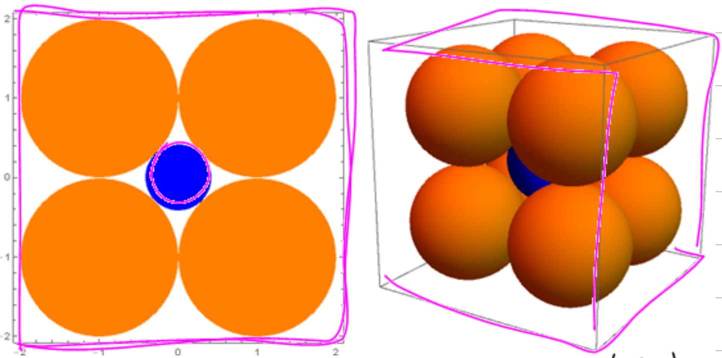
We're online!

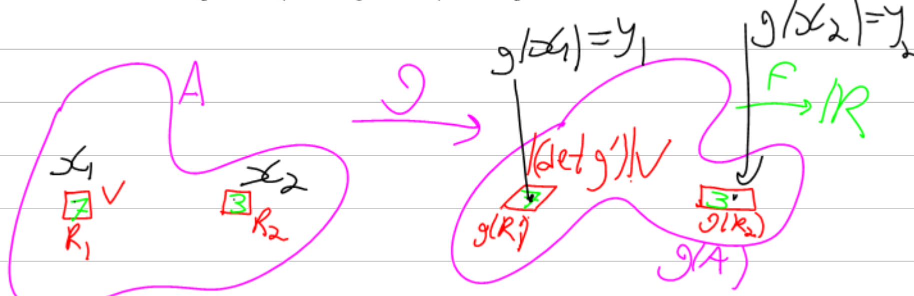
| Good | Bad | Evil |
| --- | --- | --- |
| Easier to hear & see | Glitches | No |
| Full video records | Distractions | Joy |
| Full "BB" records | Tempting to fall behind | |

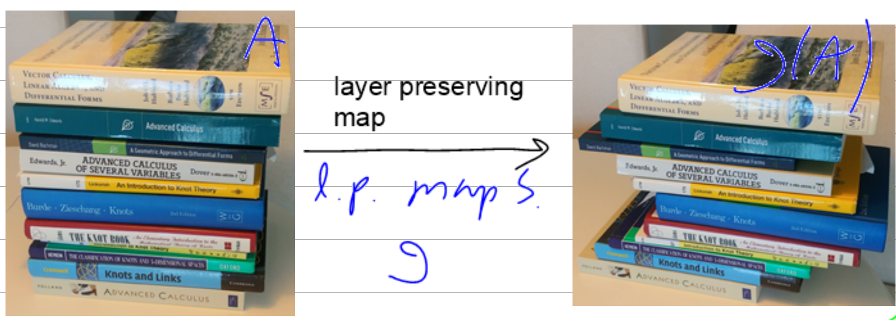
Hour 37 MAT257 Analysis II on January 10, 2022: Proof of the COV formula.
Read Along: Spivak 66-74.
Riddle Along: Riddle: Compute lim_{n\to\infty}Vol(B_n)/Vol(C_n), where B_n is the largest ball bounded by 2^n balls of radius ones with centers at {-1,1}^n and C_n is the smallest cubes bounding same balls. Promise: You will learn something very surprising if you solve this riddle.

```
GraphicsRow[{
  Graphics[{Orange, Disk /@ Tuples[{1, -1}, 2], Blue, Disk[{0, 0}, √2 - 1]}, Frame -> True],
  Graphics3D[{Orange, Ball /@ Tuples[{1, -1}, 3], Blue, Ball[{0, 0, 0}, √3 - 1]}]
}, ImageSize → 720]
```



**Thm** (Change of Variables, "COV")
Let $A \subset \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ cont. diff'ble, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $F: g(A) \to \mathbb{R}$ is integrable, (open) then
$$\int_{g(A)} F = \int_A (F\circ g)|\det g'|$$




**Debts**
1. $cov(g), cov(h) \Rightarrow cov(g\circ h)$
2. $cov(n-1) \Rightarrow cov(n)$ for l.p. maps. Fubini!
3. Pf that every g is a composition of l.p. maps (locally) IFT!
4. local ⇒ global
   cov(small sets) ⇒ cov(large sets) PO1
5. cov(1D) 157
6. maybe more...

**Lemma**
$$\begin{tikzcd} A \arrow[r, "h"] \arrow[rr, bend right, "g\circ h"] & h(A) \arrow[r, "g"] & g(h(A)) \arrow[r, "F"] & \mathbb{R} \end{tikzcd}$$
$$cov(g), cov(h) \Rightarrow cov(g\circ h)$$

"g is l.p.": $g: \mathbb{R}^n \to \mathbb{R}^n$
$$g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \\ \} \\ x_n \end{pmatrix} \qquad g = \begin{pmatrix} g_1 \\ \vdots \\ g_n \end{pmatrix}$$
$$g_n(x_1 \ldots x_n) = x_n$$

$g : \mathbb{R}^n_{x_i} \to \mathbb{R}^n_{y_i}$

$y_1 = g_1(x_1 \dots x_n)$

$\vdots \qquad \vdots$

$y_n = g_n(x_1 \dots x_n)$

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \xrightarrow[\text{IFT}]{\substack{l.p.' \\ \alpha}} \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_n \end{pmatrix} \xrightarrow[l.p.]{\beta} \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

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<u>Lemma!</u>

$\text{cov}(h), \text{cov}(g) \Rightarrow \text{cov}(g \circ h)$

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[rr, bend left, "g \circ h"] & h(A) = B \arrow[r, "g"] \arrow[d, "\bar{F}"] & g(B) \arrow[r, "F"] & \mathbb{R} \\ & \mathbb{R} & & \end{tikzcd}$$

<u>NTS</u>

$$\int_{(g\circ h)(A)} F = \int_A (F \circ g \circ h) \left| \det((g \circ h)') \right|$$

Edit

# Term Test 2.

Jan 11 at 3:05pm

Dror Bar-Natan

All Sections

Dear All,

Here's what I'll say in class tomorrow about Term Test 2.

- The test will take place on Tuesday January 18, 5-7PM (Toronto time), on Crowdmark (you will get a link by email about one minute before the official starting time). Other than documented accessibility matters, no exceptions!
- Shuyang and I will hold our regular office hours: Shuyang on Thursday at 1-2pm on **Shuyang's Zoom** (https://utoronto.zoom.us/j/83428997680) (password vchat), and I on Tuesday at 9:30-10:30, at **http://drorbn.net/vchat** (http://drorbn.net/vchat) .
- The TAs and I will hold extra pre-test office hours, in our usual zoom rooms. The details will be announced tomorrow mid-day.
- I will be available to answer questions throughout the exam, at my usual office (**http://drorbn.net/vchat** (http://drorbn.net/vchat) , but I'll add a waiting room). I will also be monitoring my regular email address (**drorbn@math.toronto.edu** (mailto:drorbn@math.toronto.edu) ) throughout the exam.
- There will be mishaps! I just hope that not too many. If you encounter one, document everything with specific details, times, and screen shots, and send me a message by Wednesday January 19 at 7PM. I will deal with these situations on a case by case basis.
- Don't let unanswered questions and/or mishaps paralyze you! If you need an answer but for whatever reason you cannot reach me, think hard, come up with what you think is the most reasonable answer/resolution, document as best as you can (for example, by adding a note on your submission), and act following your conclusions.
- Material: Everything up to and including tomorrow's material, with greater emphasis on the material that was not included in Term Test 1 (meaning, starting with countability, measure 0, and integration).
- Open book(s) and open notes but you can only use the internet (during the exam) to read the exam, to submit the exam, and to connect with the instructor/TAs to ask clarification questions. No contact allowed with other students or with any external advisors, online or in person.
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "5 of 5".
- You will be required to copy in your handwriting and sign an academic integrity statement and submit it on Crowdmark along with the rest of your exam. If you wish, you may save time by preparing the academic integrity statement in advance as in **this sample** (http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT1-LastTask.png) .
- You will be given an extra 25 minutes at the end of the exam to upload it and to copy/sign the academic integrity statement.
- The vast majority of students will do honest work, and I appreciate that. Out of respect for the honest students I will do my best to pursue and punish any cheating that may occur. I'm more experienced than you! If you plan to be dishonest, think again.
- To prepare: Do last year's **TT2** (http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/2021-257-TT2.pdf) and last year's "**TT2 rejects** (http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT2-Rejects.pdf) " (I may add some further "reject" questions later **here** (http://drorbn.net/AcademicPensieve/Classes/2122-257-AnalysisII/TT2-Rejects.pdf) ). But more important: make sure that you understand **every single bit** of class material so far!
- It is not the test I want! Class material and HW are important, but there won't be questions straight from class/HW. Many things in the last couple of years are not as we want them.

Best,

Dror.

Please, please, please, video on! (if you can)

**Thm** (Change of Variables, "COV") Let $A\subset\mathbb{R}^n$ be open, $g:A\to\mathbb{R}^n$ cont. diffable, 1-1, and s.t. $\forall x\in A$ $g'(x)$ is invertible. If $F:g(A)\to\mathbb{R}$ is integrable, then

$$\int_{g(A)} F = \int_A (F\circ g)|\det g'|$$


$x_n = z$

layer preserving map

$g:\mathbb{R}^n\to\mathbb{R}^n$

$g=\begin{pmatrix}g_1\\ \vdots\\ g_n\end{pmatrix}$

$g_n(x_1,\dots,x_n)=x_n$


**Lemma 1** cov(h), cov(g) $\Rightarrow$ cov(g∘h)

**NTS**

$$\int_{(g\circ h)(A)} F = \int_A (F\circ g\circ h)\,|\det((g\circ h)')|$$

$$A \xrightarrow{h} h(A) \xrightarrow{g} (g\circ h)(A) \xrightarrow{F} \mathbb{R}, \qquad h(A)\xrightarrow{\bar F}\mathbb{R}$$

$$\int_{(g\circ h)(A)} F \overset{cov(g)}{=} \int_{h(A)} \overbrace{(F\circ g)\,|\det g'|}^{\bar F}$$

$$\overset{cov(h)}{=} \int_A (\bar F\circ h)\cdot|\det(h')|$$

$$= \int_A (F\circ g\circ h)\cdot|\det g'\circ h|\cdot|\det h'| = \int_A (F\circ g\circ h)\,|\det((g'\circ h)\cdot h')|$$

$$= \int_A F\circ g\circ h\,|\det(g\circ h)'|$$

☺

**Debts**

1. cov(g), cov(h) $\Rightarrow$ cov(g∘h)
2. cov(n−1) $\Rightarrow$ cov(n) for l.p. maps. Fubini!
3. Pf that every g is a composition of l.p. maps (locally) IFT!
4. local $\Rightarrow$ global PO1
   cov(small sets) $\Rightarrow$ cov(large sets)
5. cov(ID) 157
6. RCOV $\Rightarrow$ COV
7. maybe more...

<u>Lemma 2</u> Assume COV(n−1). Let $g : \underset{\substack{\text{open}\\ \text{bndd.}}}{U} \to \mathbb{R}^n$

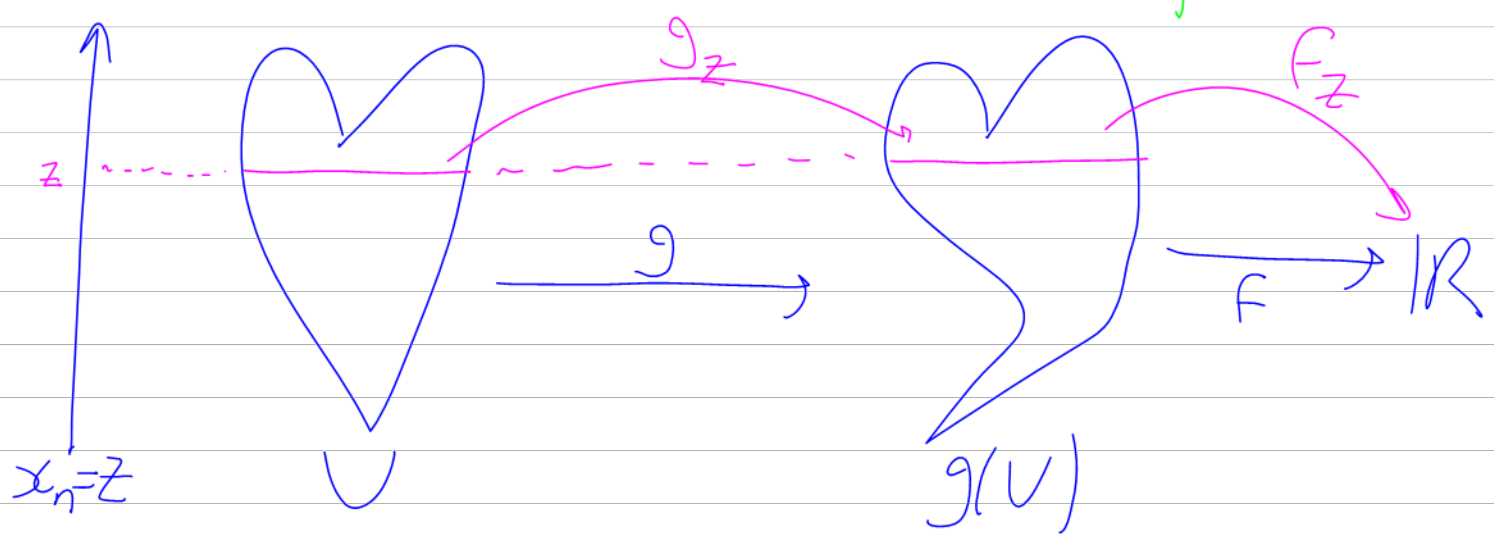
be a layer-preserving map s.t. $g(U)$ is bndd.

(meaning $g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \text{---} \\ \text{---} \\ \text{---} \\ x_n \end{pmatrix}$ or $g_n(x_1, \dots, x_n) = x_n$ )

then a restricted cov(g) holds: If $F : g(U) \to \mathbb{R}$ is cont. & $\operatorname{supp} F \subset g(U)$ then

$$\int_{\cancel{\mathbb{R}^n}\ \cancel{g(U)}} F = \int_{\cancel{U}\ \cancel{\mathbb{R}^n}} (F \circ g)\,|\det g'|$$

$$U \xrightarrow{g} g(U) \xrightarrow{F} \mathbb{R}$$




For $z \in \mathbb{R}$ define $g_z : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by

$$g_z(x) = \begin{pmatrix} g_1(x, z) \\ \vdots \\ g_{n-1}(x, z) \end{pmatrix} \qquad F_z : \mathbb{R}^{n-1} \to \mathbb{R}$$

by $F_z(x) = F(x, z)$.

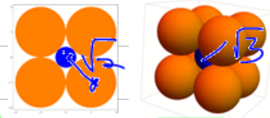
$$\int_{\cancel{\mathbb{R}^n}} F = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} dx\, F(x, z) = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} F_z$$

$$\underline{\underline{cov(n-1)}} \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} (f_z \circ g_z) \cdot |\det g_z'| =$$

As12x

$$g' = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n-1}} & \frac{\partial g_1}{\partial z} \\ \vdots & g_z' & \vdots & \\ \partial g_{n-1}/\partial x_1 & \cdots & \frac{\partial g_{n-1}}{\partial x_{n-1}} & \frac{\partial g_{n-1}}{\partial z} \\ 0 & 0 \quad 0 & 0 & 1 \end{pmatrix}$$

Read Along: Spivak 66-74.
TT2: Understand everything! And only ~~then~~ solve old problems.



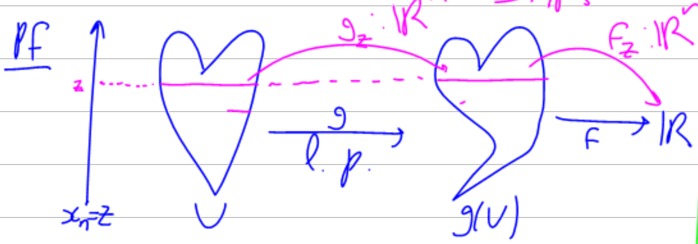

$\lim_{n\to\infty} \frac{Vol(B_n)}{Vol(C_n)} = ?$

COV: $\int_{g(A)} F = \int_A (F\circ g)\,|\det g'|$ $\quad g$ is 1–1.

Debts:
1. ~~COV(g), COV(h) ⇒ COV(g∘h)~~
2. ~~RCOV(n−1) ⇒ RCOV(n) Fubini!! for l.p. maps~~
3. ~~Pf that every g is a composition of l.p. maps (locally)~~ IFT!
4. local ⇒ global PO1 COV(small sets) ⇒ COV(large sets)
5. COV(1D) 157
6. RCOV ⇒ COV
7. COV(Coord swaps $T_{ij}$).

**Lemma 2** RCOV(n−1) → RCOV(n)

Pf




$$\int F = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} (F_z \circ g_z)\,|\det g_z'| = \#$$

$$g_n = z = x_n$$

$$g' = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n-1}} & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_{n-1}}{\partial x_1} & & \frac{\partial g_{n-1}}{\partial x_{n-1}} & \frac{\partial g_{n-1}}{\partial x_n} \\ 0 & & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} g_z' & ? \\ 0 & 1 \end{pmatrix} \qquad \det(g') = \det(g_z')$$

$$\# = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} (F\circ g)\,|\det g'| \overset{\text{Fub.}}{=} \int (F\circ g)\,|\det g'| \quad \square$$

**Lemma 3** For every $a \in A$ there is some nbd $U$ (open $U \ni a$) s.t. on $U$ $g$ is a composition of l.p. maps & coordinate swaps.

$T_{ij} : \mathbb{R}^n \to \mathbb{R}^n \quad T_{ij}(x_1 \ldots x_i \ldots x_j \ldots x_n) = (x_1 \ldots x_j \ldots x_i \ldots x_n)$

Pf Let $\alpha_k$:

$$\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \mapsto \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ g_k(x_1 \dots x_n) \end{pmatrix}$$

$y_i = g_i(x_1 \dots x_n)$

$$\begin{tikzcd} \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \arrow[r, "\alpha_k"] \arrow[rr, bend right, "g"'] & \begin{pmatrix} x_1 \\ \vdots \\ x_{n-1} \\ y_k \end{pmatrix} \arrow[r, "\beta_k"] & \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \end{tikzcd}$$

$$\alpha_k' = \left(\begin{array}{cccc|c} \multicolumn{4}{c}{\overbrace{\hphantom{1\ 0\ 0\ \dots}}^{n-1}} & \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ & & & & 0 \\ 0 & 0 & 0 & 1 & 0 \\ \hline \frac{\partial g_k}{\partial x_1} & \dots & & & \frac{\partial g_k}{\partial x_n} \end{array}\right)$$

invertible iff $\frac{\partial g_k}{\partial x_n} \neq 0$

assume $\frac{\partial g_k}{\partial x_n} = 0 \ \forall k \Rightarrow g' = \begin{pmatrix} & \frac{\partial g_1}{\partial x_n} \\ & 0 \\ & \vdots \\ & \frac{\partial g_n}{\partial x_n} \end{pmatrix}$

contradicting $g'(a)$ is invertible.

$\Rightarrow$ For some $k$, $\frac{\partial g_k}{\partial x_n} \neq 0$

Fix such value for $k$.

Now $\alpha_k'$ is invertible at $a$, so by IFT $\alpha_k$ is invertible near $a$.

Near $a$, set $\beta_k = g \circ \alpha_k^{-1}$ & Now (near $a$)

$$g = \beta_k \circ \alpha_k$$

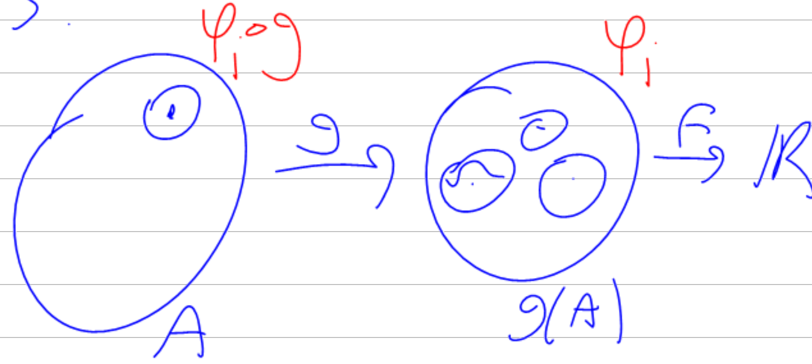
$$= \tau_{kn} \circ \underbrace{\tau_{kn} \circ \beta_k} \circ \tau_{1n} \circ \underbrace{\tau_{1n} \circ \alpha_k \circ \tau_{1n}} \circ \tau_{1n}$$

$$= \tau_{kn} \circ (\tau_{kn} \circ \beta_k) \circ \tau_{1n} \circ (\tau_{1n} \circ \alpha_k \circ \tau_{1n}) \circ \tau_{1n}$$

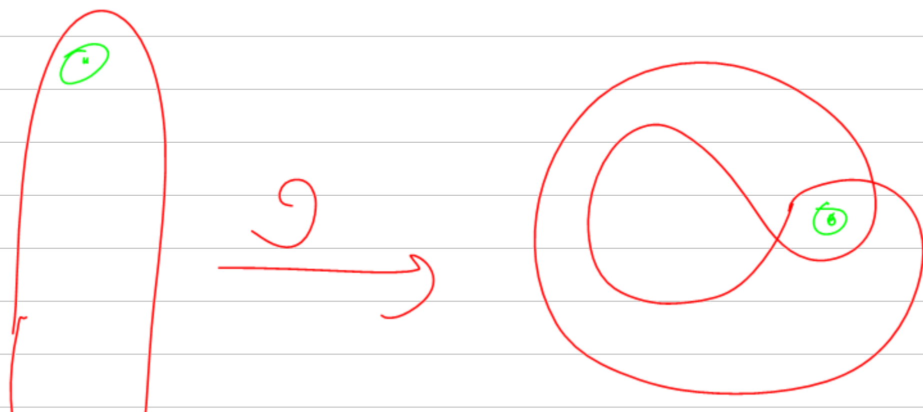
c.s. l.p. c.s. l.p. c.s.

<u>Lemma 4</u> Local cov $\Rightarrow$ Global cov for cont. F's $\square$

<u>Pf</u>




$$\operatorname{supp}(\varphi_i \circ g) = g^{-1}(\operatorname{supp} \varphi_i)$$




but $g$ is 1-1?