Do not open this notebook until instructed.

Math 257 Analysis II

Term Test 1

University of Toronto, November 2, 2021

Solve all 4 problems on this booklet.
The problems are of equal weight.
You have an hour and fifty minutes to write this test.

Notes

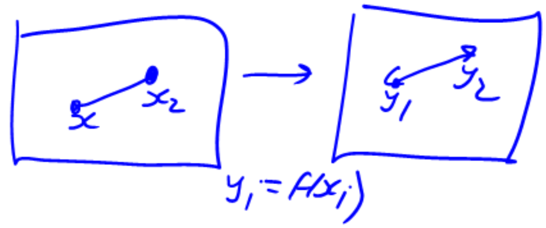
- No outside material allowed other than stationary, minimal hydration and snacks, and stuffed animals.
- Write your solution of each problem on the problem page and on the back of the problem page. If you run out of space you may continue into the scratch pages, but you **must** indicate this on the problem page or else the scratch pages will not be read.
- **Neatness counts! Language counts!** The *ideal* written solution to a problem looks like a page from a textbook; neat and clean and consisting of complete and grammatical sentences. Definitely phrases like "there exists" or "for every" cannot be skipped. Lectures are mostly made of spoken words, and so the blackboard part of proofs given during lectures often omits or shortens key phrases. The ideal written solution to a problem does not do that.

Good Luck!

Problem 1. A continuous function $f\colon \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$|(x_1 - x_2) - (\underbrace{f(x_1)}_{y_1} - \underbrace{f(x_2)}_{y_2})| \le \frac{1}{7}\,|x_1 - x_2|$$

"All scale Fidelity ;



$y_i = f(x_i)$

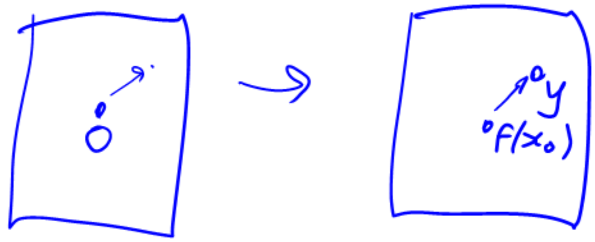
for every x_1 and x_2 in $\mathbb{R}^n$. Prove that for every $y \in \mathbb{R}^n$ there is an $x \in \mathbb{R}^n$ such that $f(x) = y$.

Tip. Don't start working! Read the whole exam first. You may wish to start with the questions that are easiest for you.

Tip. You may want to start by writing "draft solutions" on the last pages of this notebook and only then write the perfected versions in the space left here for solutions.

$f:\mathbb{R}^n \to \mathbb{R}$

Problem 2. Show that if a function f is defined near a point $a \in \mathbb{R}^n$ and has continuous partial derivatives near a, then it is differentiable at a.



Tip. In math exams, "show" means "prove".

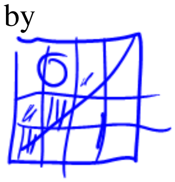
$x_0 = 0$
$x_1 = x_0 + (y - f(x_0))$
$\vdots$
$x_n = x_{n-1} + (y - f(x_{n-1}))$
$x_{n+1} = x_n + (y - f(x_n))$
$|x_{n+1} - x_n| = |(x_n - x_{n-1}) - (f(x_n) - f(x_{n-1}))|$
$< \frac{1}{7}|x_n - x_{n-1}|$

Problem 3. A set $A \subset [0, 1]$ is a (possibly infinite) union of open intervals and it contains all the rational numbers in $(0, 1)$. Show that the boundary of A is $[0, 1] \setminus A$.

Problem 4. Let $f\colon [0, 1] \times [0, 1] \to \mathbb{R}$ be given by

$$U(f,P) - L(f,P) = \frac{2n+1}{n^2} \to 0$$

$$f(x, y) = \begin{cases} 1 & x > y \\ 0 & x \le y \end{cases}.$$



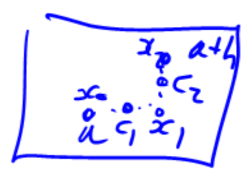
Prove that f is integrable on $[0, 1] \times [0, 1]$ directly by using partitions (namely, without using theorems about continuity and integrability).

Q2 $\quad f' = (\partial_1 f \ldots \partial_n f)$

$\underbrace{f(a+h) - f(a)} - \sum \underbrace{\partial_i f(a)}_{\#} \cdot h_i \in o(h)$

$x_k = (a_1 + h_1 \ldots a_k + h_k, a_{k+1} \ldots a_n)$

$x_0 = a$
$x_n = a + h$



$f(a+h) - f(a) - \# = f(x_n) - f(x_0) - \#$

$= \sum_{k=1}^{n} f(x_k) - f(x_{k-1}) - \sum \partial_k f(a) \cdot h_k = \sum ((\partial_k f)(c_k) - \partial_k f(a)) h_k$