M_{n\times n}} \in \mathcal{T}^n V$

$(v_1 \dots v_n)$ basis of $V \Rightarrow \exists!\ (\varphi_1 \dots \varphi_n)$ basis of V^*, w/ $\varphi_i(v_j) = \delta_{ij}$

$\mathcal{T}^k \times \mathcal{T}^l \longrightarrow \mathcal{T}^{k+l}$ $\quad (T_1, T_2) \longmapsto T_1 \otimes T_2 = T_1 T_2$

$(T_1 T_2)(u_1 \dots u_{k+l}) = T_1(u_1 \dots u_k) \cdot T_2(u_{k+1}, \dots u_{k+l})$



$\otimes$ is associative, distributive, non-commutative

$$\Big(\overset{\substack{\mathcal{T}^k \\ \Downarrow}}{T_1}\big(\overset{\substack{\mathcal{T}^l \\ \Downarrow}}{T_2} \cdot \overset{\substack{\mathcal{T}^m \\ \Downarrow}}{T_3}\big)\Big)(u_1 \dots u_{k+l+m})$$

$$= T_1(u_1 \dots u_k) \cdot (T_2 \cdot T_3)(u_{k+1} \dots u_{k+l+m})$$

$$= T_1(u_1 \dots u_k) \cdot T_2(u_{k+1} \dots u_{k+l})\, T_3(u_{k+l+1} \dots u_{k+l+m})$$

$((T_1 \cdot T_2)T_3)(u_1 \dots u_{k+l+m}) = \dots =$ Same.

$(T_1+T_2)T_3 = T_1 T_3 + T_2 T_3$ in $\mathcal{T}^{k+l}$

$T_1, T_2 \in \mathcal{T}^k$ $\quad T_3 \in \mathcal{T}^l$

$(lhs)(u_1 \dots u_{k+l}) = \dots\dots$

$(rhs)(u_1 \dots u_{k+l}) = \dots\dots$ $\quad \supset$!

$T_1(T_2+T_3) = T_1 T_2 + T_1 T_3$

$\otimes$ is bilinear ✓

$(\alpha T_1 + \beta T_2) T_3 = \dots\dots$

$T_1(\alpha T_2 + \dots) = \dots$

$T_1 \cdot T_2 \neq T_2 \cdot T_1$ in general.

counter-example: $V = \mathbb{R}^2$ $\quad e_1 \ e_2$

$V^* \ni \varphi_1 \ \varphi_2 \in \mathcal{T}^1(V)$

$(\varphi_1 \varphi_2)(e_1, e_2) = \varphi_1(e_1) \cdot \varphi_2(e_2) = 1 \cdot 1 = 1$

$\neq$

$(\varphi_2 \varphi_1)(e_1, e_2) = \varphi_2(e_1) \cdot \varphi_1(e_2) = 0 \cdot 0 = 0$

<u>Notation</u> $\underline{n} = \{1, \dots, n\}$

$\underline{n}^k = \{ i = I = (i_1 \dots i_k) : i_\nu \in \underline{n} \}$ $\quad |\underline{n}^k| = n^k$

$(v_j)_{j=1}^{n} \in V^n$ $\quad J \in \underline{n}^k$ "multi-index"

$$v_J = (v_{j_1}, v_{j_2}, \dots\ v_{j_k})$$

$\varphi_i \in V^*$ $\quad i = 1, \dots, n$ $\quad I \in \underline{n}^k$

$$\varphi_I = \varphi_{i_1} \cdot \dots \cdot \varphi_{i_k}$$

Eg $\quad \varphi_1 \cdot \varphi_2 = \varphi_{(1,2)} \qquad \varphi_{(1,2)}(e_{(1,2)}) = 1$

$\quad \varphi_2 \cdot \varphi_1 = \varphi_{(2,1)} \qquad \varphi_{(2,1)}(e_{(1,2)}) = 0$

Suppose V is a v.s. w/ basis $v_1 \dots v_n$ and dual basis $\varphi_1 \dots \varphi_n$ & $I, J \in \underline{n}^k$ $\quad I = (i_1 \dots\ i_k)$, $J = (j_1 \dots\ j_k)$

$$\underset{\overset{\uparrow}{T^k}}{\varphi_I}(v_J) = (\varphi_{i_1} \cdot \dots \cdot \varphi_{i_k})(v_{j_1}, \dots\ v_{j_k})$$

$$= \prod_{\alpha=1}^{k} \varphi_{i_\alpha}(v_{j_\alpha})$$

$$= \prod_{\alpha=1}^{k} \delta_{i_\alpha j_\alpha} = \begin{cases} 1 & I = J \\ 0 & I \neq J \end{cases}$$

$$\overset{//}{\varphi_I}(v_J) = \delta_{IJ}$$

Thm V w/ basis $v_1 \dots v_n$ & dual basis $\varphi_1 \dots \varphi_n$. Then

$$\{\varphi_I : I \in \underline{n}^k\} \text{ is a basis}$$

of $T^k V$. Hence $\dim T^k V = n^k$

Proof LI If $T_1, T_2 \in T^k$, then

$$T_1 = T_2 \iff \forall I \quad T_1(\overset{v_{i_1} \dots v_{i_k}}{v_I}) = T_2(v_I)$$

pf LI $\Rightarrow$ ✓

$\Leftarrow$ Assume $\forall I \; T_1(v_I) = T_2(v_I)$

Set $T = T_1 - T_2$

$$T(u_1 \dots u_k) = T\Big(\sum_{i_1=1}^{n} a_{1 i_1} v_{i_1}, \sum a_{2 i_2} v_{i_2}, \dots, \sum a_{k i_k} v_{i_k}\Big)$$

$$= \sum \sim T(v_{i_1}, \sum \dots \sum)$$

$$= \sum \sim T(v_{i_1}, v_{i_2}, v_{i_3} \dots v_{i_k})$$

$$= \sum \sim T(v_I) = \sum \sim \big(\underset{\overset{\shortparallel}{0}}{T_1(v_I) - T_2(v_I)}\big)$$

$= 0 \implies T = 0 \implies T = T_2,$

L2 $\{\varphi_I\}$ spans $\mathcal{T}^k V$.

Given $T \in \mathcal{T}^k$

$$T \overset{?}{=} \sum_I a_I \varphi_I \quad \text{eval on } v_J$$

$$T(v_J) = \sum_I a_I \varphi_I(v_J)$$

$$= \sum_I a_I \delta_{IJ} = a_J$$

Given $T \in \mathcal{T}^k$, set $a_I = T(v_I)$

& then $T = \sum a_I \varphi_I$

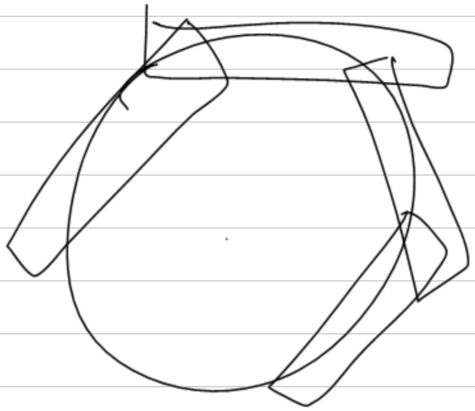
N.T.S.

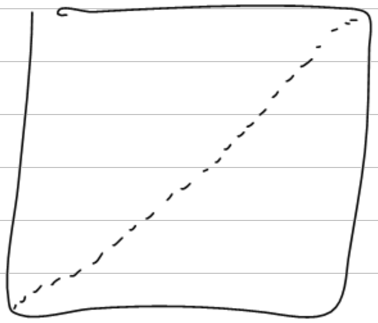
$$\begin{array}{ccc} T(v_J) & = & \left(\sum a_I \varphi_I\right)(v_J) \\ \| & & \| \\ a_J & \underset{?}{=} & a_J \end{array}$$

$\square$

Exercise

$$\mathcal{T}^k V = \underbrace{V^* \otimes V^* \otimes \cdots \otimes V^*}_{k\text{-times}}$$





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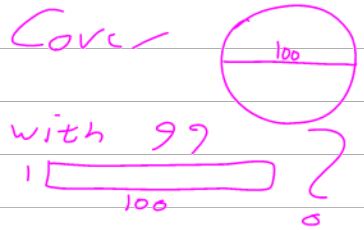
Hour 44 MAT257 Analysis II on January 26, 2022: k-tensors (3).
Read Along: Spivak 75-78. TT2 returned!



Notation $\underline{n}=\{1,\dots,n\}$

$J=I\in\underline{n}^k$ means $I=J=(j_1,\dots,j_k)$

If $v_j\in V$, $v_I=(v_{j_1}\dots v_{j_k})\in V^k$
$j=1\dots n$

If $\varphi_j\in V^*$, $\varphi_I=\varphi_{j_1}\otimes\dots\otimes\varphi_{j_k}\in\mathcal{T}^k(V)$

Thm V w/ basis $v_1\dots v_n$; $\varphi_1\dots\varphi_n$ the dual basis

Then $\{\varphi_I : I\in\underline{n}^k\}$ is a basis of $\mathcal{T}^k(V)$.

Hence $\dim\mathcal{T}^k(V)=n^k$.

pf 1. $\varphi_I(v_J)=\delta_{IJ}$. 2. $T_1=T_2$ in $\mathcal{T}^k(V)$ iff $\forall I\ T_1(v_I)=T_2(v_I)$.

3. span Given $T\in\mathcal{T}^k(V)$, $T=\sum_{I\in\underline{n}^k}a_I\varphi_I$ w/ $a_I=T(v_I)$. 4. LI?

Claim The φ_I are lindependent.

pf Assume $a_I\in\mathbb{R}$ s.t. $\sum a_I\varphi_I=0$.

So for every $J\in\underline{n}^k$,

$$\left(\sum_I a_I\varphi_I\right)(v_J)=0(v_J)$$

$$\begin{array}{c} \sum_I a_I\varphi_I(v_J) = 0 \\ \| \\ a_J=\sum_I a_I\delta_{IJ} \end{array}$$

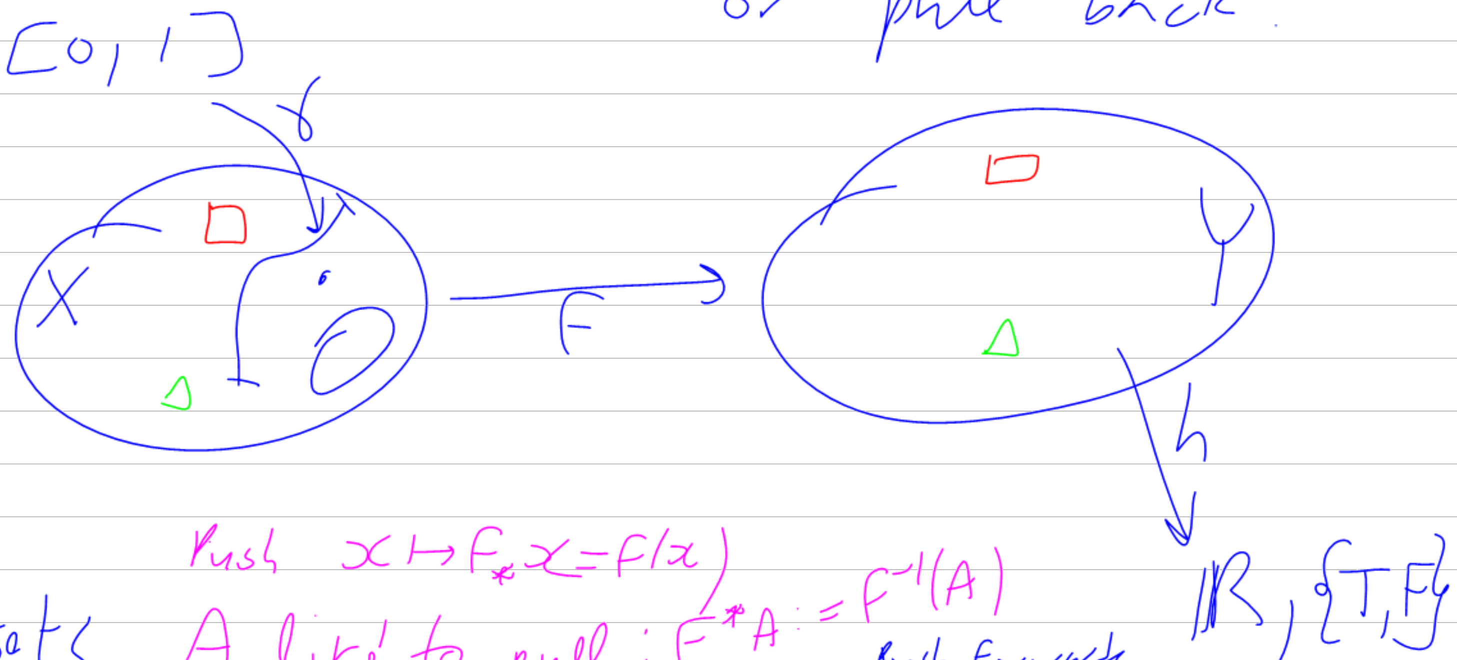
$\dim\mathcal{T}^k(V)=n^k$

$\Uparrow$

$\Rightarrow \forall J\ a_J=0$ $\boxed{\text{lind}}$ $\boxed{\text{Thm}}$

Philosophical interlude.

Theme: In math, things either push forward or pull back.



pts Push $x \mapsto F_* x = F(x)$

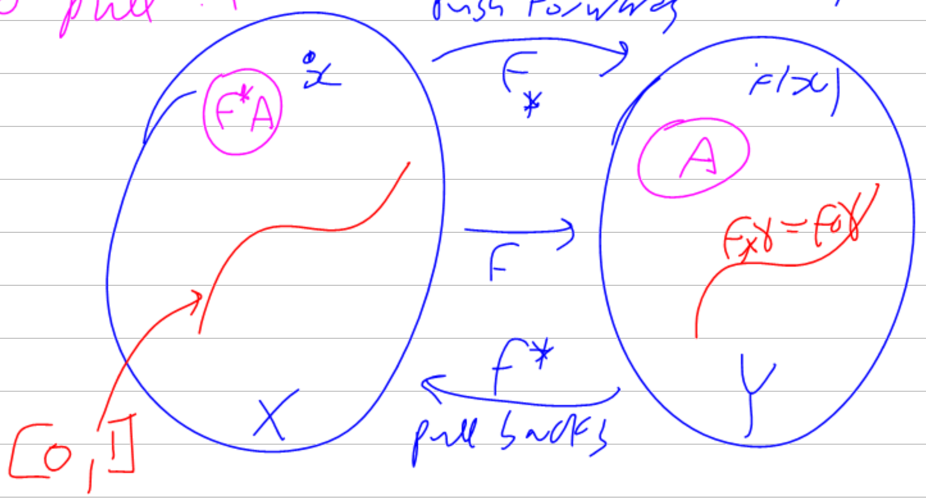
subsets A like to pull: $F^* A := F^{-1}(A)$

paths push

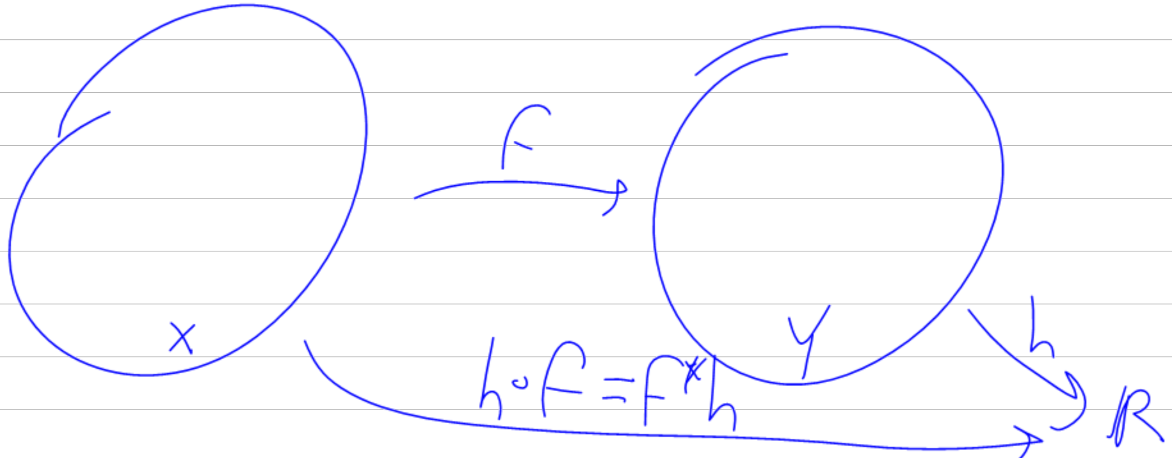
functions

linear functional

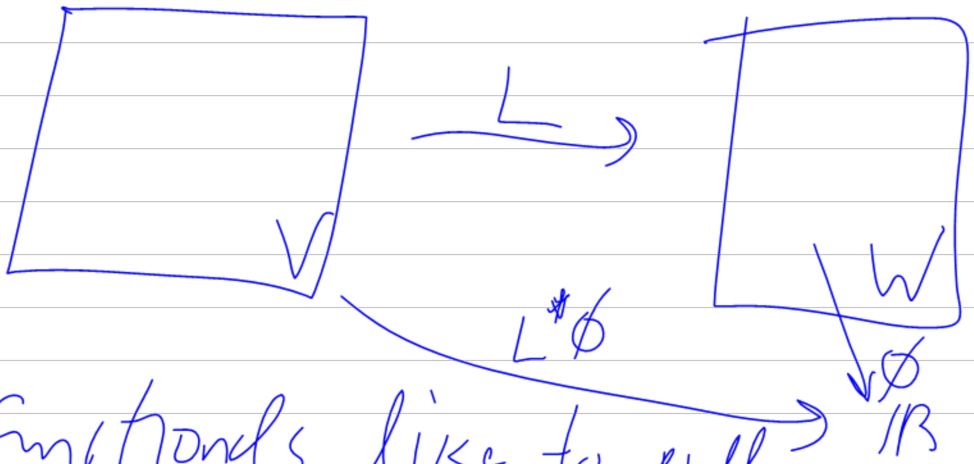
k-tensors



$X \supset A$ $F(A)$ $A \subset Y$ $\underline{\underline{F^{-1}(A)}}$



Function like to be pulled.



lin functionals like to pull.

Aside: $L^* : W^* \longrightarrow V^*$ "pullback" "adjoint"

Suppose $V \xrightarrow{L} W$ is linear

Then $\exists$ $\mathcal{T}^k(V) \xleftarrow{L^*} \mathcal{T}^k(W)$ defined

by $T \in \mathcal{T}^k W$ $\quad u_i \in V$

$$\mapsto \boxed{(L^* T)(u_1 \dots u_k) = T(Lu_1, \dots, Lu_k)}$$

<u>Claim 1.</u> If $T \in \mathcal{T}^k W$ then $L^* T \in \mathcal{T}^k V$

(namely $L^* T$ is multi-lin)

Pf $(L^*T)(u_1, \ldots, \overset{\alpha}{\downarrow}u_i' + \overset{\beta}{\downarrow}u_i'', \ldots u_K)$

$$= T(Lu_1, \ldots, L(u_i'+u_i''), \ldots Lu_K)$$

$$\underset{\text{Linear}}{\overset{L \text{ is}}{=}} T(Lu_1, \ldots, Lu_i' + Lu_i'', \ldots Lu_K)$$

$$\underset{\text{mult. line}}{\overset{T \text{ is}}{=}} T(Lu_1 \ldots Lu_i' \ldots Lu_K) + T(Lu_1 \ldots Lu_i'' \ldots Lu_K)$$

$$= (L^*T)(u_1 \ldots u_i', \ldots u_K) + (L^*T)(u_1 \ldots u_i'' \ldots u_K)$$

□

Claim 2 $L^* : \underset{(\text{a v.s.})}{\mathcal{T}^k W} \longrightarrow \underset{(\text{a v.s})}{\mathcal{T}^k V}$

is linear.

NTS if $T_1, T_2 \in \mathcal{T}^k W$ then

$$L^*(\alpha T_1 + \beta T_2) = \alpha(L^*T_1) + \beta(L^*T_2)$$

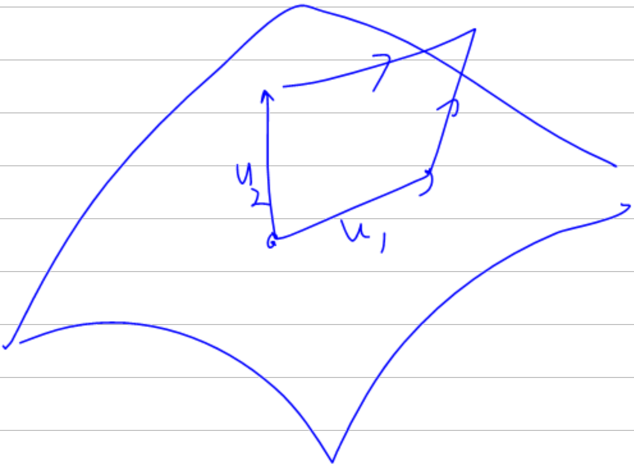
Claim 3 L^* is compatible w/ $\otimes$.

If $T_1 \in \mathcal{T}^k W$ & $T_2 \in \mathcal{T}^l W$

then

$$\underbrace{L^*(\overset{\overset{\mathcal{T}^{k+l}W}{\downarrow}}{T_1 \cdot T_2})}_{\mathcal{T}^{k+l}V} = \underbrace{(L^*\overset{\overset{\mathcal{T}^{k}V}{\downarrow}}{T_1}) \cdot (L^*\overset{\overset{\mathcal{T}^{l}V}{\downarrow}}{T_2})}_{\mathcal{T}^{k+l}V}$$

Pf Trace the defs & see that it works. □

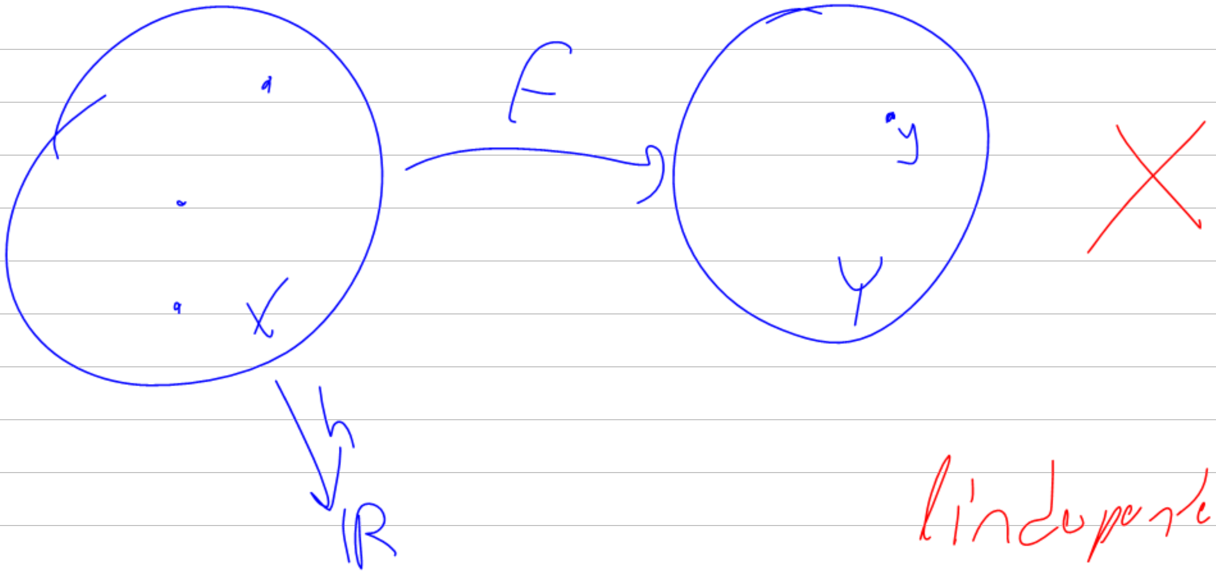


$u_1 = u_2$

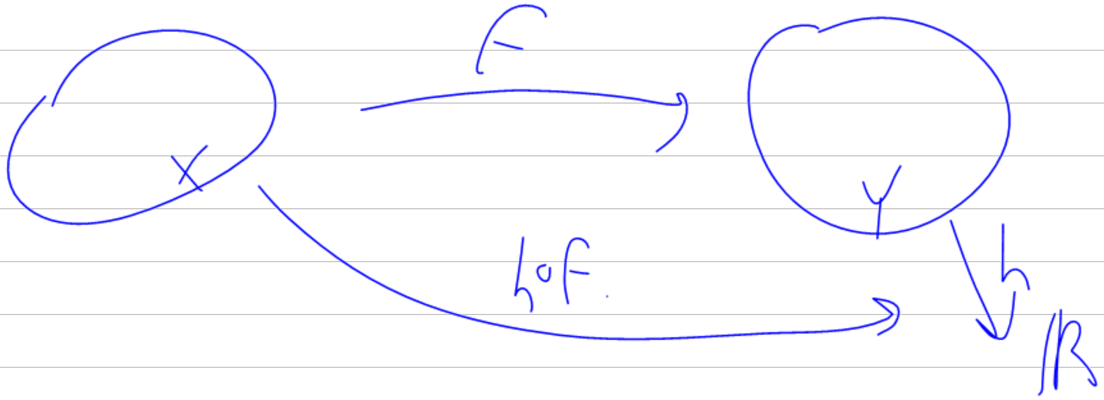
$T \in \mathcal{T}^k V$ "kills repetitions"

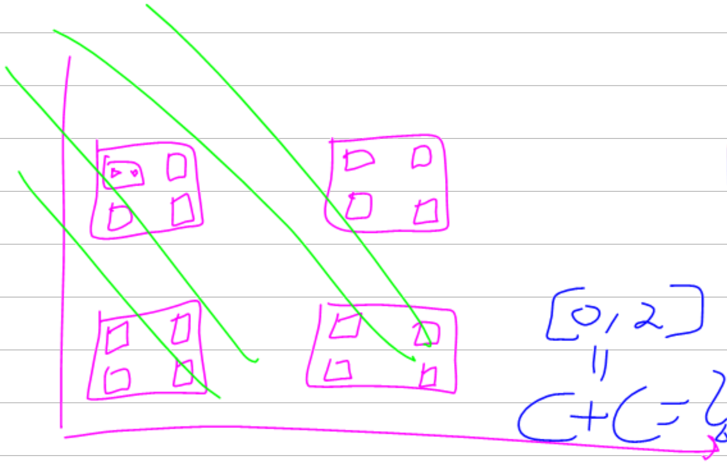
if $T(u_1, \ldots u, \ , u \quad u_k) = 0$

$T(u_1 \ldots u_k) = 0$ whenever
$u_i = u_j$ for
some i, j

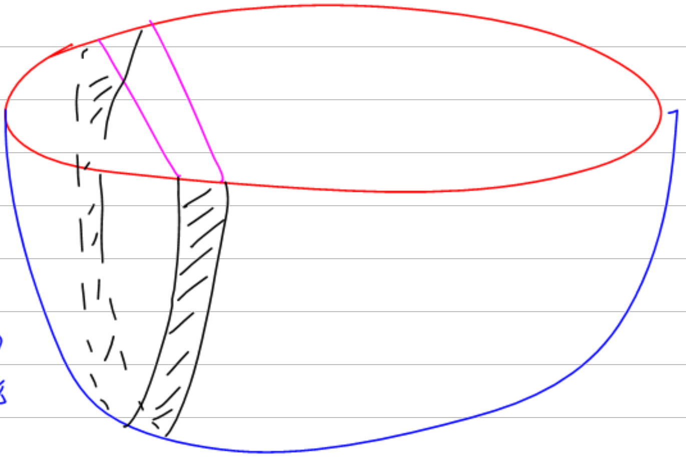


l'independent,





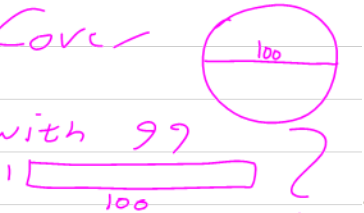
$[0,2]$
$=$
$C+C=?$



Hour 45 MAT257 Analysis II on January 28, 2022: Alternating tensors.
Read Along: Spivak 78-85.
HW12 on web by midnight.

$C+C$
$=$
?

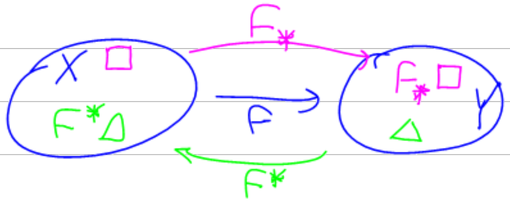
Cover



with 99 ?

V w/ basis $(v_i)_{i=1}^n$ & dual basis $(\varphi_i)_{i=1}^n$

Thm $\{\varphi_I\}_{I\in \underline{n}^k}$ is a basis of $\mathcal{T}^k V$.



Points, paths push

Functions, sets, k-tensors pull

} w/ good properties.

Def $T\in\mathcal{T}^kV$ "kills repetitions": $T(\dots u,\dots,u,\dots)\equiv 0$.

Def $T\in\mathcal{T}^k$ is "alternating"

if $T(\dots u\dots w\dots) = -T(\dots w\dots u\dots)$

other sources: Λ^kV^*

$\Lambda^k(V) := \{T\in\mathcal{T}^kV : T \text{ is alternating}\}$

Comment $\Lambda^k(V)$ is a subspace of $\mathcal{T}^kV$ □

Prop $T\in\mathcal{T}^k$ then T kills repetitions iff T is alternating.

Pf Suppose T is alternating,

$T(\dots u\dots u\dots) = -T(\dots u\dots u\dots)$

$\Rightarrow T(\dots u\dots u\dots)=0$ □

Suppose T kills repetitions:

$$0 = T(\dots u+w, \dots u+w \dots) = \cancel{T(\dots u \dots u \dots)}^{0} + T(\dots u \dots w \dots) + T(\dots w \dots u \dots) + \cancel{T(\dots w \dots w \dots)}_{0}$$

$$\Rightarrow T(\dots u \dots w \dots) + T(\dots w \dots u \dots) = 0$$

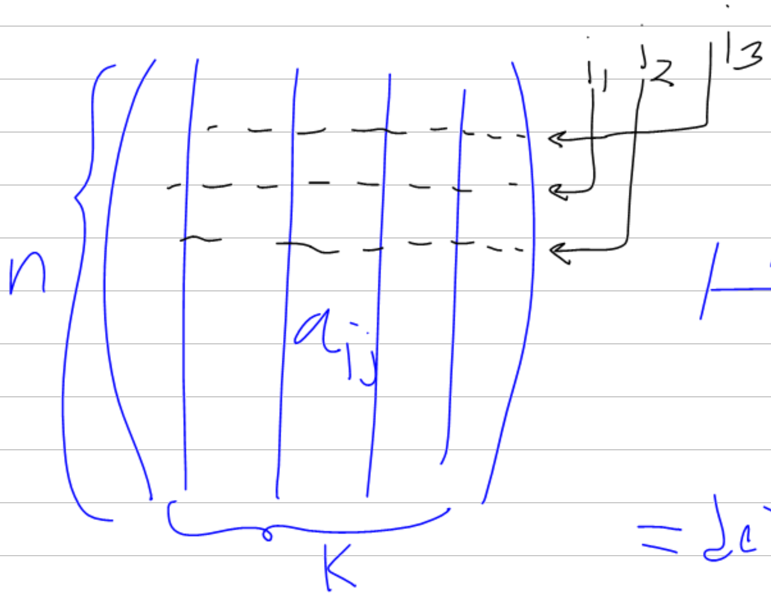
$\square$

$\in T^n \mathbb{R}^n$

Examples 1. det $\in \Lambda^n \mathbb{R}^n$

on $n \times n$ matrices

2. Suppose $k \leq n$ $\quad \lambda_I \in \Lambda^k \mathbb{R}^n \quad I \in \underline{n}^k$, $I = (i_1 \dots i_k)$



$\xmapsto{\lambda_I} \det\left(\begin{matrix} k \times k \text{ matrix} \\ \text{highlighted} \\ \text{by rows } i_1 \dots i_k \end{matrix}\right)$

$$= \det \begin{pmatrix} a_{i_1 1} & \dots & a_{i_1 k} \\ \vdots & & \\ a_{i_k 1} & \dots & a_{i_k k} \end{pmatrix}$$

1. Clearly λ_I is altern.

2. Pointless to look at I's in which an index is repeating.

3. $\lambda_{(1753)} = -\lambda_{(1735)} = +\lambda_{(1375)} = -\lambda_{(1357)}$

If we want to understand all the λ_I's, it is enough to look at I s.t.

$$\{I \in \underline{n}^k : i_1 < i_2 < \dots < i_k\} \overset{\text{temp.}}{=\!=} : \underline{n}^k_a$$

"ascending"

Q. $|\underline{n}^k_a| = ?$

$$\boxed{|\underline{n}^k| = n^k \qquad \binom{\underline{n}}{k} = \binom{n}{k}}$$

e.g. $n=8$, $k=3$

$$1 \quad 2 \quad \textcircled{3} \quad 4 \quad \textcircled{5} \quad \textcircled{6} \quad 7 \quad 8 \longleftrightarrow 3,5,6$$

$$\Rightarrow |\underline{n}^k_a| = \left(\begin{array}{l}\text{\# of ways of} \\ \text{choosing } k \text{ of} \\ n \text{ objects}\end{array}\right) = \binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

So we rename $\underline{n}^k_a$ to $\binom{\underline{n}}{k}$ = collection of ways of choosing k from $\underline{n}$

= set of ascending seqs...

$w \in \Lambda^k V \quad u_1 \dots u_k$

$$\sigma : \underline{k} \to \underline{k}$$

$$w(u_{\sigma 1} \dots u_{\sigma k}) = \pm w(u_1 \dots u_k)$$

Def A permutation of order k is

a map $\sigma : \underline{k} \longrightarrow \underline{k}$ which is a bijection.

$$S_k = \{ \text{Permutations } \sigma : \underline{k} \to \underline{k} \}$$

$$|S_k| = k!$$

$\sigma = (\sigma_1\, \sigma_2\, \sigma_3 \ldots \sigma_k)$.

$\uparrow \quad \uparrow \quad \uparrow$

$k\cdot(k-1)(k-2)\ldots) = k!$

S_k is a "group".

$$\sigma, \tau \in S_k \Rightarrow \sigma\tau = \sigma \circ \tau \in S_k$$

$\iota \in S_k$ defined by $\iota(i) = i$

Properties

1. $(\sigma\tau)\lambda = \sigma(\tau\lambda)$ "Assoc"
2. $\sigma \cdot \iota = \iota \cdot \sigma = \sigma$ "id"
3. $\sigma \cdot \sigma^{-1} = \iota$ "inverses"

S_k is a "group".

Thm $\exists!$ sign: $S_k \longrightarrow \{\pm 1\}$ s.t.

1. sign(σ