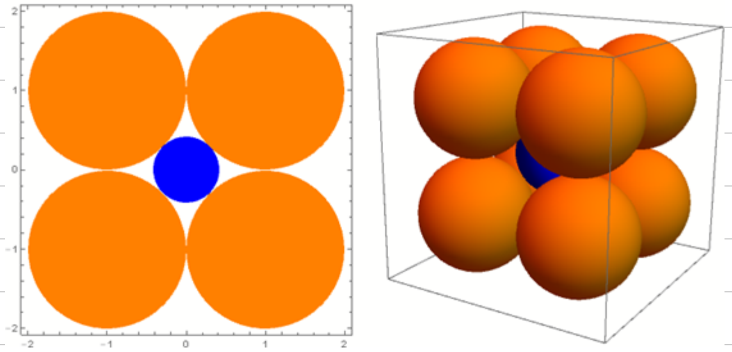
2021-22 MAT 257Y Analysis II
Dror's Open Private Notebook
Second Semseter

Hour 37 MAT257 Analysis II on January 10, 2022: Proof of the COV formula.

Read Along: Spivak 66-74.

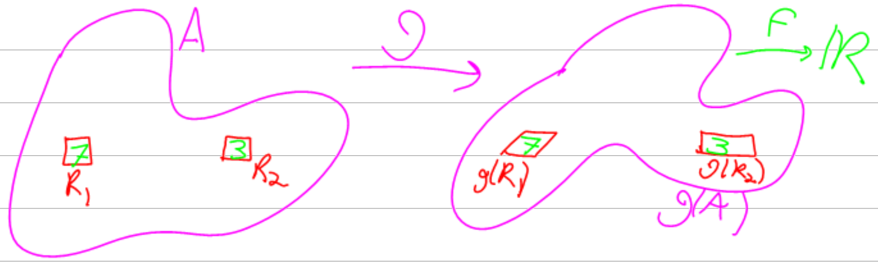
Riddle Along: Riddle: Compute lim_{n\to\infty}Vol(B_n)/Vol(C_n), where B_n is the largest ball bounded by 2^n balls of radius ones with centers at {-1,1}^n and C_n is the smallest cubes bounding same balls. Promise: You will learn something very surprising if you solve this riddle.

```
GraphicsRow[{
  Graphics[{Orange, Disk /@ Tuples[{1, -1}, 2], Blue, Disk[{0, 0}, √2 - 1]}, Frame -> True],
  Graphics3D[{Orange, Ball /@ Tuples[{1, -1}, 3], Blue, Ball[{0, 0, 0}, √3 - 1]}]
}, ImageSize → 720]
```

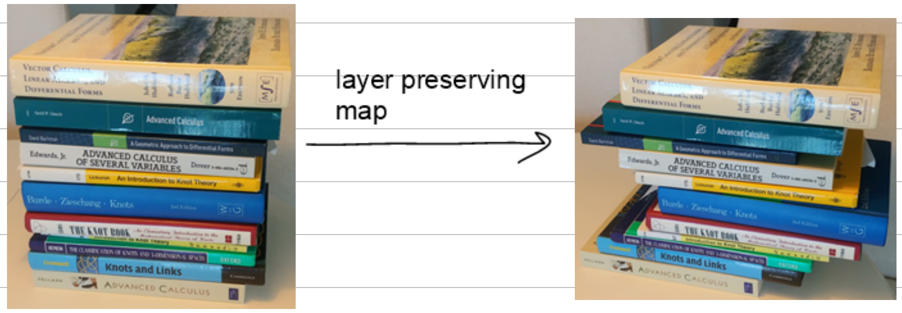


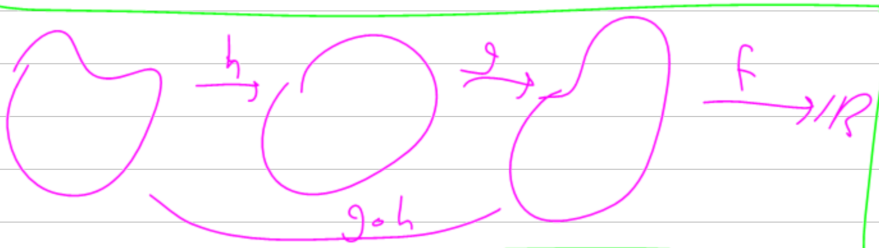
Thm (Change of Variables, "COV") Let $A \subset \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $f: g(A) \to \mathbb{R}$ is integrable, then

$$\int_{g(A)} f = \int_A (f\circ g)|\det g'|$$



Debts.





$$y_1 = g_1(x_1 \dots x_n) \qquad \begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \longmapsto \begin{matrix} x_1 \\ \vdots \\ x_{n-1} \\ y_n \end{matrix} \longmapsto \begin{matrix} y_1 \\ \vdots \\ y_n \end{matrix}$$

$$\vdots$$

$$y_n = g_n(x_1 \dots x_n)$$

Will work only locally!

Lemma 1 COV(g), COV(h) $\Rightarrow$ COV($g\circ h$)

Debts

1. COV(g), COV(h) $\Rightarrow$ COV($g\circ h$)
2. COV(n-1) $\Rightarrow$ COV(n) for layer-preserving maps.
3. PF that every g is a composition of layer-preserving maps.
4. COV$\left(\begin{matrix}\text{small}\\\text{sets}\end{matrix}\right) \Rightarrow$ COV$\left(\begin{matrix}\text{large}\\\text{sets}\end{matrix}\right)$
5. Prove COV(1D)
6. Maybe more.

Edit

Term Test 2.

Dror Bar-Natan

All Sections

Jan 11 at 3:05pm

Dear All,

Here's what I'll say in class tomorrow about Term Test 2.

- The test will take place on Tuesday January 18, 5-7PM (Toronto time), on Crowdmark (you will get a link by email about one minute before the official starting time). Other than documented accessibility matters, no exceptions!
- Shuyang and I will hold our regular office hours: Shuyang on Thursday at 1-2pm on **Shuyang's Zoom** (https://utoronto.zoom.us/j/83428997680) (password vchat), and I on Tuesday at 9:30-10:30, at **http://drorbn.net/vchat** (http://drorbn.net/vchat) .
- The TAs and I will hold extra pre-test office hours, in our usual zoom rooms. The details will be announced tomorrow mid-day.
- I will be available to answer questions throughout the exam, at my usual office (**http://drorbn.net/vchat** (http://drorbn.net/vchat) , but I'll add a waiting room). I will also be monitoring my regular email address (**drorbn@math.toronto.edu** (mailto:drorbn@math.toronto.edu)) throughout the exam.
- There will be mishaps! I just hope that not too many. If you encounter one, document everything with specific details, times, and screen shots, and send me a message by Wednesday January 19 at 7PM. I will deal with these situations on a case by case basis.
- Don't let unanswered questions and/or mishaps paralyze you! If you need an answer but for whatever reason you cannot reach me, think hard, come up with what you think is the most reasonable answer/resolution, document as best as you can (for example, by adding a note on your submission), and act following your conclusions.
- Material: Everything up to and including tomorrow's material, with greater emphasis on the material that was not included in Term Test 1 (meaning, starting with countability, measure 0, and integration).
- Open book(s) and open notes but you can only use the internet (during the exam) to read the exam, to submit the exam, and to connect with the instructor/TAs to ask clarification questions. No contact allowed with other students or with any external advisors, online or in person.
- The format will be "Solve 7 of 7", or maybe "6 of 6" or "5 of 5".
- You will be required to copy in your handwriting and sign an academic integrity statement and submit it on Crowdmark along with the rest of your exam. If you wish, you may save time by preparing the academic integrity statement in advance as in **this sample** (http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT1-LastTask.png) .
- You will be given an extra 25 minutes at the end of the exam to upload it and to copy/sign the academic integrity statement.
- The vast majority of students will do honest work, and I appreciate that. Out of respect for the honest students I will do my best to pursue and punish any cheating that may occur. I'm more experienced than you! If you plan to be dishonest, think again.
- To prepare: Do last year's **TT2** (http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/2021-257-TT2.pdf) and last year's "**TT2 rejects** (http://drorbn.net/AcademicPensieve/Classes/2021-257-AnalysisII/TT2-Rejects.pdf) " (I may add some further "reject" questions later **here** (http://drorbn.net/AcademicPensieve/Classes/2122-257-AnalysisII/TT2-Rejects.pdf)). But more important: make sure that you understand **every single bit** of class material so far!
- It is not the test I want! Class material and HW are important, but there won't be questions straight from class/HW. Many things in the last couple of years are not as we want them.

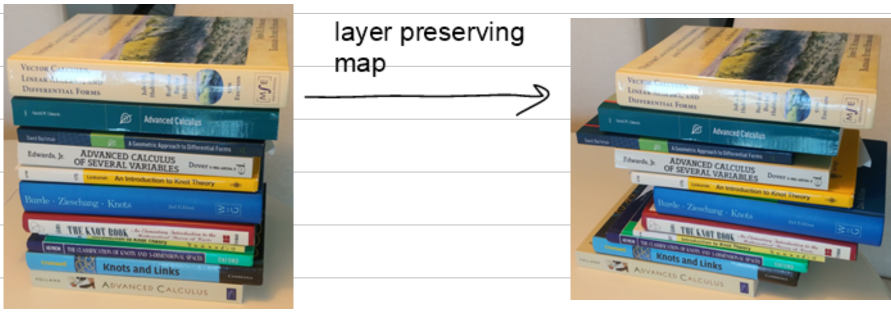
Best,

Dror.

Hour 38 MAT257 Analysis II on January 12, 2022: Proof of the COV formula (2).
Read Along: Spivak 66-74.
TT2 details.

Thm (Change of Variables, "COV") Let $A\subset\mathbb{R}^n$ be open, $g: A\to\mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x\in A$ $g'(x)$ is invertible. If $f: g(A)\to\mathbb{R}$ is integrable, then

$$\int_{g(A)} f = \int_A (f\circ g)|\det g'|$$



Lemma 1 $\operatorname{cov}(h), \operatorname{cov}(g) \Rightarrow \operatorname{cov}(g\circ h)$

NTS

$$\int_{(g\circ h)(A)} f = \int_A (f\circ g\circ h)\left|\det((g\circ h)')\right|$$

Debts 1. $\operatorname{cov}(g), \operatorname{cov}(h) \Rightarrow \operatorname{cov}(g\circ h)$

2. $\operatorname{cov}(n-1) \Rightarrow \operatorname{cov}(n)$ for l.p. maps. Fubini!

3. Pf that every g is a composition of l.p. maps (locally) IFT!

4. local ⇒ global PO1

$\operatorname{cov}(\text{small sets}) \Rightarrow \operatorname{cov}(\text{large sets})$

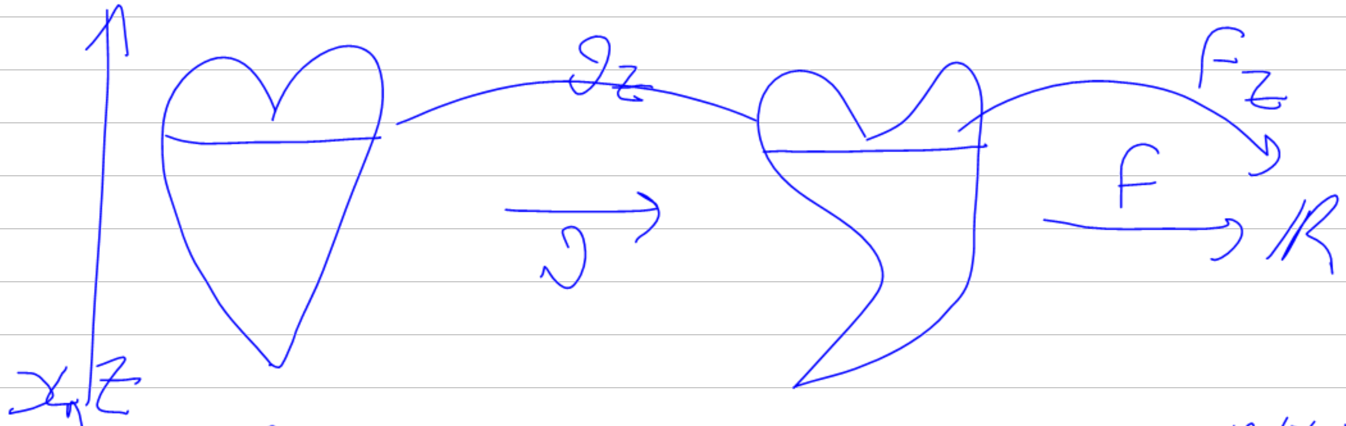
5. cov(1D) 157.

6. maybe more...

Lemma 2 Assume COV(n-1). Let $g: U \to \mathbb{R}^n$ (open & bndd) be layer-preserving (namely $g\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} \equiv \\ \equiv \\ x_n \end{pmatrix}$, or $g_n(x_1 \ldots x_n) = x_n$. & s.t. $g(U)$ is bndd) Then a restricted COV(g) holds: If $F: g(U) \to \mathbb{R}$ is cont., & bndd and $\text{supp}(F) \subset U$ then $\int_{\mathbb{R}^n\,\cancel{g(U)}} F = \int_{\cancel{U}/\mathbb{R}^n} (F\circ g)\cdot|\det g'|$

Debt 6:
RCOV(cont.)
⇒ RCOV(intg)

For simplicity, write all integrals on $\mathbb{R}^n/\mathbb{R}^{n-1}$, extending the integrands by 0 as necessary.



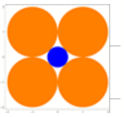
PF For $z \in \mathbb{R}$ define $g_z: \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$ by $g_z(x) = \begin{pmatrix} g_1(x,z) \\ \vdots \\ g_{n-1}(x,z) \end{pmatrix}$ & $F_z: \mathbb{R}^{n-1} \to \mathbb{R}$ by $F_z(y) = F(y,z)$.

$$\int_{\mathbb{R}^n} F = \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} dx\, F(x,z) = \int_{\mathbb{R}^n} dz \int_{\mathbb{R}^{n-1}} F_z$$

$$g' = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n-1}} & \frac{\partial g_1}{\partial x_n} \\ \frac{\partial g_{n-1}}{\partial x_1} & & \frac{\partial g_{n-1}}{\partial x_{n-1}} & \frac{\partial g_{n-1}}{\partial x_n} \\ 0 & & 0 & 1 \end{pmatrix}$$

$$= \int_{\mathbb{R}} dz \int_{\mathbb{R}^{n-1}} (F_z \circ g_z)|\det(g_z')| = \int_{\mathbb{R}^n} (F\circ g)|\det g'|$$

Hour 39 MAT257 Analysis II on January 14, 2022: Proof of the COV formula (3).
Read Along: Spivak 66-74.
TT2: Understand everything! And only thesn solve old problems.

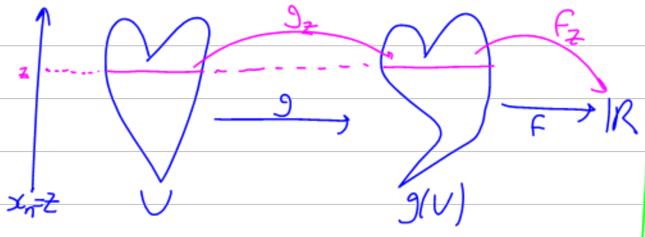


$$\lim_{n\to\infty}\frac{Vol(B_n)}{Vol(C_n)}=?$$

COV: $$\int_{g(A)} f=\int_A (f\circ g)\,|\det g'|$$

Lemma 2 RCov(n-1) → RCov(n)

Pf



$$\int f=\int_{\mathbb{R}} dz\int_{\mathbb{R}^{n-1}}\left((f_z\circ g_z)\cdot|\det g_z'|\right.$$

$$g'=\begin{pmatrix}\frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_{n-1}} & \frac{\partial g_1}{\partial x_n}\\ \frac{\partial g_{n-1}}{\partial x_1} & & \frac{\partial g_{n-1}}{\partial x_{n-1}} & \frac{\partial g_{n-1}}{\partial x_n}\\ 0 & & 0 & 1\end{pmatrix}$$

Debts

1. ~~cov(g), cov(h) ⇒ cov(g∘h)~~
2. RCov(n-1) ⇒ RCov(n) for l.p. maps. Fubini!
3. Pf that every g is a composition of l.p. maps. (locally) IFT!
4. local ⇒ global PO1

   cov(small sets) ⇒ cov(large sets)
5. cov(ID) 157.
6. RCOV ⇒ COV
7. maybe more...

Lemma 3 For every $a\in A$ there is some open $U\ni a$ s.t. on U g is a composition of l.p. maps & coordinate swaps.

Pf Let $\alpha_k:\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}\longmapsto\begin{pmatrix}x_1\\ \vdots\\ x_{n-1}\\ g_k(x_1\dots x_n)\end{pmatrix}$

$y_i=g_i(x)$

$$\begin{tikzcd}\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix} \arrow[r, mapsto] \arrow[r, bend right, "\alpha_k"'] \arrow[rr, bend left, "g"] & \begin{pmatrix}x_1\\ x_{n-1}\\ y_k\end{pmatrix} \arrow[r, mapsto] \arrow[r, bend right, "\beta_k"'] & \begin{pmatrix}y_1\\ \vdots\\ y_n\end{pmatrix}\end{tikzcd}$$

then $$\alpha_k'=\left(\begin{array}{ccc|c}1 & 0 & & 0\\ 0 & I_{n-1} & & 0\\ 0 & & & 0\\ & & & 0\\ \hline \frac{\partial g_k}{\partial x_1} & & \frac{\partial g_k}{\partial x_{n-1}} & \frac{\partial g_k}{\partial x_n}\end{array}\right).$$

For at least on k, $\frac{\partial g_k}{\partial x_n}\neq 0$, pick that k.

PM: Emphasize that β_k isn't $\begin{pmatrix}x_1\\ x_{n-1}\\ y_k\end{pmatrix}\longmapsto\begin{pmatrix}g_1(x\dots)\\ \vdots\\ g_{n-1}(x\dots)\\ g_n(x\dots)\end{pmatrix}$.

Near a, α_K is invertible, set $\beta_K = g\circ\alpha_K^{-1}$

Let τ_{ij} be the (ij) swap. Then

$$g = \beta_K\circ\alpha_K = \tau_{Kn}\circ\underbrace{\tau_{Kn}\circ\beta_K}\circ\tau_{1n}\circ\underbrace{\tau_{1n}\circ\alpha_K\circ\tau_{1n}}\circ\tau_{1n}$$

Lemma 4 Local COV ⇒ global COV.

Find a cover $\mathcal{V}=\{V\}$ of $g(A)$ by bndd open sets s.t. $\forall V\in\mathcal{V}$ $g^{-1}(V)$ is bndd & on it g is a composition of O.P. maps & coord. swaps.

here we use g is a bijection!

Let $\{\psi_i\}$ be a PO1 for $g(A)$ sub to $\mathcal{V}$.

Then $\{\varphi_i = \psi_i\circ g\}$ is a PO1 for A sub to $\mathcal{U}=\{g^{-1}(V)\}$

So

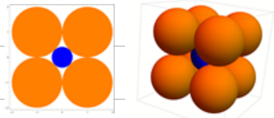
$$\int_{g(A)} F = \sum_i \int_{\mathbb{R}^n}\psi_i F = \sum_i\int_{\mathbb{R}^n}(\psi_i\circ g)(F\circ g)|\det g'| = \dots$$

Lemma 5 COV hold if $n=1$.

pf WLOG, $A=(a,b)$. g is 1-1, so g is monotone. So $g(A) = g((a,b)) = \begin{cases}(g(a),g(b))\\(g(b),g(a))\end{cases}$

$$\int_{g(A)} F \overset{\text{if } g \text{ is decreasing}}{=} \int_{g(b)}^{g(a)} F = \int_b^a (F\circ g)\,g' = +\int_a^b (F\circ g)\underset{\det g'}{\underset{\parallel}{|g'|}}$$ □

Hour 40 MAT257 Analysis II on January 17, 2022: Proof of the COV formula (4).
Read Along: Spivak 66-74.
TT2: Understand everything! And only thesn solve old problems. And good luck!



$\lim_{n\to\infty} \frac{Vol(B_n)}{Vol(C_n)} = ?$

COV: $\int_{g(A)} F = \int_A (F\circ g)|\det g'|$

4. local ⇒ global PO1
$COV\left(\begin{matrix}\text{small}\\ \text{sets}\end{matrix}\right) \Rightarrow COV\left(\begin{matrix}\text{large}\\ \text{sets}\end{matrix}\right)$ as expected w/ tiny complication

5. COV(1D) 157 dull.

6. RCOV ⇒ COV "<"

7. $COV(T_{ij})$ So dull will skip.

Debts: ~~1. $cov(g), cov(h) \Rightarrow cov(g\circ h)$~~

~~2. $RCov(n-1) \Rightarrow RCov(n)$ for l.p. maps~~ ~~Fubini!~~

~~3. Pf that every g is a composition of l.p. maps (locally)~~ ~~IFT!~~

Lemma 4 Local COV ⇒ global COV:

Find a cover $\mathcal{V} = \{V\}$ of $g(A)$ by bndd open sets s.t. $\forall V \in \mathcal{V}$ $g^{-1}(V)$ is bndd & on it g is a composition of l.p. maps & coord. swaps.

here we use g is a bijection!

Let $\{\psi_i\}$ be a PO1 for $g(A)$ sub to $\mathcal{V}$.

Then $\{\varphi_i = \psi_i \circ g\}$ is a PO1 for A sub to $\mathcal{U} = \{g^{-1}(V)\}$

So $\int_{g(A)} F = \sum_i \int_{\mathbb{R}^n} \psi_i F = \sum_i \int_{\mathbb{R}^n} (\psi_i \circ g)(F\circ g)|\det g'| = \ldots$

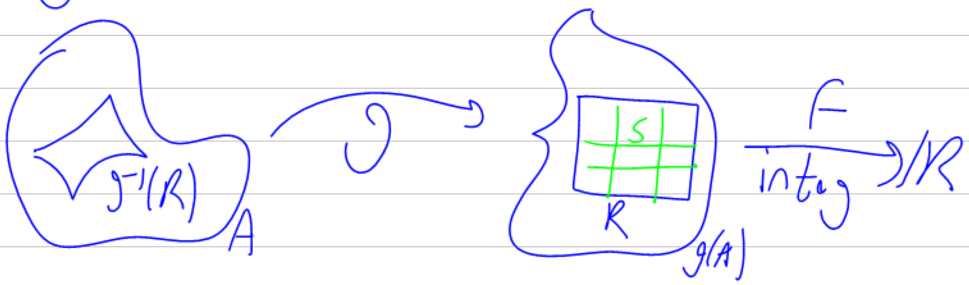
Lemma 5 COV holds if $n=1$.

Pf WLOG, $A = (a,b)$. g is 1-1, so g is monotone. So $g(A) = g((a,b)) = \begin{cases}(g(a),g(b))\\ (g(b),g(a))\end{cases}$

$\int_{g(A)} F \overset{\substack{\text{if } g \text{ is}\\ \text{decreasing}}}{=} \int_{g(b)}^{g(a)} F = \int_b^a (F\circ g)\, g' = +\int_a^b (F\circ g)\underset{\det g'}{\underset{\shortparallel}{|g'|}}$ □

Lemma 6 Suppose COV holds for ~~cont~~ const functions f.
Then it also holds for arbitrary integrable f.

Pf We'll prove a local version which can be globalized as before



NTS:

$$\int_R f = \int_{g^{-1}(R)} (f\circ g)|\det g'|$$

Pf $L(f,P) = \sum_{S\in P} V(S)m_S(f) = \sum_S \int_S m_S(f) =$

$= \sum_S \int_{g^{-1}(S)} m_S(f)|\det g'| \le \sum_S \underline{\int}_{g^{-1}(S)} (f\circ g)|\det g'|$

$= \sum_S \underline{\int}_{g^{-1}(R)} \chi_S (f\circ g)\cdot|\det g'|$

$\le \underline{\int}_{g^{-1}(R)} \sum_S \chi_S (f\circ g)|\det(g')|$

$= \underline{\int}_{g^{-1}(R)} (f\circ g)|\det(g')| \le \int \_\_$

$\ldots \le U(f,P)$ $\square$

Aside $\underline{\int} h_1 + \underline{\int} h_2 \le \underline{\int}(h_1+h_2)$

Pf $L(h_1+h_2, P) = \sum_S V(S) m_S(h_1+h_2)$

$\ge \sum_S V(S)(m_S(h_1)+m_S(h_2))$

$= L(h_1,P)+L(h_2,P)$

Lemma 7 COV holds for coord swaps T_{ij}.

Pf NTS $\int_{T_{ij}(A)} f = \int_A f\circ T_{ij}$ E.g., $\int_{TA} f(x,y) = \int_A f(y,x)$ ....

$\boxed{\text{COV}}$

I hope TT2 went well!
Hour 41 MAT257 Analysis II on January 19, 2022:
Sketchy coordinate swaps and Sard's Theorem,
Detailed k-tensors.
Read Along: Spivak 66-74, 75-78.

Riddle Along: Let f be a distance-non-inreasing function from the plane to the plane (d(x,y) >= d(f(x),f(y)), for all x,y), and let R be a rectangle in the plane. Is it always true that the length of the boundary of R is greater or equal to the length of the boundary of f(R)?

COV: $\int_{g(A)} f = \int_A (f\circ g)|\det g'|$ Debt 7 COV(T_{ij}) So dull will skip.

Lemma 7 COV holds for coord swaps T_{ij}.

PF NTS $\int_{T_{ij}(A)} f = \int_A f\circ T_{ij}$ E.g., $\int_{TA} f\circ T = \int_A f$

Q. How do you write the proof of something so disturbingly obvious?

A. You go back to the defs.

PF Given $A=[a_1,b_1]\times[a_2,b_2]$; $TA=[a_2,b_2]\times[a_1,b_1]$

$P=((a_1=t_{10},t_{11},\ldots,t_{1n}=b_1),(a_2=t_{20},\ldots,t_{2n}=b_2))$ $TP=\ldots$

$L(f\circ T,\underset{\text{of } TA}{P}) = \sum_{S\in P} v(S)m_S(f\circ T) = \sum_{S\in P} v(TS)m_{TS}(f) =$

$\sum_{TS\in TP} \_\_ = \sum_{S\in TP} \_\_ = L(f,TP)$ ...... $\boxed{7}\,\boxed{COV}$

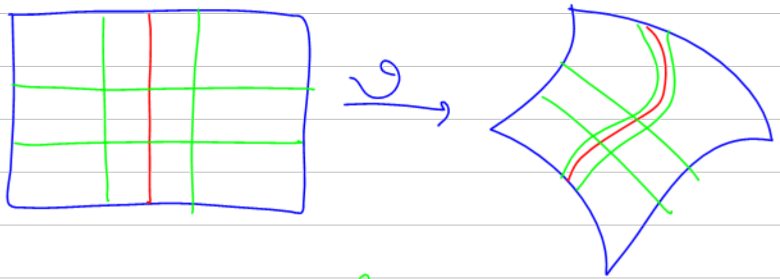
The Baby Sard Theorem. $A\subset\mathbb{R}^n$ open, $g: A\to\mathbb{R}^n$ cont. diffable, $C=\{x\in A: \det(g'(x))=0\}$. Then $g(C)$ is of meas-0.

Corollary COV holds even w/o the condition "$g'(x)$ is invertible".

The Adult Sard Thm (Way harder!) $g: \overset{\text{open} \subset \mathbb{R}^n}{A} \to \mathbb{R}^m$ k-times cont. diffable where $k=\max(n-m+1,1)$, $C:=\{x: \text{rank } g'(x)<m\}$. Then $g(C)$ is meas-0.

Baby Proof of Baby Stokes

Full proof in 2021-257/210115



done line

Remember, $\int_C d\omega = \int_{\partial C} \omega$

warning: Elsewhere $\mathcal{T}^k(V^*)$

Def $T : V^k \longrightarrow \mathbb{R}$ is "multi-linear" if . . .
k-linear

Examples 2. Inner products   1. V^*

3. dets   0. $\mathbb{R}$

Def $\mathcal{T}^k(V) =$ "k-tensors on V" $= \left\{\begin{matrix} k\text{-linear maps} \\ V^k \longrightarrow \mathbb{R} \end{matrix}\right\}$

$\mathcal{T}^k$ is a vector space

Also, $\mathcal{T}^k \times \mathcal{T}^l \to \mathcal{T}^{k+l}$ via $(T_1, T_2) \longmapsto T_1 \otimes T_2$

$\mathcal{T} = \bigoplus_k \mathcal{T}^k$ is a graded non-commutative algebra w/ unit

$\mathcal{T}^1(V) = V^*$ $\left[\text{E.g. } (\mathbb{R}^2)^* \overset{\text{ex}}{=} \left[\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix}\right]^* = \begin{matrix}(-2 & \frac{3}{2}) \\ (1 & -\frac{1}{2})\end{matrix} \right]$

Thm V w/ basis $v_1 \ldots v_n$; $\varphi_1 \ldots \varphi_n$ the dual basis

Then $\{\varphi_{i_1} \otimes \ldots \otimes \varphi_{i_k} : 1 \leq i_j \leq n\}$ is a basis of $\mathcal{T}^k(V)$. Hence $\dim \mathcal{T}^k(V) = n^k$.

Pf . . .

Hour 42 MAT257 Analysis II on January 21, 2022: k-tensors.
Read Along: Spivak 75-78.
HW11 on web by midnight! (Sorry, much of it will only make sense on Monday)
Riddle Along: The Cantor set C is of measure 0. Is the same true for C+C={x+y: x,y in C}?

Remember, $\int_C d\omega = \int_{\partial C} \omega$

Def $T: V^k \longrightarrow \mathbb{R}$ is "multi-linear" if . . .
k-linear

Examples 2. Inner products   1: V^*

n. dets   0: $\mathbb{R}$

Def $\mathcal{T}^k(V)$ = "k-tensors on V" = $\left\{ \begin{matrix} \text{k-linear maps} \\ V^k \longrightarrow \mathbb{R} \end{matrix} \right\}$

warning: Elsewhere $\mathcal{T}^k(V^*)$

Examples:

$\mathcal{T}^0(V) = \mathbb{R}$ $\mathcal{T}^1(V) = V^*$

Bases & dual bases   Eg. $(\mathbb{R}^2)^* = \left[\begin{pmatrix}1\\2\end{pmatrix}, \begin{pmatrix}3\\4\end{pmatrix} \right]^* = \begin{pmatrix} -2 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}$

Claim $\mathcal{T}^k$ is a vector space

Also, $\mathcal{T}^k \times \mathcal{T}^l \to \mathcal{T}^{k+l}$ via $(T_1, T_2) \longmapsto T_1 \otimes T_2$

$\mathcal{T} = \bigoplus_k \mathcal{T}^k$ is a graded non-commutative algebra w/ unit

---

Notation $\underline{n} = \{1, \dots, n\}$ $J = I \in \underline{n}^k$ means $I = J = (j_1, \dots, j_k)$

If $v_j \in V$, $V_I = (v_{j_1} \dots v_{j_k}) \in V^k$

If $\varphi_j \in V^*$, $\varphi_I = \varphi_{j_1} \otimes \dots \otimes \varphi_{j_k} \in \mathcal{T}^k(V)$

Thm V w/ basis $v_1 \dots v_n$; $\varphi_1 \dots \varphi_n$ the dual basis

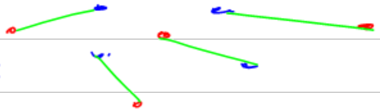
Then $\{\varphi_I : I \in \underline{n}^k\}$ is a basis of $\mathcal{T}^k(V)$.

Hence $\dim \mathcal{T}^k(V) = n^k$.

1. $T_1 = T_2$ in $\mathcal{T}^k(V)$ iff $\forall I$ $T_1(V_I) = T_2(V_I)$.   2. $\varphi_I(V_J) = \delta_{IJ}$.

3. span Given $T \in \mathcal{T}^k(V)$, we want $T = \sum_{I \in \underline{n}^k} a_I \varphi_I$ 4. LI.

Hour 43 MAT257 Analysis II on January 24, 2022: k-tensors (2).
Read Along: Spivak 75-78.
Riddle Along: n red dots and n blue dots are placed in the plane with no 3 on the same line. Prove that it is possible to pair them up using n straight line segments so that no two of the segments will intersect.



Def $\mathcal{T}^k V =$ "k-tensors on V" $= \left\{ \begin{matrix} k\text{-linear maps} \\ V^k \to \mathbb{R} \end{matrix} \right\}$. A v.s!

(Elsewhere $\mathcal{T}^k(V^*)$)

$\mathcal{T}^0 V = \mathbb{R}$ $\quad \mathcal{T}^1 V = V^*$ $\quad \langle , \rangle \in \mathcal{T}^2 V$ $\quad \det|_{M_{n\times n}} \in \mathcal{T}^n V$

$(v_1 \dots v_n)$ basis of $V \Rightarrow \exists!\ (\varphi_1 \dots \varphi_n)$ basis of V^*, w/ $\varphi_i(v_j) = \delta_{ij}$

$$\mathcal{T}^k \times \mathcal{T}^l \longrightarrow \mathcal{T}^{k+l} \quad (T_1, T_2) \longmapsto T_1 \otimes T_2 = T_1 T_2$$

$$(T_1 T_2)(u_1 \dots u_{k+l}) = T_1(u_1 \dots u_k)\, T_2(u_{k+1}, \dots u_{k+l})$$

$\otimes$ is associative, distributive, yet non-commutative. So $\mathcal{T} = \bigoplus_k \mathcal{T}^k$ is a graded non-commutative algebra w/ unit

---

Notation $\underline{n} = \{1, \dots, n\}$ $\quad J = I \in \underline{n}^k$ means $I = J = (j_1, \dots, j_k)$

If $v_j \in V$, $\quad v_I = (v_{j_1} \dots v_{j_k}) \in V^k$

If $\varphi_j \in V^*$, $\quad \varphi_I = \varphi_{j_1} \otimes \dots \otimes \varphi_{j_k} \in \mathcal{T}^k(V)$

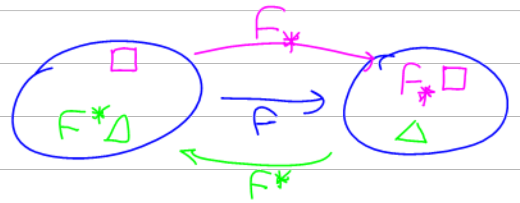
Thm V w/ basis $v_1 \dots v_n$; $\varphi_1 \dots \varphi_n$ the dual basis

Then $\{\varphi_I : I \in \underline{n}^k\}$ is a basis of $\mathcal{T}^k(V)$.

Hence $\dim \mathcal{T}^k(V) = n^k$.

1. $T_1 = T_2$ in $\mathcal{T}^k(V)$ iff $\forall I$ $T_1(v_I) = T_2(v_I)$. $\quad$ 2. $\varphi_I(v_J) = \delta_{IJ}$.

3. Span Given $T \in \mathcal{T}^k(V)$, we want $T = \sum_{I \in \underline{n}^k} a_I \varphi_I$ $\quad$ 4. LI.



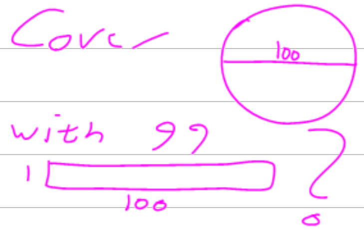
Given $F: V \to W$, $F^*: \mathcal{T}^k W \to \mathcal{T}^k V$

Linear, respects $\otimes$.

Example If T is an "inner-product",

Gram-Schmidt: $\exists F$ s.t. $F^* T = \langle\ \rangle$

Hour 44 MAT257 Analysis II on January 26, 2022: k-tensors (3).
Read Along: Spivak 75-78. TT2 returned!

Cover  with 99 ?

Notation $\underline{n} = \{1, \dots, n\}$

$J = I \in \underline{n}^k$ means $I = J = (j_1, \dots, j_k)$

If $v_j \in V$, $V_I = (v_{j_1} \dots v_{j_k}) \in V^k$

If $\varphi_j \in V^*$, $\varphi_I = \varphi_{j_1} \otimes \dots \otimes \varphi_{j_k} \in T^k(V)$

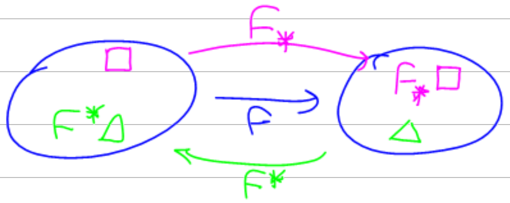
Thm V w/ basis $v_1 \dots v_n$; $\varphi_1 \dots \varphi_n$ the dual basis

Then $\{\varphi_I : I \in \underline{n}^k\}$ is a basis of $T^k(V)$.

Hence $\dim T^k(V) = n^k$.

Pf 1. $\varphi_I(V_J) = \delta_{IJ}$. 2. $T_1 = T_2$ in $T^k(V)$ iff $\forall I \; T_1(V_I) = T_2(V_I)$.

3. span Given $T \in T^k(V)$, $T = \sum_{I \in \underline{n}^k} a_I \varphi_I$ w/ $a_I = T(V_I)$. 4. LI ?



Given $F : V \to W$, $F^* : T^k W \to T^k V$

Linear, respects $\otimes$.

Example If T is an "inner-product",

Gram-Schmidt: $\exists F$ s.t. $F^* T = \langle \; \rangle$

} Skipped! move to HW?

Claim Alternating $\Leftrightarrow$ kills repetitions

Def $\Lambda^k(V)$ a sub-V.S. of $T^k(V)$ Warning: $\Lambda^k(V^*)$

Examples: Determinants, minors.

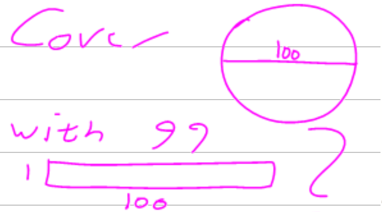
Permutations [make a graph S_k] signs [Axiomatics, strand diagrams, $\prod$ sign$(\sigma(j)-\sigma(i))$ det]

$T \in \Lambda^k \Leftrightarrow T \circ \sigma = (-1)^\sigma T$

Hour 45 MAT257 Analysis II on January 28, 2022: Alternating tensors.
Read Along: Spivak 78-85.
HW12 on web by midnight.

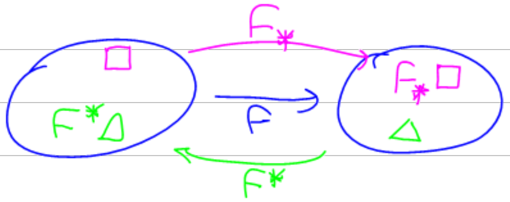
C+C = ?

Cover



V w/ basis $(v_i)_{i=1}^n$ & dual basis $(\varphi_i)_{i=1}^n$

Thm $\{\varphi_I\}_{I\in\underline{n}^k}$ is a basis of $\mathcal{T}^k V$.



Points, paths push
functions, sets, k-tensors pull } w/ good properties.

Def $T\in\mathcal{T}^kV$ "kills repetitions": $T(\ldots u,\ldots,u,\ldots)\equiv 0$.

Claim Alternating $\Longleftrightarrow$ kills repetitions

Def $\Lambda^k(V)$ ~ sub-v.s. of $\mathcal{T}^k(V)$ Warning: $\Lambda^k(V^*)$

Examples: Determinants, minors (λ_I, $I\in\underline{n}^k$, really $\underline{n}^k_a$)

$$\left|\underline{n}^k_a\right| = \binom{n}{k} \text{ so rename } \underline{n}^k_a \text{ to } \binom{\underline{n}}{k}.$$

Permutations [make a graph S_k] signs [Axiomatics, strand diagrams $\prod \operatorname{sign}(\sigma(j)-\sigma(i))$ det +]

$T\in\Lambda^k \Longleftrightarrow T\circ\sigma^* = (-1)^\sigma T$

Aside: Why the * here? Because really, it's a pullback.

$W_I = \sum_{\sigma\in S_k}\varphi_I\circ\sigma^*$
basis for $\Lambda^k(V)$
$\dim\Lambda^k(V)$ } by imitating $\mathcal{T}^k(V)$!

If time, count non-dec seqs.

$(\mathrm{Alt}\,T)(v_1\ldots v_k) := \frac{1}{k}\sum_{\sigma\in S_k}\frac{1}{k!}T(v_\sigma)$

Prop 1. $\mathrm{Im}\,\mathrm{Alt}\subset\Lambda^k$, so $\mathrm{Alt}:\mathcal{T}^k\to\Lambda^k$.
2. If $w\in\Lambda^k$, $\mathrm{Alt}(w)=w$
3. $\mathrm{Alt}\circ\mathrm{Alt}=\mathrm{Alt}$

$w\wedge\eta := \frac{(k+l)!}{k!\,l!}\mathrm{Alt}(w\otimes\eta)$

Bilinear, associative, super-commutative. "commutes" with pullbacks.

$\sigma \in S_k \quad \sigma : \underline{k} \to \underline{k}$

$\sigma^* : V^k \to \underbrace{V^k}_{\shortparallel}$

$\{\underline{k} \to V\}$

$$\begin{tikzcd} \underline{k} \arrow[r, "\sigma"] \arrow[dr, bend right, "u \circ \sigma"] & \underline{k} \arrow[d, "u"] \\ & V \end{tikzcd}$$

$(u_1 \ldots u_k) \mapsto (u_{\sigma 1} \ldots u_{\sigma k})$

$\sigma^* \circ \tau^*$

$= (\tau \circ \sigma)^*$

```
GraphicsGrid[Partition[Table[
   Graphics[Table[
     x = Sum[a⟦i⟧ 3^-i, {i, n}]; y = Sum[b⟦i⟧ 3^-i, {i, n}];
     Rectangle[{x, y}, {x, y} + 3^-n],
     {a, Tuples[{0, 2}, n]}, {b, Tuples[{0, 2}, n]}]],
   {n, 0, 5}], 3], Frame → All]
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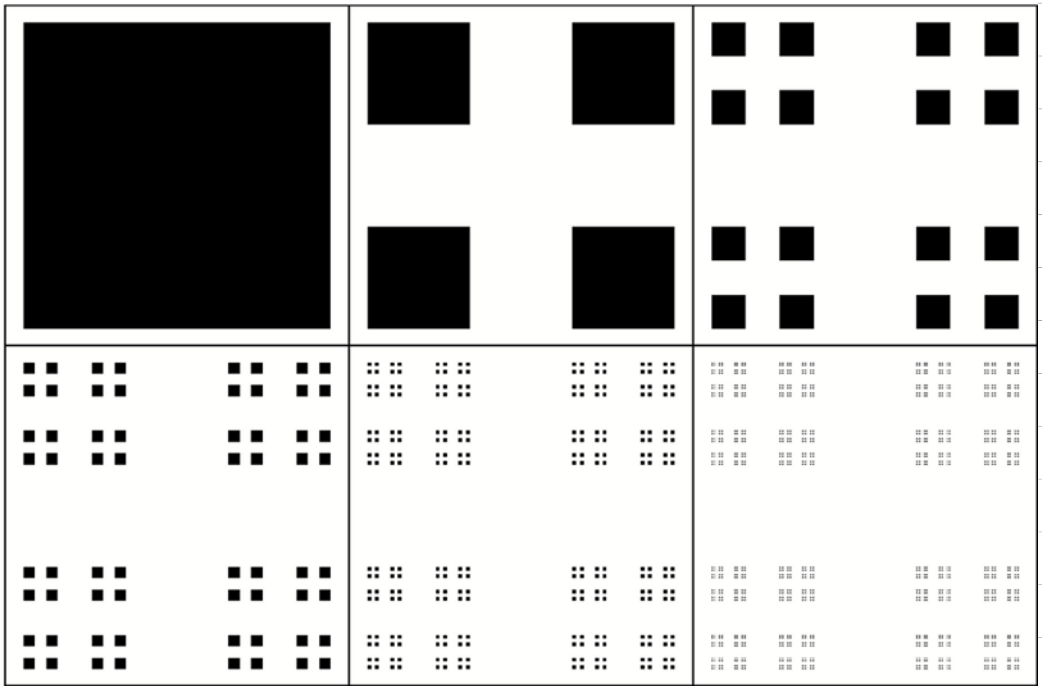


Hour 46 MAT257 Analysis II on January 31, 2022: Alternating tensors (2).
Read Along: Spivak 78-85.

$C+C=?$

Standing Assumption V w/ basis $(v_i)_{i=1}^n$ & dual basis $(\varphi_i)_{i=1}^n$

Thm $\{\varphi_I\}_{I\in\underline{n}^k}$ is a basis of T^kV

Claim Alternating $\iff$ kills repetitions

mention pullbacks

Def $\Lambda^k(V)$ ~ sub-v.s. of $T^k(V)$ warning: $\Lambda^k(V^*)$

$\underline{n}^k_\wedge = \binom{\underline{n}}{k} = \{(i_1,\dots,i_k)\in\underline{n}^k : i_1<\dots<i_k\}$ $\quad \left|\binom{\underline{n}}{k}\right| = \binom{n}{k}$

$S_k = \{\text{bijections } \sigma:\underline{k}\to\underline{k}\}$ A group! 1. Associative 2. Identity 3. Inverses

"permutations"

The "permutation group"

Non-commutative

signs [Axiomatics, strand diagrams $\quad \prod \operatorname{sign}(\sigma j-\sigma i)$, det]

$T\in\Lambda^k \iff T\circ\sigma^* = (-1)^\sigma T$

Aside: Why the * here? Because really, it's a pullback.

$\omega_I = \sum_{\sigma\in S_k}(-1)^\sigma\varphi_I\circ\sigma^*$ basis for $\Lambda^k(V)$

$\dim\Lambda^k(V)$

by imitating $T^k(V)$?!

$(\mathrm{Alt}\,T)(v_1\dots v_k) := \frac{1}{k}\sum_{\sigma\in S_k}\frac{1}{k!}T(v_\sigma)$

Prop 1. $\operatorname{Im}\mathrm{Alt}\subset\Lambda^k$, so $\mathrm{Alt}: T^k\to\Lambda^k$.
2. If $\omega\in\Lambda^k$, $\mathrm{Alt}(\omega)=\omega$
3. $\mathrm{Alt}\circ\mathrm{Alt}=\mathrm{Alt}$

$\omega\wedge\eta := \frac{(k+l)!}{k!\,l!}\mathrm{Alt}(\omega\otimes\eta)$

Bilinear, associative, super-commutative. "commutes" with pullbacks.

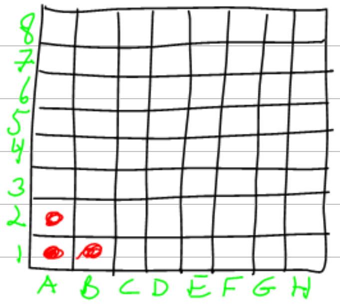
Aside $\underline{n}^k_{nd} = \{1\le i_1\le i_2\le i_3\le\dots\le i_k\le n\}$

$n=5 \quad k=7$: $1223555 \longleftrightarrow 1\boxed{*2**3*45***}$

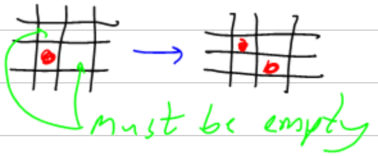
$$|\underline{n}^k_{nd}| = \binom{n+k-1}{k}$$

Chars in green box: $n+k-1$ of those k are $*$'s.

Hour 47 MAT257 Analysis II on 2/2/2022: Alternating tensors (3).
Read Along: Spivak 78-85.
Riddle Along: On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied. Can you clear the original 3 pawns?



A move:



$S_k = \{\text{bijections } \sigma : \underline{k} \longrightarrow \underline{k}\}$

$T \in \Lambda^k \iff T \circ \sigma^* = (-1)^\sigma T$ $\quad \text{Sign}(\sigma) = (-1)^\sigma = (-1)^{\#\text{transpositions to make } \sigma}$

$W_I = \sum_{\sigma \in S_k} (-1)^\sigma \varphi_I \circ \sigma^*$ $\quad \sigma^* : V^k \longrightarrow V^k$ "pullback"

$\sigma^*(v_1 \dots v_k) = (v_{\sigma 1} \dots v_{\sigma k})$

---

basis for $\Lambda^k(V)$, $\dim \Lambda^k(V)$ by immitating $\mathcal{T}^k(V)$

Aside $\quad \Omega^k_{nd} = \{1 \leq i_1 \leq i_2 \leq i_3 \leq \dots \leq i_k \leq n\}$

$n=5 \quad k=7$: $\quad 1223555 \longleftrightarrow 1\boxed{*2**3*45***}$

chars in green box: $n+k-1$ of those k are $*$'s.

$|\Omega^k_{nd}| = \binom{n+k-1}{k}$

Thm $\exists!$ $(\lambda, \eta) \longmapsto \lambda \wedge \eta$, $\Lambda^k \times \Lambda^l \to \Lambda^{k+l}$ s.t.

0. bilinear

2. Super-Commutative: $(\lambda \wedge \eta) = (-1)^{kl} \eta \wedge \lambda$

1. Assoc. $(\lambda \wedge \eta) \wedge \phi = \lambda \wedge (\eta \wedge \phi)$

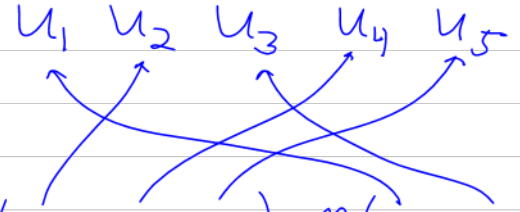
3. $W_I = \varphi_{i_1} \wedge \dots \wedge \varphi_{i_k}$

Existence set $(\lambda \wedge \eta)(u_1 \dots u_{k+l}) = \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma (\lambda \otimes \eta)(\sigma^*(u_1 \dots u_{k+l}))$

$= \frac{1}{k!\,l!} \sum_{\sigma \in S_{k+l}} (-1)^\sigma \lambda(u_{\sigma 1} \dots u_{\sigma k}) \eta(u_{\sigma(k+1)} \dots u_{\sigma(k+l)})$

$$= \sum_{\substack{\sigma \in S_{k+l} \\ \sigma 1 < \dots < \sigma k \\ \sigma(k+1) < \dots < \sigma(k+l)}} (-1)^\sigma \lambda(u_{\sigma 1} \dots u_{\sigma k}) \eta(u_{\sigma(k+1)} \dots u_{\sigma(k+l)})$$

$k=3$ $l=2$:

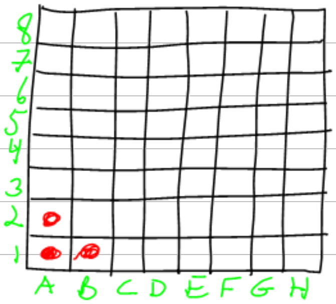


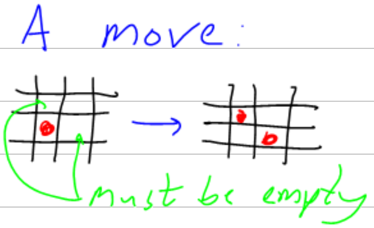
$\sum_{\text{"good"}} (-1)^\sigma \lambda(\_\ \_\ \_)\ \eta(\_\ \_)$

"σ splitting" $\quad \sigma = [24513]$

on to the proofs of $\in \Lambda^{k+l}$ & 1, 2, 3.

Hour 48 MAT257 Analysis II on Feb 4, 2022: Alternating tensors (4).
Starting on Monday we're back to in-person!!
HW12 due and HW13 on web by 11:59pm!
Read Along: Spivak 78-85.
Riddle Along: On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied. Can you clear the original 3 pawns?





Reminder $\{W_I=\sum_{\sigma\in S_k}(-1)^\sigma \varphi_I\circ\sigma^*\}_{I\in n_a^k}$ make a basis of $\Lambda^k V$, so $\dim=\binom{n}{k}$

Thm $\exists!$ $(\lambda,\eta)\longmapsto\lambda\wedge\eta$, $\Lambda^k\times\Lambda^l\to\Lambda^{k+l}$ s.t.

0. bilinear

2. Super-commutative: $(\lambda\wedge\eta)=(-1)^{kl}\eta\wedge\lambda$

1. Assoc. $(\lambda\wedge\eta)\wedge\phi=\lambda\wedge(\eta\wedge\phi)$

3. $W_I=\varphi_{i_1}\wedge\dots\wedge\varphi_{i_k}$

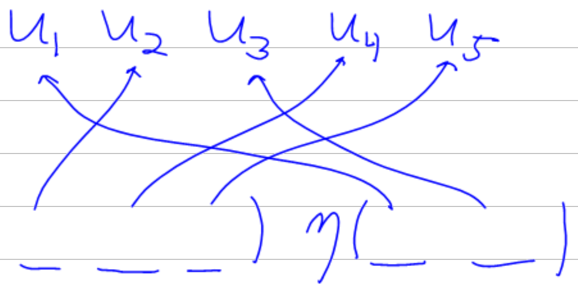
Existence set $(\lambda\wedge\eta)(u_1\dots u_{k+l})=\frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}(-1)^\sigma(\lambda\circ\eta)(\sigma^*(u_1\dots u_{k+l}))$

$$=\frac{1}{k!l!}\sum_{\sigma\in S_{k+l}}(-1)^\sigma\lambda(u_{\sigma 1}\dots u_{\sigma k})\eta(u_{\sigma(k+1)}\dots u_{\sigma(k+l)})$$

$$=\sum_{\substack{\sigma\in S_{k+l}\\ \sigma 1<\dots<\sigma k\\ \sigma(k+1)<\dots<\sigma(k+l)}}(-1)^\sigma\lambda(u_{\sigma 1}\dots u_{\sigma k})\,\eta(u_{\sigma(k+1)}\dots u_{\sigma(k+l)})$$

on to the proofs of $\in\Lambda^{k+l}$ & 1, 2, 3.

$k=3$ $l=2$:



$\sum_{\text{"good"}}(-1)^\sigma\lambda(\_\ \_\ \_)\,\eta(\_\ \_)$

"a splitting"

$\sigma=[24513]$

Pullbacks [compatible w/ all ops]

$\dim V=n$ $\Lambda^n(V)=\Lambda^{top}(V)=\{\text{volume elements}\}\ni\omega$

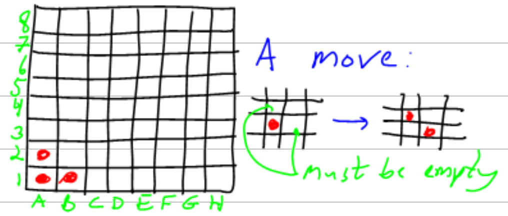
$L:V\to V\Rightarrow L^*\omega=(\det A)\omega$

Orientation 1. A basis, up to a positive-det C.O.B.

2. A volume form, up to a pos. scalar

Both pushes & pulls!

Hour 49 MAT257 Analysis II on Feb 7, 2022: Alternating tensors (5), orientations.
Read Along: Spivak 78-85.
Riddle Along: On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied. Can you clear the original 3 pawns?



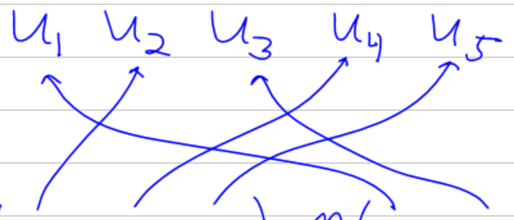
Thm $\exists! \; (\lambda,\eta) \longmapsto \lambda\wedge\eta, \quad \Lambda^k \times \Lambda^l \to \Lambda^{k+l}$ s.t.

0. bilinear 2. Super-commutative: $(\lambda\wedge\eta) = (-1)^{kl}\eta\wedge\lambda$

1. Assoc. $(\lambda\wedge\eta)\wedge\phi = \lambda\wedge(\eta\wedge\phi)$ 3. $\omega_I = \varphi_{i_1}\wedge \dots \wedge\varphi_{i_k}$

Existence Set $(\lambda\wedge\eta)(u_1 \dots u_{k+l}) = \frac{1}{k!\,l!}\sum_{\sigma\in S_{k+l}} (-1)^\sigma (\lambda\otimes\eta)(\sigma^*(u_1\dots u_{k+l}))$

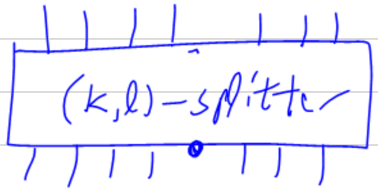
$$= \sum_{\substack{\sigma\in S_{k+l}\\ \sigma 1<\dots<\sigma k\\ \sigma(k+1)<\dots<\sigma(k+l)}} (-1)^\sigma \lambda(u_{\sigma 1}\dots u_{\sigma k})\,\eta(u_{\sigma(k+1)}\dots u_{\sigma(k+l)})$$



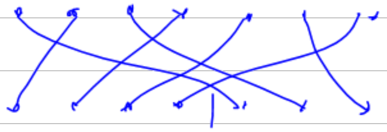
$k=3\;\; l=2$: $\sum_{\substack{\text{"good"}\\ \text{"}\sigma\text{ splitting"}}} (-1)^\sigma \lambda(\_\;\_\;\_)\,\eta(\_\;\_)$ $\sigma = [2\,4\,5\,1\,3]$

board line

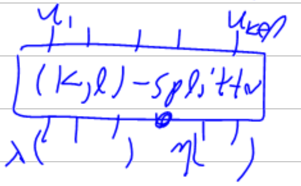
Pictorial proofs of $\in\Lambda^{k+l}$ & 0, 1, 2, 3.



$= \text{"}\sum_{\substack{\sigma\in S_{k+l}\\ \sigma 1<\dots<\sigma k\\ \sigma(l+1)<\dots<\sigma(l+k)}}\text{"} (-1)^\sigma$



So $\lambda\wedge\eta(u_1\dots u_{k+l}) =$



$\in$: trivial.

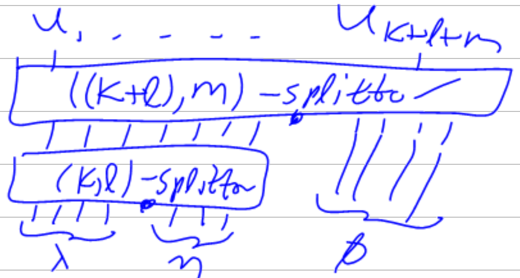
0 : easy

1:

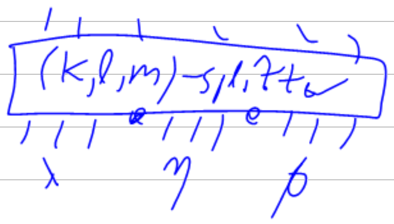
$((\lambda\wedge\eta)\wedge\phi)(u_1\dots u_{k+l+m}) =$

$\uparrow \;\; \uparrow \;\; \uparrow$

$\Lambda^k \;\; \Lambda^l \;\; \Lambda^m$

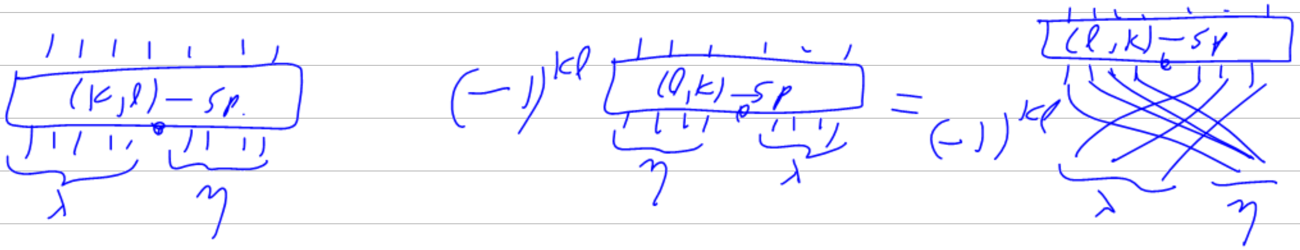


$=$



3. Easy.

2. Super-commutativity: $\lambda\wedge\eta=(-1)^{kl}\eta\wedge\lambda$



Pullbacks [compatible w/ all ops]

dim V=n $\Lambda^n(V)=\Lambda^{top}(V)=\left\{\begin{matrix}\text{Volume}\\\text{elements}\end{matrix}\right\}\ni\omega$

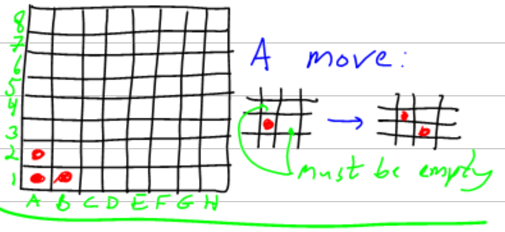
$L:V\longrightarrow V \Rightarrow L^*\omega=(\det A)\cdot\omega$

Orientation 1. A basis, up to a positive-det C.O.B.

2. A volume form, up to a pos. scalar

Both pushes & pulls!

Hour 50 MAT257 Analysis II on Feb 9, 2022: orientations.
Read Along: Spivak 78-85.
Riddle Along: On a chessboard, there are three pawns at the lower left (at A1, A2, and B1). On each move, pick up one pawn, remove it and place one new pawn to the right and one new pawn above, but only if these squares are unoccupied. Can you clear the original 3 pawns?



dim $V=n$ $\quad \Lambda^n(V) = \Lambda^{top}(V) = \left\{\begin{matrix}\text{Volume}\\ \text{elements}\end{matrix}\right\} \ni W$

$L: V \to V \Rightarrow L^*W = (\det A)W$ [A: the matrix representing L]

board line

pf $L^*W = \delta(A)W$ where 1. δ is multilinear 2. known behavior over column ops

3. $\delta(I)=1$

Orientation 1. A volume form, up to a pos. scalar

2. A basis, up to a positive-det C.O.B

Equivalence! Both pushes & pulls!

$\mathbb{R}^n_p \sim T_p\mathbb{R}^n = \{(p,v)\} = \{v_p\}$

a v.s., inner product

Vector fields & component fnctns. (cont., diff'ble)

$F(p) = \sum F^i(p)\,(p, e_i)$

Sums, inner products.

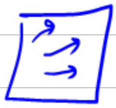
Diff. forms:

$$\lambda(p) = \sum_{I \in n^k_\wedge} \lambda_I(p)\, W_I(p)$$

Pushing & pulling.

Hour 51 MAT257 Analysis II on Feb 11, 2022: Vector Fields and Differential Forms
Read Along: Spivak 86-92.
HW13 due and HW14 on web by 11:59pm!
Riddle Along: A unit cube in R^3, the area of its projection on any plane is equal to the length of its projection on a perpendicular line to that plane.

$\int_M d\omega = \int_{\partial M} \omega$

Today: , , $\Omega^k(\mathbb{R}^n)$

The tangent space $\mathbb{R}^n_p \sim T_p\mathbb{R}^n := \{\overbrace{(p,v)}^{\xi} : v \in \mathbb{R}^n\} = \{v_p\}$ ← A tangent vector

a v.s., inner product, pushes!

Direction derivatives $D_{(p,v)}$ "bi"linear, Leibnitz.

compatibility w/ push, pull.

$F : \mathbb{R}^n \longrightarrow \bigcup_{p \in \mathbb{R}^n} T_p\mathbb{R}^n$ s.t. $F(p) \in T_pV$ is a "Vector field"

Can add, scale, inner-multiply, but not push or pull.

$$F(p) = \sum F^i(p)\,(p, e_i)$$

$$D_F = \sum F^i(p)\frac{\partial}{\partial x_i} = \sum F^i(p)\,\partial_i$$

E.g. $F_1(\overset{\overset{(x,y)\in\mathbb{R}^n}{\parallel}}{p}) = x\partial_x + y\partial_y$ $\quad F_2(p) = -y\partial_x + x\partial_y$

Diff. forms, $\lambda(p) = \sum_{I \in \Omega^k_n} \lambda_I(p)\,\omega_I(p)$

$+, \cdot, \wedge$, Pushing & pulling, compatibilities.

$\Omega^k(\mathbb{R}^n)$: Cont. diff'able k-forms.

$d : \Omega^0(\mathbb{R}^n) \longrightarrow \Omega^1(\mathbb{R}^n)$

$dx^i \qquad dF$

Compatibility with pull-backs.

Hour 52 MAT257 Analysis II on Feb 14, 2022: Differential Forms
Read Along: Spivak 86-92.
Riddle Along: On Z x Z, a visible roach R starts at (0, 0) and once a minute jumps to the northeast, up to a distance of 10. Meanwhile, an exterminator E can poison one grid point per minute, away from R. Can E trap R?

$$\int_M d\omega = \int_{\partial M} \omega$$

The tangent space $\mathbb{R}^n_p \sim T_p\mathbb{R}^n := \{\overbrace{(p,v)}^{\xi} : v \in \mathbb{R}^n\} = \{v_p\}$ ← A tangent vector

$\wedge$ v.s., $\langle\, , \rangle$, pushes!

Directional derivatives D_ξ: "bi"linear, Leibnitz.

$F : \mathbb{R}^n \to \bigcup_{p\in\mathbb{R}^n} T_p\mathbb{R}^n$ s.t. $F(p) \in T_pV$ is a "Vector field"

Can add, scale, inner-multiply, Also, $F : \mathbb{R}^n \to \mathbb{R}^n$, but...

$$F(p) = \sum F^i(p)\,(p, e_i) = \sum F^i(p)\frac{\partial}{\partial x_i} = \sum F^i(p)\partial_i$$

E.g. $F_1(\overset{\overset{(x,y)\in\mathbb{R}^2}{\parallel}}{P}) = x\partial_x + y\partial_y$ $\qquad F_2(P) = -y\partial_x + x\partial_y$

$\exists D_F$; do not push or pull!

Diff. forms, $\lambda(p) = \sum_{I \in \Delta^k_\wedge} \lambda_I(p)\,\omega_I(p)$

$+, \cdot, \wedge$, Pushing & pulling, compatibilities.

$\Omega^k(\mathbb{R}^n)$: Cont. diff'able k-forms.

$d : \Omega^0(\mathbb{R}^n) \longrightarrow \Omega^1(\mathbb{R}^n)$ $\quad dx^i \quad dF$

Compatibility with pull-backs. E.g. $\phi : \mathbb{R}^2_{r,\theta} \to \mathbb{R}^2_{x,y}$ by $\binom{r}{\theta} \mapsto \binom{r\cos\theta}{r\sin\theta}$ compute $\phi^*(dx \wedge dy)$.

Define $d : \Omega^k \to \Omega^{k+1}$ by $d\omega = \sum_{i=1}^n dx_i \wedge \frac{\partial\omega}{\partial x_i}$

Properties 1. linear 2. Leibnitz 3. $d^2 = 0$

4. pullbacks

Hour 53 MAT257 Analysis II on Feb 16, 2022: d
Read Along: Spivak 86-92.
Riddle Along: What with the roach?
Can you write the function 1+xy+(xy)^2 as a sum f(x)g(y)+h(x)k(y) of two products of functions that depend on just one variable?

$$\int_M d\omega = \int_{\partial M} \omega$$

$\Omega^k(\mathbb{R}^n)$: smooth(?) k-forms.

$\cup$

$\lambda(p) = \sum_{I\in\Omega^k_\wedge} \lambda_I(p)\, \omega_I(p)$ +, ·, ∧

Aside.

$\Omega^0(\mathbb{R}^n) = C^\infty(\mathbb{R}^n)$

$f\wedge\omega = f\cdot\omega$

If $g:\mathbb{R}^n\to\mathbb{R}^m$, $g^*:\Omega^k(\mathbb{R}^m)\to\Omega^k(\mathbb{R}^n)$ by

$$(g^*\omega)(p)(\xi_1\ldots\xi) = \omega(g_*p)(g_*\xi_1,\ldots g_*\xi_k)$$

Compatible w/ +, ·, ∧; contra-variant.

---

proof of compatibility w/ ∧.

$d:\Omega^0(\mathbb{R}^n)\to\Omega^1(\mathbb{R}^n)$ dx^i [hence $\omega_I = dx_I$]

$df = \sum \frac{\partial f}{\partial x_i}dx_i$

Compatibility with pull-backs. E.g. $\phi:\mathbb{R}^2_{r,\theta}\to\mathbb{R}^2_{x,y}$ by $\begin{pmatrix}r\\ \theta\end{pmatrix}\mapsto\begin{pmatrix}r\cos\theta\\ r\sin\theta\end{pmatrix}$ compute $\phi^*(dx\wedge dy)$.

Define $d:\Omega^k\to\Omega^{k+1}$ by $d\omega = \sum_{i=1}^n dx_i\wedge\frac{\partial\omega}{\partial x_i}$

Properties 1. linear 2. Leibnitz 3. $d^2=0$
4. pullbacks

TA meeting notes.

$$\mathbb{R}^2_{r,\theta} \xrightarrow[\binom{r}{\theta}\mapsto\binom{r\cos\theta}{r\sin\theta}]{g} \mathbb{R}^2_{x,y}$$

$$\omega = dx\wedge dy \in \Omega^2(\mathbb{R}^2_{x,y})$$

$$df = \sum \frac{\partial f}{\partial x_i} dx_i$$

Compute $g^*\omega = g^*(dx\wedge dy)$

$$= (g^*dx)\wedge(g^*dy)$$

$$= d(g^*x)\wedge d(g^*y)$$

$$(\underset{k}{\omega}\wedge\underset{\ell}{\eta}) = (-1)^{k\ell}\eta\wedge\omega$$

$$= d(r\cos\theta)\wedge d(r\sin\theta)$$

$$= \left(\frac{\partial r\cos\theta}{\partial r}dr + \frac{\partial r\cos\theta}{\partial\theta}d\theta\right)\wedge(\ . \ . \ . \ )$$

$$= (\underline{\cos\theta}\, dr - r\,\underline{\sin\theta\, d\theta})\wedge(\sin\underline{\theta\, dr} + r\cos\underline{\theta\, d\theta})$$

$$= \cos\theta\cdot r\cos\theta\, dr\wedge d\theta - r\sin\theta\,\underline{d\theta}\wedge\sin\theta\,\underline{dr}$$

$$= (r\cos^2\theta + r\sin^2\theta)dr\wedge d\theta = r\,dr\wedge d\theta$$

Hour 54 MAT257 Analysis II on Feb 18, 2022: d (2)
HW14 due, HW15 on web by 11:59pm! Read Along: Spivak 86-92.
Next week is break. Have fun! After the break classes will no longer be taped(') and tutorials will be in person!
Riddle Along: What with the roach? And the bifactorization? Also,
Players A writes the numbers 1-18 on the faces of three blank dice. Players B chooses one of the 3, players A chooses one of the remaining 2 and trashes the third. They then play "dice war" 1000 times, on money.
Whom would you rather be, A or B?

$$\int_M d\omega = \int_{\partial M} \omega$$

$d: \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ by $(df)(\xi) = D_\xi f$

$dx_i = \varphi_i = \omega_{(i)}$ $\quad dx_I := dx_{i_1} \wedge \dots \wedge dx_{i_k} = \omega_I$ $\quad df = \sum_{i=1}^n \frac{\partial f}{\partial x_i} dx_i$

$g: \mathbb{R}^2_{r,\theta} \longrightarrow \mathbb{R}^2_{x,y}$ by $\binom{x}{y} = g\binom{r}{\theta} = \binom{r\cos\theta}{r\sin\theta}$. $\quad g^*(dx \wedge dy) = ?$

Ans: $g^*(dx \wedge dy) = d(g^*x) \wedge d(g^*y) = d(r\cos\theta) \wedge d(r\sin\theta)$

$= (\cos\theta\, dr - r\sin\theta\, d\theta) \wedge (\sin\theta\, dr + r\cos\theta\, d\theta) = \dots$

Define $d: \Omega^k \to \Omega^{k+1}$ by $d\omega = \sum_{i=1}^n dx_i \wedge \frac{\partial \omega}{\partial x_i}$

Properties 1. linear 2. Leibnitz 3. $d^2 = 0$

im ⊂ ker
exact is closed.

4. pullbacks

Thm $(d\omega)(\xi_1, \dots, \xi_{k+1}) =$ where $\xi_i = (p, v_i)$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \Big[\omega(p + \epsilon v_i)(\epsilon v_1, \dots, \widehat{\epsilon v_i}, \dots, v_{k+1}) - \omega(p)(\epsilon v_1 \dots \widehat{\epsilon v_i}, \dots, v_{k+1}) \Big)$$

@$k=0$: ....

geom meaning, relation w/ $\int_C d\omega = \int_{\partial C} \omega$.

Proof Enough to take $\omega = f \cdot \lambda$, where λ has constant coeffs. Then....

Hour 55 MAT257 Analysis II on Feb 28, 2022: d (3)
Read Along: Spivak 86-92.
Riddle Along: What with the roach? And the bifactorization? And the two players?
Also, the game of 15.

$$\int_M d\omega = \int_{\partial M} \omega$$

$d^{old}: \Omega^0(\mathbb{R}^n) \to \Omega^1(\mathbb{R}^n)$ by $(d^{old} f)(\xi) = D_\xi f$

$d: \Omega^k \to \Omega^{k+1}$ by $d\omega = \sum_{i=1}^{n} dx_i \wedge \frac{\partial \omega}{\partial x_i}$

Properties 0. At $k=0$, $d^{old}=d$.

1. linear $\square$ 2. Leibnitz: $d(\omega\wedge\eta) = d\omega\wedge\eta + (-1)^{\omega}\omega\wedge d\eta$

3. $d^2=0$ 4. $d(g^*\omega) = g^* d\omega$

Thm $(d\omega)(\xi_1,\dots,\xi_{k+1}) =$ where $\xi_i = (p, v_i)$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon^{k+1}}\sum_{i=1}^{k+1}(-1)^{i-1}\Big[\omega(p+\epsilon v_i)(\epsilon v_1,\dots,\widehat{\epsilon v_i},\dots,v_{k+1})$$
$$-\omega(p)(\epsilon v_1,\dots,\widehat{\epsilon v_i},\dots,v_{k+1})$$

@ $k=0$: ....

geom meaning, relation w/ $\int_C d\omega = \int_{\partial C} \omega$.

Proof Enough to take $\omega = f\cdot\lambda$, where λ has constant coeffs. Then....

Singular k-cube in $A\subset\mathbb{R}^n$: Cont. $c:[0,1]^k\to A$.
(0-cubes, 1-cubes)

(The space of k-chains in A) := The free Abelian group generated by all k-cubes $= \left\{\sum_{i=1}^{m} a_i c_i\right\}$

1. order immaterial
2. $0\cdot c_i$ can be omitted
3. repeating cubes can be merged"

"shopping lists"
"inventories"

Can add! $(2c_1+3c_2)+(c_4+c_1-3c_2)= \dots$

Can multiply by integers! $(0,t)+(t,0)\neq(t,t)$.

Has inverses! (Has "0")

Hour 56 MAT257 Analysis II on March 2, 2022: The geometric meaning of d, chains.
Read Along: Spivak 91-110 (but skip Poincare's lemma).
TT3 is in less than a week, on Tuesday March 8 at 5pm at EX320.
Material: everything to and including Friday's class. Format: Like TT1, solve n of n, a mix of class/HW/fresh.
Preparation: Fully understand *everything*. More by email later today.

$$\int_{\cancel{M}C} d\omega = \int_{\partial \cancel{M}C} \omega$$

$d : \Omega^k \to \Omega^{k+1}$ by $d\omega = \sum_{i=1}^{n} dx_i \wedge \frac{\partial \omega}{\partial x_i}$

Thm $(d\omega)(\xi_1, \dots, \xi_{k+1}) =$ where $\xi_i = (p, v_i)$

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{k+1}} \sum_{i=1}^{k+1} (-1)^{i-1} \Big[\omega(p+\epsilon v_i)(\epsilon v_1, \dots \widehat{\epsilon v_i}, \dots v_{k+1})$$
$$- \omega(p)(\epsilon v_1, \dots \widehat{\epsilon v_i}, \dots, v_{k+1})$$

@ $k=0$: . . . .

geom meaning, relation w/ $\int_C d\omega = \int_{\partial C} \omega$.

Proof Enough to take $\omega = f \cdot \lambda$, where λ has constant coeffs. Then . . . .

Singular k-cube in $A \subset \mathbb{R}^n$: Cont. $c : [0,1]^k \to A$.
(0-cubes, 1-cubes)

$C_k(A) =$ (The space of k-chains in A) := The free Abelian group generated by all k-cubes $= \left\{ \sum_{i=1}^{m} a_i c_i \right\}$

1. order immaterial
2. $0c_i$ can be omitted
3. repeating cubes can be merged"

"shopping lists"
"inventories"

Can add! $(2c_1 + 3c_2) + (c_4 + c_1 - 3c_2) = \dots$

Can multiply by integers! $(0,t)+(t,0) \neq (t,t)$.

Has inverses! (Has "0")

Hour 57 MAT257 Analysis II on March 4, 2022: chains and boundaries.
Read Along: Spivak 91-110 (but skip Poincare's lemma).
TT3 on Tuesday! *Truly understand everything*

$$\int_{\cancel{M}\,c} dw = \int_{\partial \cancel{M}\,c} w$$

Thm $(dw)(\epsilon\xi_1,\dots\epsilon\xi_{k+1}) \sim$ where $\xi_j=(p,v_j)$

$$\sum_{j=1}^{k+1}(-1)^{j-1}\left[\begin{array}{l} w(p+\epsilon v_j)(\epsilon v_1,\dots\widehat{\epsilon v_j},\dots v_{k+1}) \\ -w(p)(\epsilon v_1,\dots\widehat{\epsilon v_j},\dots,v_{k+1}) \end{array}\right]$$

Singular k-cube in $A\subset\mathbb{R}^n$: Cont. $c:[0,1]^k\to A$.

$$C_k(A)=\left(\begin{array}{c}\text{The space}\\ \text{of } k\text{-chains}\\ \text{in } A\end{array}\right) := \begin{array}{c}\text{The free Abelian}\\ \text{group generated}\\ \text{by all } k\text{-cubes}\end{array} = \left\{\sum_{j=1}^{m} a_j c_j\right\}$$

1. order immaterial
2. $0\cdot c_j$ can be omitted
3. repeating cubes can be "merged"

"shopping lists"
"inventories"

Can add! $(2c_1+3c_2)+(c_4+c_1-3c_2)=\dots\dots$

Can multiply by integers!

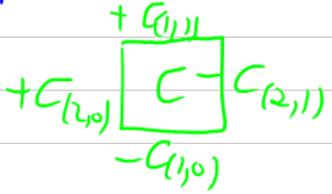
Has inverses! (Has "0")

Example: compute $(0,t)+(t,0)-(t,t)$

$I^k_{(j,\alpha)}: I^{k-1}_{y_1..y_{k-1}} \longrightarrow I^k$ $\quad c_{(j,\alpha)} := c\circ I^k_{(j,\alpha)}$ "face"

$$\partial c := \sum_{j=1}^{k}(-1)^{j+\alpha}c_{(j,\alpha)}$$ Extends to k-chains!

Thm $\partial^2=0$ Pf skip, but draw pictures.



Push or pull? Compatibility w/ ∂.

singular C^1 cubes/chains, C^1_k, from now: this is the standing assumption.

Hour 58 MAT257 Analysis II on March 7, 2022: Integration on chains and Stokes' theorem.
Read Along: Spivak 97-108.
TT3 on tomorrow! *Truly understand everything*

$$\int_c d\omega = \int_{\partial c} \omega$$

$$C^1_k(A) = \begin{pmatrix} \text{cont. diffable singular} \\ k\text{-chains in } A\subset\mathbb{R}^n \end{pmatrix} = \left\{ \sum \alpha_i c_i \;:\; \begin{matrix} \alpha_i \in \mathbb{Z} \\ c_i : I^k \underset{\text{cont. diffable}}{\longrightarrow} A \end{matrix} \right\} \Big/ \sim$$

$$I^k_{(j,\alpha)} : I^{k-1}_{y_1 \dots y_{k-1}} \longrightarrow I^k \qquad (y_1 \dots y_{k-1}) \longmapsto (y_1 \dots y_{j-1}, \underset{j}{\alpha}, y_j \dots y_{k-1})$$

$$\partial c = \sum_{j=1}^{k} \sum_{\alpha\in\{0,1\}} (-1)^{j+\alpha} \underbrace{c \circ I^k_{(j,\alpha)}}_{\text{"faces"}}$$

Extends to k-chains!

Thm $\partial^2 = 0$

Push or pull? Compatibility w/ ∂.

$\int_{I^k} f\, dx_1 \wedge \dots \wedge dx_k$ No examples!

$A \subset \mathbb{R}^n$, $c \in C_k(A)$, $\omega \in \Omega^k(A)$, $\int_c \omega$

example 1 $c = \begin{pmatrix} \cos 2\pi\theta \\ \sin 2\pi\theta \end{pmatrix}$ $\omega = \frac{-y\,dx}{x^2+y^2} + \frac{x\,dy}{x^2+y^2}$

example 0 $c = p$, $\omega = f$ $\int_p f\omega = f(p)$

prop. Suppose $c : I^2 \to \mathbb{R}^2_{x,y}$ is 1-1 cube w/ $\det(c') > 0$ & $A = c(I^2)$. Then $\int_c dx\wedge dy = \dots$

Prop Given $c : I^k \to A \subset \mathbb{R}^n$, $\omega \in \Omega^k(A)$ and $r : I^k \to I^k$ 1-1, onto, w/ $\det r' > 0$, $\int_c \omega = \int_{c\circ r} \omega$

Compatibility of push forwards & pull backs.

Hour 59

Bring a ball!

$\mathbb{R}^3$ $\quad \Omega^0 \quad \Omega^1 \xrightarrow{d} \Omega^2 \quad \Omega^3$

$f \quad F \quad G \quad g$

$F_1 dx_1 + \ldots \qquad G_1 dx_2 \wedge dx_3$

$$\operatorname{curl} F = \begin{pmatrix} \partial_2 F_3 - \partial_3 F_2 \\ \vdots \end{pmatrix}$$

---

geom meaning. div

curl

grad.

Gauss $\displaystyle\int_A \operatorname{div} G = \int_{\partial A} (G \cdot n)\, dA$

Stokes $\displaystyle\int_S (\operatorname{curl} F)\cdot n\, dA = \int_{\partial S} (F \cdot T)\, dl$

F.T. of calc.

Green A $\displaystyle\int_{\partial D} F \cdot n = \int_D \partial_1 F_1 + \partial_2 F_2$

Green B $\displaystyle\int_{\partial D} F \cdot T = \int_D \partial_1 F_2 - \partial_2 F_1$

TA Meeting

$$\begin{tikzcd}
\begin{matrix} x_1 \\ \vdots \\ x_n \end{matrix} \arrow[r, "g_1"'] \arrow[rr, bend left] & \begin{matrix} x_1 \\ \vdots \\ x_{k-1} \\ y_k \\ x_{k+1} \\ \vdots \\ x_n \end{matrix} \arrow[r] \arrow[l, bend right] & \begin{matrix} y_1 \\ \vdots \\ y_n \end{matrix}
\end{tikzcd}$$

$$\begin{pmatrix} 1 & & & & & & \\ & 1 & & & & & \\ & & 1 & & & & \\ \partial_1 f_k & \cdots & \cdots & \boxed{\partial_k f_k} & \cdots & \partial_n f_k \\ & & & & 1 & & \\ & & & & & 1 \end{pmatrix}$$

$$\partial_k f_k \neq 0.$$

$$\exists i \quad \frac{\partial f_i}{\partial x_i} \neq 0.$$

Hour 60 MAT257 Analysis II on March 11, 2022: Stokes' theorem for chains.
Read Along: Spivak 97-108.
No news on TT3. HW16 on web by midnight.
Our final was just scheduled: Wed Apr 27 7-10pm @EX200 :(
I'll be in Texas this weekend! See http://drorbn.net/waco22.

Riddle Along: A rectangle R has sides that are (both) not integers, and is tiled with rectangles R_i. Show that at least one of the R_i's has sides that are (both) not integer.

$$\int_c d\omega = \int_{\partial c} \omega$$

$I^k_{(j,\alpha)} : I^{k-1}_{y_1 \dots y_{k-1}} \longrightarrow I^k$ $\quad (y_1 \dots y_{k-1}) \mapsto (y_1 \dots y_{j-1}, \underbrace{\alpha}_{j}, y_j \dots y_{k-1})$

$$\partial c = \sum_{j=1}^{k} \sum_{\alpha\in\{0,1\}} (-1)^{j+\alpha}\, \underbrace{c \circ I^k_{(j,\alpha)}}_{\text{"faces"}}$$

chains push or pull?
Compatibility w/ ∂.

$\int_{I^k} f\, dx_1 \wedge \dots \wedge dx_k := \int_{I^k} f \qquad \int_c \omega := \int_{I^k} c^* \omega$

Prop 2 Given $c : I^k \to A \subset \mathbb{R}^n$, $\omega \in \Omega^k(A)$ and $r : I^k \to I^k$ 1-1, onto, w/ $\det r' > 0$, $\int_c \omega = \int_{c \circ r} \omega$

prop. 1 Suppose $c : I^k \to \mathbb{R}^k_{x_i}$ is 1-1 cube w/ $\det(c') > 0$ & $A = c(I^k)$. Then $\int_c f\, dx_1 \wedge \dots \wedge dx_k = \int_A f$

Compatibility of push forwards & pull backs.

Thm $c \in C_k(A \subset \mathbb{R}^n)$, $\omega \in \Omega^{k-1}(A) \Rightarrow \int_c d\omega = \int_{\partial c} \omega$

Pf WLOG, $c : I^k \to A$ is a single cube.

$$\int_c d\omega = \int_{I^k} c^*(d\omega) = \int_{I^k} d(c^*\omega) \overset{?}{=} \int_{\partial I^k} c^*\omega = \int_{c_*\partial I^k} \omega = \dots$$

then do I^k, $\omega = f\, dx_1 \dots \widehat{dx_i} \dots dx_k =: f\, dx_{\text{no } i}$

(use y_i for the variables inside faces)

lhs = ... $\quad (-1)^{i-1}\left[\int_{\mathbb{R}^{k-1}_{y_1 \dots y_{k-1}}} f(y_1 \dots, \underset{i}{1}, \dots y_{k-1}) - \int f(\dots 0 \dots)\right]$

rhs = ...

Hour 61 MAT257 Analysis II on March 14, 2022: Proof of Stokes, manifolds.
Read Along: Spivak 109-115.
No news on TT3 :(.

$$\int_{4M} d\omega = \int_{\partial 4M} \omega$$

Riddle: How far can you tilt a column of n books, n v. large?

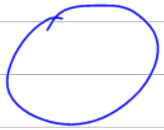
$$\partial c = \sum_{j=1}^{k} \sum_{\alpha\in\{0,1\}} (-1)^{j+\alpha} c \circ I^k_{(j,\alpha)}$$

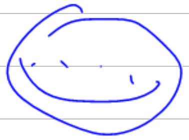
Thm $c \in C_k(A \subset \mathbb{R}^n)$, $\omega \in \Omega^{k-1}(A) \Rightarrow \int_c d\omega = \int_{\partial c} \omega$

NTS $\int_{I^k} d\omega = \int_{\partial I^k} \omega$

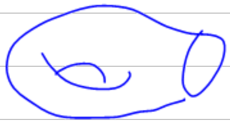
$$lhs = \dots \qquad = (-1)^{i-1}\left[\int_{\mathbb{R}^{k-1}_{y_1 \dots y_{k-1}}} f(y_1, \dots, \underset{i}{1}, \dots y_{k-1}) - \int f(\dots 0 \dots)\right]$$

$rhs = \dots$ Then manifolds.

Integration depends only on the images of the cubes. Can we throw them away? Integrate on  ?

 ?

 ?

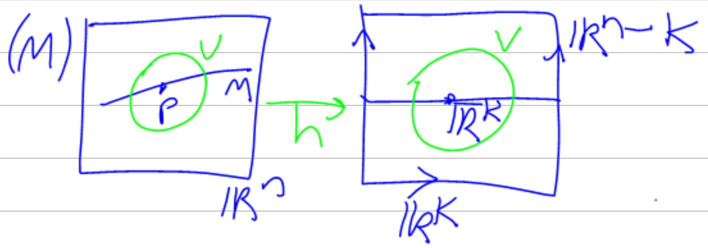
 ?

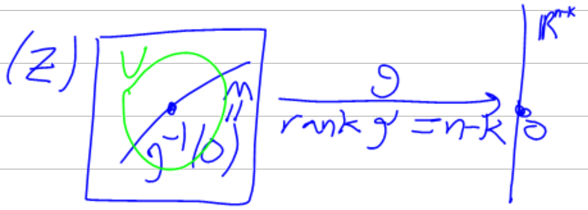
Thm Given $k \leq n$, $M \subset \mathbb{R}^n$, $x \in M$, TFAE:

(M) $\exists$ open $U \ni p$, open $V \subset \mathbb{R}^n_{\text{other}}$
manifold
& a diffeomorphism $h: U \to V$ s.t.
$h(U \cap M) = V \cap (\mathbb{R}^k \times \{0_{\mathbb{R}^{n-k}}\})$.

(Z) $\exists$ open $U \ni p$ & smooth $g: U \to \mathbb{R}^{n-k}$
zeros
s.t. $U \cap M = U \cap g^{-1}(0)$ & rank $(g') = n-k$

(C) $\exists$ open $U \ni p$, open $W \subset \mathbb{R}^k$ & smooth 1-1
coordinates.
$f: W \to \mathbb{R}^n$ s.t. ① $f(W) = M \cap U$
② $f^{-1}: M \cap V \to W$ is cont. ③ $\forall a \in W$, rank $f'(a) = k$.
why necessary?





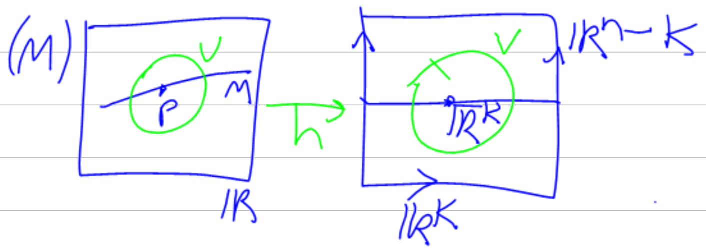
Def A k-manifold is a subset $M \subset \mathbb{R}^n$ s.t. at every $x \in M$, (MZC) holds.

Examples. $S^1 \subset \mathbb{R}^2$ in two ways. S^2 in $\mathbb{R}^3$.

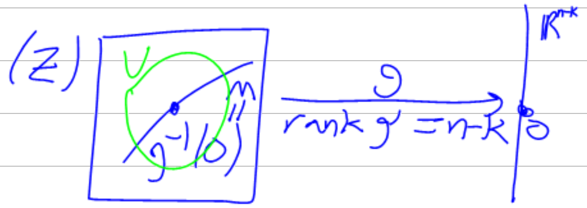
$T^2 =$  in $\mathbb{R}^3$

Hour 62 MAT257 Analysis II on March 16, 2022: M<->Z<->C.
Read Along: Spivak 109-115. TT3 results: Later today.

Thm Given $k\le n$, $M\subset\mathbb{R}^n$, $x\in M$, TFAE:



(M) ∃ open $U\ni p$, open $V\subset\mathbb{R}^n_{\text{other}}$
manifold
& a diffeomorphism $h: U\to V$ s.t.
$h(U\cap M)=V\cap(\mathbb{R}^k\times\{0_{\mathbb{R}^{n-k}}\})$.



(Z) ∃ open $U\ni p$ & smooth $g: U\to\mathbb{R}^{n-k}$
zeros
s.t. $U\cap M=U\cap g^{-1}(0)$ & rank$(g')=n-k$

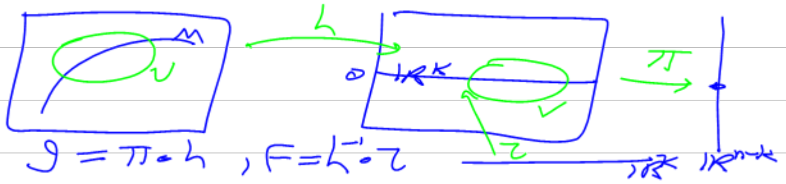
(C) ∃ open $U\ni p$, open $W\subset\mathbb{R}^k$ & smooth 1-1
coordinates
$f: W\to\mathbb{R}^n$ s.t. ① $f(W)=M\cap U$
② $f^{-1}: M\cap U\to W$ is cont. ③ $\forall a\in W$, rank $f'(a)=k$.
why necessary?

Def A k-manifold is a subset $M\subset\mathbb{R}^n$ s.t. at every $x\in M$, (MZC) holds.

Examples $S^1\subset\mathbb{R}^2$ in two ways. S^2 in $\mathbb{R}^3$. $T^2=$  in $\mathbb{R}^3$

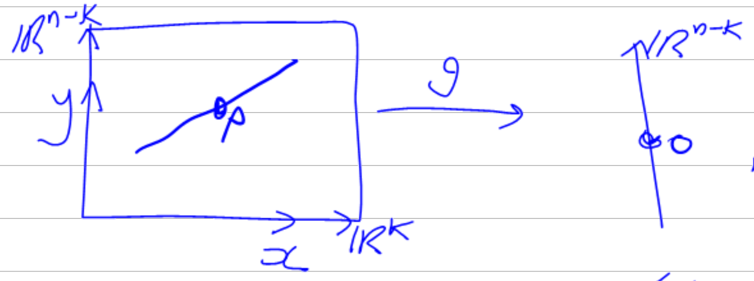
Pf of thm

$M\Rightarrow Z, C$: Trivial



$g=\pi\circ h$, $f=h^{-1}\circ\iota$

$Z\Rightarrow M$:

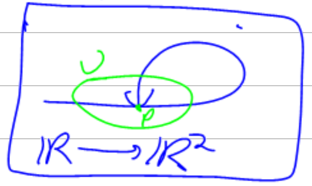


WLOG, rank $\frac{\partial g}{\partial y}=n-k$

set $h(x,y)=(x,g(x,y))$ $\quad h'(p)=\begin{pmatrix}I & 0\\ * & \frac{\partial g}{\partial y}\end{pmatrix}$

So use the IFT.

Aside In C, cond ② is necessary:

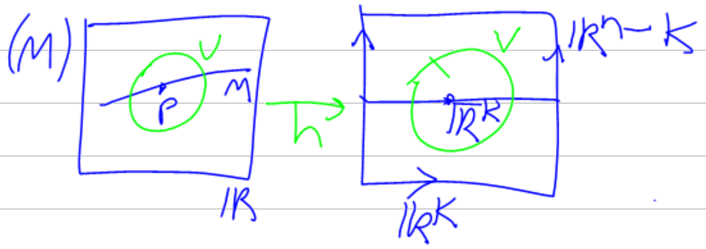


"counterexample 6"

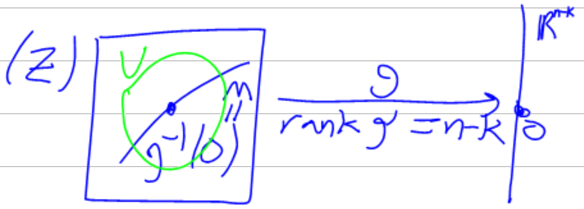
Hour 63 MAT257 Analysis II on March 18, 2022: M<->Z<->C (2).
Read Along: Spivak 109-115. HW 16 due, HW17 online by 11:59pm.

Thm Given $k\leq n$, $M\subset\mathbb{R}^n$, $x\in M$, TFAE:

(M) $\exists$ open $U\ni p$, open $V\subset\mathbb{R}^n_{\text{other}}$ & a diffeomorphism $h:U\to V$ s.t. $h(U\cap M)=V\cap(\mathbb{R}^k\times\{0_{\mathbb{R}^{n-k}}\})$.
manifold



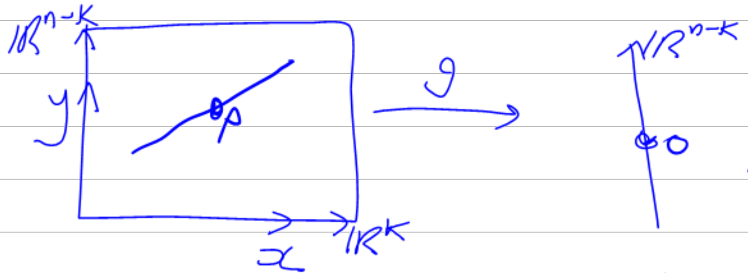
(Z) $\exists$ open $U\ni p$ & smooth $g:U\to\mathbb{R}^{n-k}$ s.t. $U\cap M=U\cap g^{-1}(0)$ & $\operatorname{rank}(g')=n-k$
zeros



(C) $\exists$ open $U\ni p$, open $W\subset\mathbb{R}^k$ & smooth 1-1 $f:W\to\mathbb{R}^n$ s.t. ① $f(W)=M\cap U$
coordinates
② $f^{-1}:M\cap U\to W$ is cont. ③ $\forall a\in W$, $\operatorname{rank}f'(a)=k$.
why necessary?

Def. A k-manifold.

$Z\Rightarrow M$:

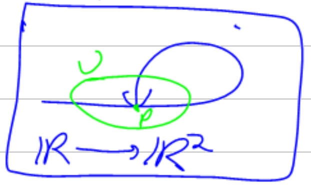


WLOG,
$\operatorname{rank}\frac{\partial g}{\partial y}=n-k$

set $h(x,y)=(x,g(x,y))$ $h'(p)=\begin{pmatrix} I & 0 \\ * & \frac{\partial g}{\partial y}\end{pmatrix}$

So use the IFT.

Aside In C, cond ② is necessary:
"counterexample 6"



(c) ⇒ (M) For convenience, assume $(\pi_1 \circ f)' = (\pi_{1-k} \circ f)'$ is rank k

Let $\ell : W_x^{\subset \mathbb{R}^k} \times \mathbb{R}_y^{n-k} \longrightarrow \mathbb{R}^n$ be $(x,y) \longmapsto f(x) + \begin{pmatrix} 0 \\ y \end{pmatrix}$

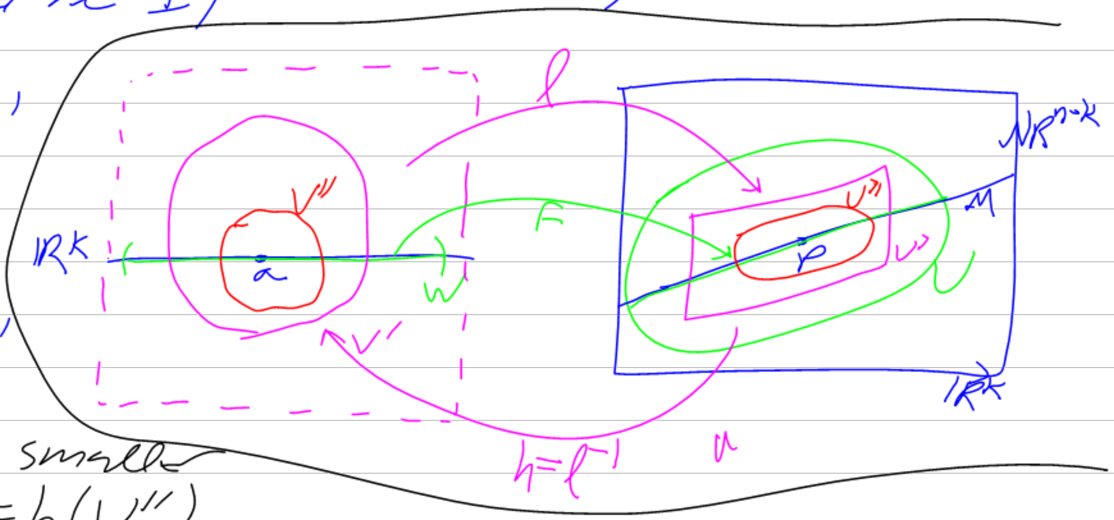
then $\ell' = \begin{pmatrix} \partial f_1/\partial x & 0 \\ \partial f_2/\partial x & I \end{pmatrix}$ is invertible, so so is

$\ell|_{V'} : V' \to U'$,

where $V' \subset W \times \mathbb{R}^{n-k}$

& $U' \subset U$

Let $h = \ell^{-1} : U' \to V'$



NTS. For an even smaller $U'' \subset U'$, w/ $V'' = h(U'')$

$h(U'' \cap M) = V'' \cap (\mathbb{R}^k \times \{0\}) \iff \ell(V'' \cap \mathbb{R}^k) = U'' \cap M \iff f(V'' \cap \mathbb{R}^k) = U'' \cap M$

Aside $\overset{\mathbb{R}^n}{\underset{}{}}$ $\overset{\mathbb{R}^m}{}$

$f : \overset{\cup\!\!\!\!\!\ \mathbb{R}^n}{A} \longrightarrow \overset{\mathbb{R}^m}{B}$ cont

$\iff \tilde{V} \subset \mathbb{R}^m$ open $\Rightarrow \exists$ open $\tilde{U}$ in $\mathbb{R}^m$ s.t. $f^{-1}(\tilde{V}) = A \cap \tilde{U}$

$f^{-1} : M \cap U \to W$ cont

$f : W \to M \cap U$ is open

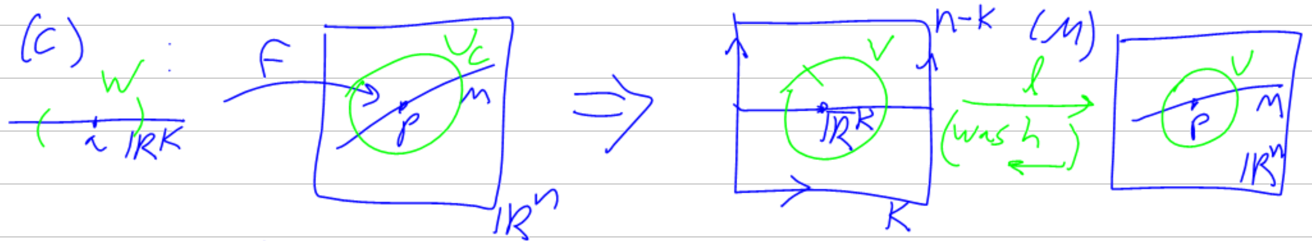
$f(V' \cap \mathbb{R}^k)$ is open in $M \cap U$

so it is $M \cap U''$ □

Hour 64 MAT257 Analysis II on March 21, 2022:
C->M, manifolds with boundary, TM..
Read Along: Spivak 109-122.

Can you hang a medal on 2 nails so that if you remove any one of them, the medal will fall? Same with 3

$\frac{1}{4}$ Thm (c)



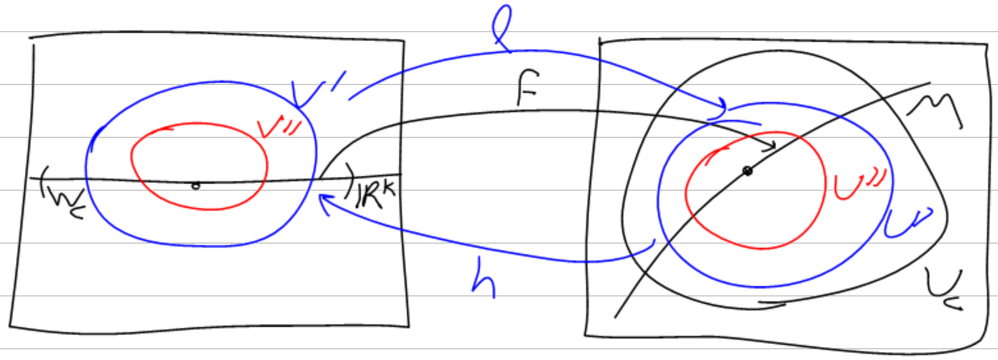
① $F(W)=M\cap U_c$ ② $F^{-1}: M\cap U_c \to W$ is cont.

$l(V\cap\mathbb{R}^k\times\{0\}) = U\cap M$

③ rank $F'(a)=k$

∃ diffeo $l: V'\subset W\times\mathbb{R}^{n-k}$
$\to U'\subset U$

N.T.S. For smaller open $U''\subset U'$, $V''\subset V'$,



$l(V''\cap\mathbb{R}^k)=U''\cap M \Longleftrightarrow F(V''\cap\mathbb{R}^k)=U''\cap M$

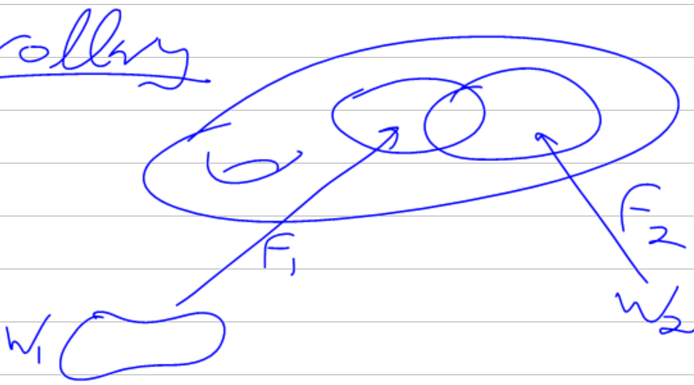
Aside $F: \underset{\mathbb{R}^n}{\overset{}{A}} \to \underset{}{B}$ cont ($A\subset\mathbb{R}^n$, $B\subset\mathbb{R}^m$)
$\Longleftrightarrow$ $\tilde{V}\subset\mathbb{R}^m$ open $\Rightarrow \exists$ open $\tilde{U}$ in $\mathbb{R}^n$ s.t. $F^{-1}(\tilde{V})=A\cap\tilde{U}$

$F^{-1}: M\cap U_c\to W$ cont
$F: W\to M\cap U_c$ is open
$F(V'\cap\mathbb{R}^k)$ is open in $M\cap U_c$
So it is $M\cap U''$ $\square$

Def If $M^k\subset\mathbb{R}^n$ is a mfld, the F's of (c) are called "coord. patches"

①,②,③ $\forall a\in W$ $F'(a)$ is 1-1.

Corollary



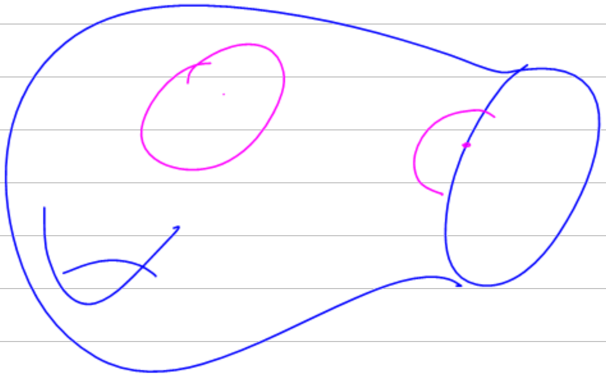
$f_2^{-1}\circ f_1 : f_1^{-1}f_2(W_2)\to \ldots$
is a diffeo.
(smooth w/ smooth inverse)
"transition functions"

Some philosophy on mflds as seen in 367

"An object is lots of partial views, along w/ the knowledge of how to move between them"

Manifold w/ bndry

$$\begin{tikzcd} \mathbb{R}^k_{x_k \geq 0} & \mathbb{R}^n \\ W \arrow[u, phantom, "\cup"] \arrow[r, "F"', "\text{1-1 \& smooth}^*"'] & M \arrow[u, phantom, "\cup"] \end{tikzcd}$$



①

②

③

1. a pt is either "interior" or "bndry"
2. ∂M is a mfld.

Tangent space. M, T_pM [image from a patch, indep of the patch]

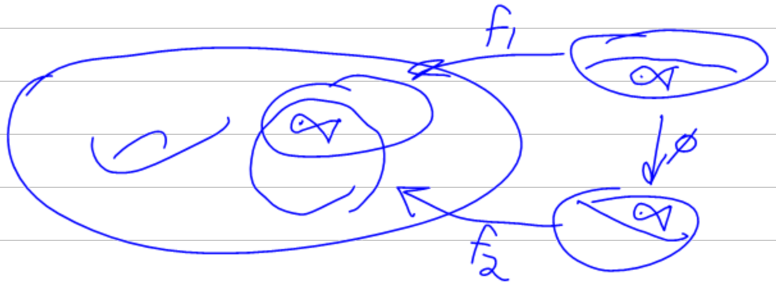
Vector fields. [smoothness]

p-forms.

d

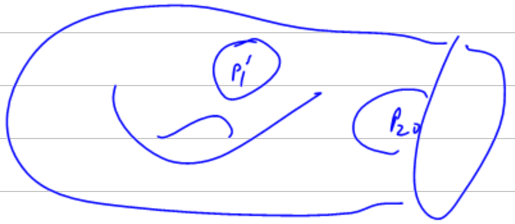
Hour 65 MAT257 Analysis II on March 23, 2022:
Manifolds with boundary, TM, forms on manifolds.
Read Along: Spivak 109-122.

Can you fold a rectangular piece of paper so that the length of the bndry increases?



a Fish* is smooth if it is smooth as seen on any chart.

*Fnctn in or out, or between.



mfld w/ bndry: For every $p \in M$ $\exists$ open $W \subset \mathbb{R}^k_+ = [x_k \geq 0]$, open $p \in U \subset M$ & smooth** 1-1 $f: W \to U$ s.t. ① ② ③

** Extends to smooth.

W/o proof 1. Any p is either "interior" or "bndry"

$\partial M :=$ {all bndry pts} warning: $\neq$ bd M!

2. ∂M is a manifold.

The tangent space $M_p = T_pM$: image from a point, independent of the patch.

Vector fields, directional derivatives.

1-forms: $+$, $\wedge$, d obey all the usual rules.

Example on S^2 $x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$

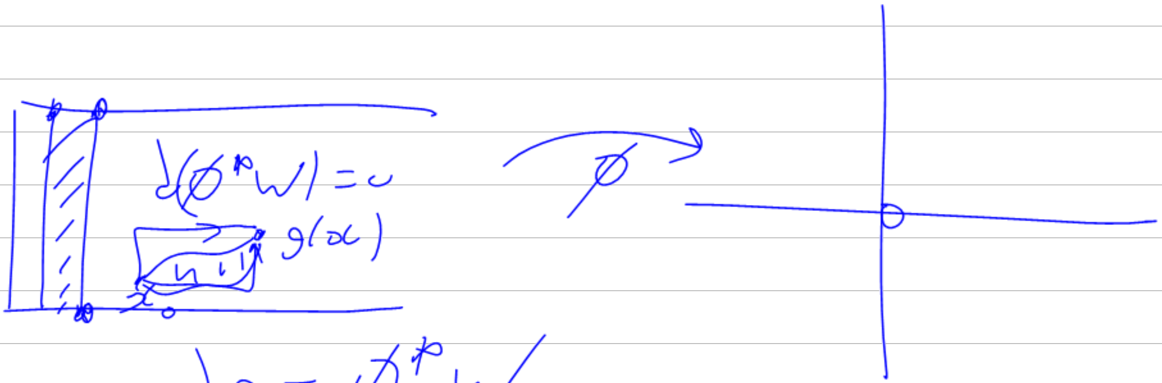
$x\,dx + y\,dy + z\,dz = 0$

pullbacks of diff. forms.

On $\mathbb{R}^2 \setminus \{0\}$, $\omega \in \Lambda^1$, $d\omega = 0$

$$\eta = \frac{y\,dx - x\,dy}{x^2+y^2} \qquad d\eta = 0$$

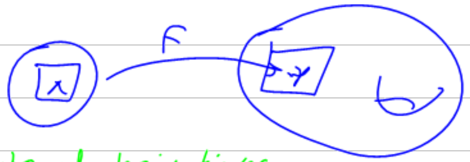
$\exists \lambda$ s.t. $\omega - \lambda\eta$ is exact.



$$\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy = dg = \phi^*\omega$$

Hour 66 MAT257 Analysis II on March 25, 2022: Forms on manifolds, orientations.
Read Along: Spivak 109-122.
Course evals due April 10!

Theme: A fish on a mfld is a fish on every coord-chart provided they agree



The tangent space $M_p = T_pM = f_* T_a\mathbb{R}^k$

Directional derivatives.

$\omega\in\Omega^l(M)$: $\omega : M \to \bigcup_{p\in M} \Lambda^l T_pM$ $\omega(p)\in\Lambda^l T_pM$ smooth.

$+, \wedge, d, \phi^*, \phi_*$ obey all the usual rules. Except no dx_I thm.

Example on S^2 $x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$

$x\,dx + y\,dy + z\,dz = 0$

Cubes, chains, $C^p(M)$, ∂, $\int_{\text{chains}}$ & Stokes Thm all make sense.

One further ingredient is necessary to integrate on manifolds

Can we choose coord. patches for the whole M (an "atlas") s.t. all trans. functions have $\det\alpha' > 0$?

Reminder orientation on V^k:

$$\frac{\text{ordered basis}}{\text{pos det changes}} \sim \frac{\eta\in\Lambda^k(V)}{\text{mult by pos scalars}}$$

orientation of M Examples S, S^2, pt, Möb

orientable, oriented.

Orienting the bdry by prepending & outward normal( ))

$\eta_{\partial M}(x) = i^*_{\partial M\to M}\, \iota_\nu \eta_M(x)$

Hour 67 MAT257 Analysis II on March 28, 2022: Orientation, integration.
Read Along: Spivak 109-122.   Course evals due April 10!

Cubes, chains, $C^p(M)$, ∂, $\int_{\text{chains}}$ & Stokes Thm all make sense.

Orientation on V^k: $\dfrac{\text{ordered basis}}{\text{pos det changes}} \sim \dfrac{\eta\in\Lambda^k(V)}{\text{mult by pos scalars}}$

Orientation of M: A continuously varying choice of an orientation for T_pM, for each $p\in M$

$=\{\text{nowhere 0 top forms on } M\}/\eta_1\sim_f\eta_2$ = An $O_p=\dfrac{\text{ordered basis}}{\text{pos det changes}}$ for each $p\in M$, s.t. $\forall p$ $\exists$ nbd $U\ni p$ & V.F $X_1 .. X_k$ s.t. $O_{p'}=(X_1 ... X_k)$ on U as orientations.

orientable, oriented.

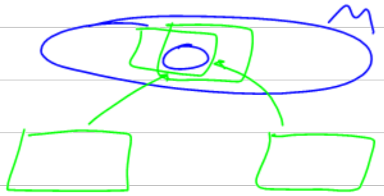
Examples S, S^2, pt, Möb

Orienting the bdry by prepending & outward normal())

Example. D^2 $\qquad \eta_{\partial M}(x) = \iota^*_{\partial M\to M}\,\iota_\nu\,\eta_M(x)$

Proposition Let C_i $(i=1,2)$ be smooth injective orientation preserving k-cubes in an oriented M^k, and let $\omega\in\Omega^k(M)$ be s.t.

$$\operatorname{supp}\omega\subset C_1(I^k)\cap C_2(I^k)$$



then $\int_{C_1}\omega=\int_{C_2}\omega$; Def Call this $\int_M\omega$

Proof & injectivity warning (Spivak has it wrong?!)

Now general integration using PO1, assuming $\sum\int|\varphi_i\omega|$ is absolutely convergent

Indep. of the PO1.

Hour 68 MAT257 Analysis II on March 30, 2022: Orientation (\eps), integration.
Read Along: Spivak 109-122.   Course evals due April 10!

Orienting M^k: $\dfrac{\text{nowhere-0 } \eta\in\Omega^k(M)}{\text{mult by } f>0}$ ~ Orienting T_pM for every $p\in M$, in a manner that can be presented locally by k cont. vector fields.

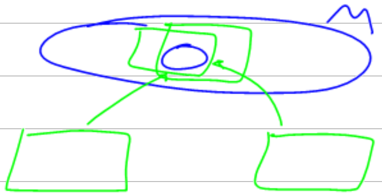
If M is oriented, the induced orientation on ∂M at $p\in\partial M$ is such that if you prepend to it the outward pointing normal $\nu_p\in T_pM$ to $T_p\partial M$, you get the orientation of M at p.

Alternatively, $\eta_{\partial M}(x) = i^*_{\partial M\to M}\,\iota_\nu\,\eta_M(x)$

Example: I & $\partial I = \{0,1\}$.

Proposition Let C_i $(i=1,2)$ be smooth injective orientation preserving k-cubes in an oriented M^k, and let $\omega\in\Omega^k(M)$ be s.t.

$$\operatorname{supp}\omega\subset C_1(I^k)\cap C_2(I^k)$$



then $\int_{C_1}\omega = \int_{C_2}\omega$; **Def** Call this $\int_M\omega$

Proof & injectivity warning (Spivak has it wrong!)

Now general integration using POU, assuming $\sum\int|\varphi_i\omega|$ is absolutely convergent

Indep. of the POU

Note 1. where it makes sense, $\int\omega$ is linear in ω.
2. $\int_{-M}\omega = -\int_M\omega$

Thm If $\omega\in\Omega^{k-1}(M^k)$, then $\int_M d\omega = \int_{\partial M}\omega$

Hour 69 MAT257 Analysis II on Friday April 1, 2022: Integration, Stokes.
Read Along: Spivak 122-126. Course evals due April 10!
HW18 due and HW19 (last) on web by 11:59pm!

Prop M^k oriented, $c_i : I^k \to M$ smooth, 1-1, c_i' injective and orientation preserving ($i=1,2$), $w \in \Omega^k(M)$ w/ $\operatorname{supp} w \subset \operatorname{Im} c_1 \cap \operatorname{Im} c_2 \Rightarrow$

$$\int_{c_1} w = \int_{c_2} w =: \int_M w$$

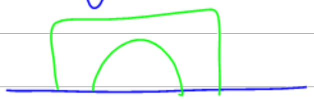
Now general integration using POI, assuming $\sum \int \varphi_i |w|$ is absolutely convergent. Indep. of the POI.

Note
1. where it makes sense, $\int w$ is linear in w.
2. $\int_{-M} w = -\int_M w$.

Stokes' Thm If M is compact and oriented & $w \in \Omega^{k-1}(M)$, then $\int_M dw = \int_{\partial M} w$

Type I

Type B



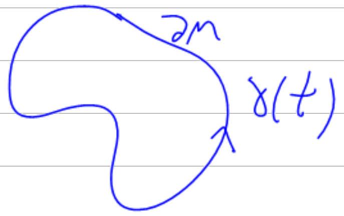
Then merge $\int_{\partial M} w = \sum \int_{\partial M} \varphi_i w$

$$= \sum \int_M d(\varphi_i w) = \sum \int_M (d\varphi_i) \cdot w + \sum \int_M \varphi_i \, dw$$

Example. The Fundamental Thm of Calculus.

Example. Green's theorem. $w = P dx + Q dy$

$M =$



$$dw = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$$

Two interpretations! with $F = \binom{P}{Q}$ this is "rotation",
with $G = \binom{-Q}{P}$ this is "divergence".

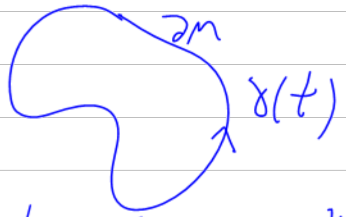
Hour 70 MAT257 Analysis II on Monday April 4, 2022: Stokes examples.
Read Along: Spivak 126-End.   Course evals due April 10!

Bring: Planimeter handouts roller.

Stokes' Thm If M is compact and oriented & $\omega\in\Omega^{k-1}(M)$, then $\int_M d\omega = \int_{\partial M}\omega$

Example. $\int_a^b f'(x)dx = f(b)-f(a)$; $\int_a^b (\text{grad} f)\cdot\dot\gamma(t)\,dt = f(b)-f(a)$

Example. Green's theorem. $\omega = P\,dx + Q\,dy$

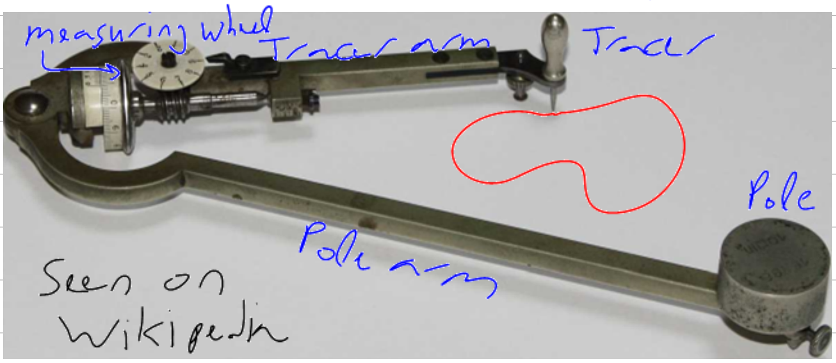
$M =$ 

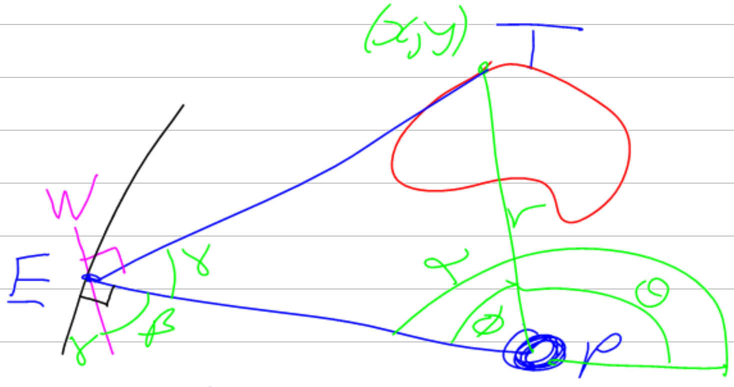
$d\omega = \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dx\wedge dy$

Two interpretations! with $F=\binom{P}{Q}$ this is "rotation",
with $G=\binom{-Q}{P}$ this is "divergence".

Example. The $x\,dy$ planimeter.

Example. The polar planimeter.





$\omega = (d\alpha)\cdot\cos\gamma$

$= \cos(\pi-2\phi)\,d(\phi+\theta) = -\cos 2\phi\,d(\phi+\theta)$

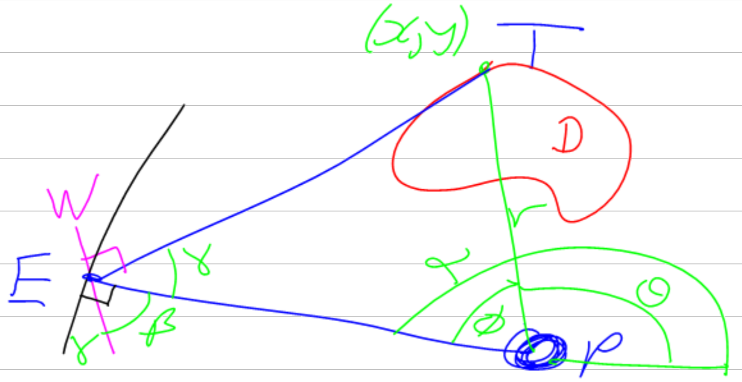
$d\omega = 2\sin 2\phi\,d\phi\wedge d\theta = \underbrace{2\cos\phi}_{r}\,\underbrace{2\sin\phi\,d\phi}_{-dr}\wedge d\theta$

$= -r\,dr\wedge d\theta = dx\wedge dy$

Hour 71 MAT257 Analysis II on Monday April 6, 2022: The Planimeter, the R^3 theorems.
Read Along: Spivak 126-End. Course evals due April 10!
Final exam info in final class.

Bring: handouts

The polar planimeter



Next task: The $\mathbb{R}^3$ Thms

$$\int_{M^3_{\text{cmpt, ori}}} \operatorname{div} \overset{\text{V.F}}{F}\, dV = \int_{\partial M^3} F\cdot \underset{\text{unit normal to } \partial M}{n}\, dA$$

$$\int_{M^2_{\text{cmpt, ori}}} (\operatorname{curl} F)\cdot \underset{\text{unit normal to } M}{n}\, dA = \int_{\partial M^2} F\cdot \underset{\text{unit tange to } \partial M}{T}\, ds$$

$$W = (d\measuredangle)\cdot\cos\gamma = \cos(\pi-2\phi)\, d(\phi+\theta) = -\cos 2\phi\, d(\phi+\theta)$$

$$dW = 2\sin 2\phi\, d\phi\wedge d\theta = \underbrace{2\cos\phi}_{r}\, \underbrace{2\sin\phi\, d\phi}_{-dr}\wedge d\theta$$

$$= -r\, dr\wedge d\theta = -dx\wedge dy$$

Next tasks... But first, what are dV, dA, ds?

M^k oriented. dV: That multiple of the orientation k-form for which $dV(u_1, \dots u_k) = 1$ if (u_i) make an positive o.n basis of T_xM

E.g. $ds(T) = 1$; on S^2, $dA_{(0,0,1)} = dx\wedge dy$

If $M^2 \subset \mathbb{R}^3$ & n is the positive unit normal,

$n: M \to \bigcup T_x\mathbb{R}^3$
0. $n(x) \in T_x\mathbb{R}^3$
1. $n(x) \perp T_xM$
2. $|n(x)| = 1$
3. (n,u,v) oriented. ...

$$dA(u,v) = \begin{vmatrix} - & u & - \\ - & v & - \\ - & n & \end{vmatrix} = \underbrace{(u\times v)}_{\text{needs reminder}}\cdot n = (n_1\, dy\, dz + n_2\, dz\, dx + n_3\, dx\, dy)(u,v)$$

$$= \pm|u\times v| = \pm|u|\cdot|v|\sin\theta \overset{\pm}{=} \sqrt{|u|^2|v|^2 - \langle u,v\rangle}$$

Example: on S^2, get $x\,dy\,dz + y\,dz\,dx + z\,dx\,dy$ as in HW

Example: a formula for $\text{Area}(M = c(I^2))$.

Hour 72: (As in the 3DTheorems handout and slides.

Recall in $\mathbb{R}^3$

$$\begin{tikzcd}
\Omega^0 \arrow[r, "d"] & \Omega^1 \arrow[r, "d"] & \Omega^2 \arrow[r, "d"] & \Omega^3 \\
\underset{\text{Fnctns}}{\{f\}} \arrow[u] \arrow[r, "\text{grad}"] & \underset{\text{V.F.}}{\{F\}} \arrow[u, "W^1_F = \dots"'] \arrow[r, "\text{curl}"] & \underset{\text{V.F}}{\{G\}} \arrow[u, "W^2_F"'] \arrow[r, "\text{div}"] & \underset{\text{Fnctns.}}{\{g\}} \arrow[u, hook, "W^3_g ="']
\end{tikzcd}$$

Claim 1. On $N^1 \subset \mathbb{R}^3$, $W^1_F = (T \cdot F)ds$

2. On $M^2 \subset \mathbb{R}^3$, $W^2_G = (G \cdot n)dA$

Pf 1 Enough to compute on T.

Pf 2 Enough to compute on u, v, a positive ON basis of TM

$W^2_G(u,v) = G \cdot (u \times v) = G \cdot n \cdot |u \times v| = G \cdot n \, dA(u,v)$ □

Thms & Geometric interpretations....

Hour 73 (bonus): Maxwell as in handout/slides.

$$f^* d(h\, dy_I) = f^*\left(\sum_{a=1}^{m} \frac{\partial h}{\partial y_a}\, dy_a \wedge dy_I\right)$$

$$= \sum_{a=1}^{m} \frac{\partial h}{\partial y_a}\circ f \; df_a \wedge f^*(dy_I) = \sum_{a=1}^{m} \frac{\partial h}{\partial y_a}\circ f \sum_{b=1}^{n} \frac{\partial f_a}{\partial x_b}\, dx_b \wedge f^* dy_I$$

$$= \sum_{b=1}^{n} \frac{\partial (h\circ f)}{\partial x_b}\, dx_b \wedge f^*(dy_I) = d(f^*h)\wedge f^* dy_I$$

$$=$$