2021-22 MAT 257Y Analysis II
Dror's Open Private Notebook
First Semseter

on board: MAT 257Y w/ Dror Bar-Natan

http://drorbn.net/2122-257 $\to$ About

Math today! Admin is on web. Yet,

1. Covid's still around! Wear masks and expect mishaps.
2. For the first two weeks, classes will be videotaped. After that we'll go in-person only.

Introduce TAs!

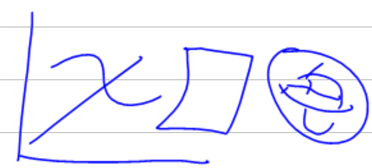
Math Intro: $\mathbb{R}^1 \nearrow \mathbb{R}^n$: LinAlg, cont., differentiability, $\int$, ...

$$\int_a^b F'(t)\,dt = F(b) - F(a) \quad \nearrow \quad \int_C d\omega = \int_{\partial C} \omega$$

F on I

F' on ∂I

"Stokes' Thm"



Ambition: E&M.

<u>LinAlg review</u> <u>Def</u> For $x, y \in \mathbb{R}^n$,

$$\langle x, y\rangle = \sum x_i y_i \qquad |x|^2 = \langle x, x\rangle \qquad |x| = \sqrt{\sum x_i^2} \geq 0$$

<u>Thm</u> If $x, y \in \mathbb{R}^n$ & $a \in \mathbb{R}$ 0. $\langle x, y\rangle$ is bilinear & symmetric; $|x|$ is semi-linear.

1. $|x| \geq 0$ & $|x| = 0$ iff $x = 0$.
2. $|\langle x, y\rangle| \leq |x| \cdot |y|$, "Cauchy-Schwarz" equality iff x, y are lin-dep.

<u>Pf</u> $0 \leq \underbrace{\left| |y|^2 x - \langle x, y\rangle y \right|}_{\text{what's that?}} = |y|^4 |x|^2 - 2|y|^2 \langle x, y\rangle^2 + |y|^2 \langle x, y\rangle^2$ (<= a bit incomplete) done line

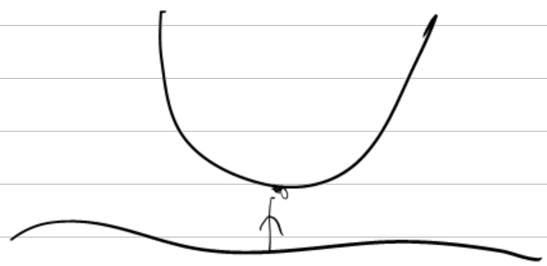
3. $|x+y| \leq |x| + |y|$ "Triangle Ineq"
4. $\langle x, y\rangle = \dfrac{|x+y|^2 - |x-y|^2}{4}$ "Polarization"

<u>Def</u> $d(x, y) = |x - y|$

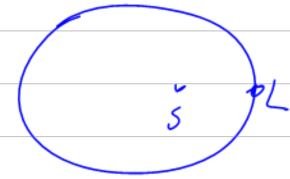
<u>Thm</u> Symmetric, Pos. def., Δ-ineq

Aside $\min |x+\lambda y|^2 = \min |x|^2 + 2\lambda \langle x, y\rangle + \lambda^2 |y|^2$

Obtained at $\underline{\lambda = -\frac{\langle x, y\rangle}{|y|^2}}$



Hour 2, September 13 2021: "About", more LinAlg.
Introduce Shuyang.
Read Along Spivak p. 1-5.
Riddle Along. Can you save ?



$V_L = 4V_S$

Go over "About" handout. See that projector works!

Def For $x, y \in \mathbb{R}^n$,

$$\langle x, y\rangle = \sum x_i y_i \qquad |x|^2 = \langle x, x\rangle \qquad |x| = \sqrt{\sum x_i^2} \geq 0$$

Thm 0. $\langle x, y\rangle$ is bilinear & symmetric
$|x|$ is semi-linear.

1. Positivity $|x| \geq 0$ & $|x| = 0$ iff $x = 0$.

1.5 Symmetry: $\langle x, y\rangle = \langle y, x\rangle$.

2. Cauchy-Schwarz: $|\langle x, y\rangle| \leq |x| \cdot |y|$, equality iff x, y are lin-dep.

3. $|x+y| \leq |x| + |y|$ "Triangle Ineq".

4. $\langle x, y\rangle = \dfrac{|x+y|^2 - |x-y|^2}{4}$ "Polarization".

Def $d(x, y) = |x - y|$

Thm Symmetric, Pos. def., Δ-ineq done Riddle

Linear $\mathbb{R}^n \to \mathbb{R}^m$

$$\left\{ \begin{matrix} T: \mathbb{R}^n \to \mathbb{R}^m \text{ s.t.} \\ T(ax+by) = aT(x) + bT(y) \end{matrix} \right\} \underset{L_A \leftarrow A}{\overset{T \to M_T}{\rightleftarrows}} M_{m\times n}(\mathbb{R}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \right\}$$

If A is a matrix, $L_A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \left(\quad \right)$

If T is a Lin Trans,

$$M_T = \left(Te_1 \,\middle|\, \cdots \,\middle|\, Te_n \right) \quad \text{where } e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th entry}$$

Hour 3, September 15 2021: More norms, more LinAlg.
Note taker needed!
Read Along Spivak p. 1-5.
Riddle Along. Can you save ??

Red eye.

$i \quad \check{s}$

$V_L = V_S$

$x \in \mathbb{R}^n \quad |x|_2 = (\sum x_i^2)^{1/2} \quad |x|_1 = \sum |x_i| \quad |x|_\infty = \max |x_i|$

Claim $|x|_1 \leq \sqrt{n}\,|x|_2 \quad |x|_2 \leq \sqrt{n}\,|x|_\infty \quad |x|_\infty \leq |x|_1$ in HW

Linear $\mathbb{R}^n \to \mathbb{R}^m$

$$\left\{ \begin{array}{l} T : \mathbb{R}^n \to \mathbb{R}^m \text{ s.t.} \\ T(ax+by) = aT(x) + bT(y) \end{array} \right\} \xrightarrow{T \to M_T} M_{m \times n}(\mathbb{R}) = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \right\}$$

$L_A \leftarrow A$

If A is a matrix, $L_A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = A \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \left(\begin{array}{c} \overline{\quad\quad} \\ \overline{\quad\quad} \\ \overline{\quad\quad} \end{array} \right)$

If T is a Lin Trans,

$$M_T = \left(Te_1 \middle| \cdots \middle| Te_n \right) \quad \text{where } e_i = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i\text{th entry}$$

Claim! $M_{L_A} = A \quad \& \quad L_{M_T} = T$

2. $M_{T+S} = M_T + M_S, \quad M_{aT} = a M_T$

$L_{A+B} = L_A + L_B, \quad L_{aA} = a L_A$

3. If $T : \mathbb{R}^n \to \mathbb{R}^m$ & $S : \mathbb{R}^m \to \mathbb{R}^p$, namely $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^p$, then $M_{S \circ T} = M_S \cdot M_T$

$X \times Y$, closed rectangles $R = \prod_{i=1}^{n} [a_i, b_i]$, open rectangles

Def $A \subset \mathbb{R}^n$ "open" means $\forall x \in A \ \exists^{\text{open rect}} R$ s.t. $x \in R \subset A$.
$B \subset \mathbb{R}^n$ "closed" means B^c is open.

Thm 1. $\emptyset, \mathbb{R}^n$ are "clopen".
2. Any union of opens is open, any intersection of closed is closed.
3. A finite intersection of opens is open.
A finite union of closed is closed

Hour 4, September 17 2021: Open and closed in R^n, compactness.
Note taker needed!
Read Along. Spivak p. 1-10.
HW1 on Crowdmark by midnight!
Remember Piazza!
Riddle along. You have two students to save. Now ask me!

closed rectangles: $R = \prod [a_i, b_i] = \{x \in \mathbb{R}^n : \forall i\ a_i \leq x_i \leq b_i\}$
open rectangles: $R = \prod (a_i, b_i) = \{x \in \mathbb{R}^n : \forall i\ a_i < x_i < b_i\}$
$A \subset \mathbb{R}^n$ "open": $\forall x \in A\ \exists$ open rect R $x \in R \subset A$

on board

can replace w/ ball in any of the norms.

Def $B \subset \mathbb{R}^n$ "closed" means B^c is open.

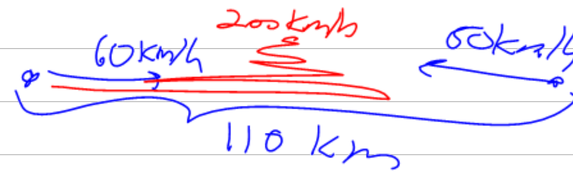
Thm 1. $\emptyset, \mathbb{R}^n$ are "clopen".
2. Any union of opens is open, any intersection of closed is closed.
3. A finite intersection of opens is open.
A finite union of closed is closed

Given $A \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, then exactly one of the following holds:

1. $\exists$ open rect. R s.t. $x \in R \subset A$. "x is in $\mathrm{int}(A)$" (an open set?)
2. $\exists$ open rect R s.t. $x \in R \subset A^c$. "x is in $\mathrm{ext}(A)$". (an open set?)
3. Every open rect containing x intersects both A & A^c "x is in $\mathrm{Bd}(A)$". (A closed set?)

Open covers & compactness.

Heine-Borel $[a,b]$ is compact.



Given $A \subset \mathbb{R}^n$, $x \in \mathbb{R}^n$, then exactly one of the following holds:

1. $\exists$ open rect. R s.t. $x \in R \subset A$. "x is in int(A)" (an open set?)
2. $\exists$ open rect R s.t. $x \in R \subset A^c$. "x is in ext(A)". (an open set?)
3. Every open rect containing x intersects both A & A^c "x is in Bd(A)". (A closed set?)

Open covers & compactness.

Examples:
* Finite sets.
* R, {(-\infty,0),(-1,1),(0,\infty)}
* R, {(n,\infty),(-\infty,n)}
* R, {(-n,n)}

<u>Heine-Borel</u> $[a,b]$ is compact.

<u>Pf</u> $G := \{g \in [a,b] : [a,g] \text{ has a finite cover}\}$

$\gamma = \sup(G)$ (makes sense?)

$\gamma \in G$; $\gamma = b$.

done line

<u>Thm</u> If $A \subset \mathbb{R}^m$ is compact & $B \subset \mathbb{R}^n$ is compact, then $A \times B \subset \mathbb{R}^{m+n}$ is compact.

<u>Cor</u> closed rect. are compact.

<u>Thm</u> A closed subset of a compact set is compact.

<u>Cor</u> $A \subset \mathbb{R}^n$ is cmpct iff it is closed & bndd.
possibly prove only $\Leftarrow$.

Hour 6, September 22 2021: Compactness.
Read Along. Spivak p. 1-10.
Riddle Along (via a former MAT257 student). Choose p uniformly at random in [0,1] and toss a p-coin n times.
What's the chance you got precisely k heads?

Compact: Every open cover has a finite subcover.

Thm $[a,b]$ is compact.

proof $G := \{g \in [a,b] : [a,g] \text{ has finite subcover}\}$

$\gamma := \sup G$ exists! $\gamma = b$ shown! on board

Now $b \in G$ & done.

Thm If $A \subset \mathbb{R}^m$ is compact & $B \subset \mathbb{R}^n$ is compact,

then $A \times B \subset \mathbb{R}^{m+n}$ is compact.

Cor closed rect. are compact.

Thm A closed subset of a compact set is compact.

Cor $A \subset \mathbb{R}^n$ is compact iff it is closed & bdd.

possibly prove only $\Leftarrow$.

Not discussed: $f: \mathbb{R}^n \to \mathbb{R}^m$, $f: A \to B$, $f(C)$, $f^{-1}(D)$, graphs,
compositions, component functions, coordinate projections

Hour 7, September 24 2021: Continuity.
Still seeking a notetaker!
HW1 due at midnight! HW2 on web by midnight!
Read Along. Spivak p. 11-14.
Riddle Along. Your turn!

Not discussed: $f:\mathbb{R}^n\to\mathbb{R}^m$, $f:A\to B$, $f(C)$, $f^{-1}(D)$, graphs, compositions, component functions, coordinate projections

$\lim_{x\to a} f(x)$ [Spivak is annoying]

continuity at a; continuity

the better notion!

Thm 1. $f:\mathbb{R}^n\to\mathbb{R}^m$ is cont. iff whenever $U\subset\mathbb{R}^m$ is open, $f^{-1}(U)\subset\mathbb{R}^n$ is open too.

state but leave as exercise

2. $f: A\subset\mathbb{R}^n\to\mathbb{R}^m$ is cont. iff whenever $U\subset\mathbb{R}^m$ is open, there is an open $V\subset\mathbb{R}^n$ s.t. $f^{-1}(U)=V\cap A$.

Important "$f^{-1}(U)$ is open in A" in general...

Thm 2 If $f:A\to\mathbb{R}^m$ is cont. and A is compact, then $f(A)$ is compact.

Hour 8, September 27 2021: Continuity, diffability.
Read Along. Spivak p. 11-14 & 15-19.
Riddle Along. Is there a cont. f:R->R s.t. f//f=cos?

$f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ cont $\Leftrightarrow \forall a \in A\ \forall \epsilon > 0\ \exists \delta > 0\ |x-a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$

Thm 1. $f: \mathbb{R}^n \to \mathbb{R}^m$ is cont. iff whenever $U \subset \mathbb{R}^m$ is open, $f^{-1}(U) \subset \mathbb{R}^n$ is open too.

Exercise { 2. $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$ is cont. iff whenever $U \subset \mathbb{R}^m$ is open, there is an open $V \subset \mathbb{R}^n$ s.t. $f^{-1}(U) = V \cap A$. "open in A"

Pf of 1. $\Rightarrow$:

$\Leftarrow$

Thm 2 If $f: A \to \mathbb{R}^m$ is cont. and A is compact, then $f(A)$ is compact.

Def $f: \mathbb{R}^n \to \mathbb{R}^m$ is diffable at $a \in \mathbb{R}^n$ if there is a lin trans $L: \mathbb{R}^n \to \mathbb{R}^m$ s.t. $f(a+h) = f(a) + L \cdot h + e(h)$, where $\lim_{h \to 0} e(h)/|h| = 0$.

Def $o(h)$, abuse of notation: $f(a+h) = f(a) + L \cdot h + o(h)$.
Comment: $o(h)$ is a vector space.
Thm If f is diffable at a, then L is unique.
Pf NTS that if $L \in o(h)$ then $L = 0$ ...

Indeed for $\alpha > 0$, $$\frac{LV}{|V|} = \frac{L(\alpha V)}{|\alpha V|} = \lim_{\alpha \to 0} \frac{L(\alpha V)}{|\alpha V|} = 0$$

Def $Df(a)$ E.g. $f: \mathbb{R} \to \mathbb{R}$, $Df(a) = (f'(a))$

Comments 1. Def makes sense for $f: A \subset \mathbb{R}^n \to \mathbb{R}^m$, A open.
2. Extend to non-open sets.
3. Diffability on A.

Hour 9, September 29 2021: Differentiability.
Read Along. Spivak p. 15-19.
Riddle Along. Can you fit 31 2x1 domino pieces on a chess board with two diagonally opposite corders removed?

Def $o_{V,m}(h) := \{e : U \to \mathbb{R}^m : U \text{ open } \& \ 0 \in U,\ e(0)=0,\ \lim_{h\to 0} e(h)/|h| = 0\}$

Comment: $o_V(h)$ is a vector space

Def $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ (open) diff'ble at $a \in A$ $\quad f(a+h) = f(a) + Lh + o(h)$
(abuse of notation)

Thm If f is diff'ble at a, then L is unique.
Pf NTS that if $L \in o(h)$ then $L = 0$ ...

Indeed for $\alpha > 0$, $$\frac{LV}{|V|} = \frac{L(\alpha V)}{|\alpha V|} = \lim_{\alpha\to 0}\frac{L(\alpha V)}{|\alpha V|} = 0$$

Def $Df(a) = f'(a)$ E.g. $f : \mathbb{R}\to\mathbb{R}$, $Df(a) = (f'(a))$

Comments 1. Extend to non-open sets. 2. Differentiability on A.

Thm The chain rule: If $f : \mathbb{R}^n \to \mathbb{R}^m$ is diff'ble at $a \in \mathbb{R}^n$, and $g : \mathbb{R}^m \to \mathbb{R}^p$ is diff'ble at $f(a)$, the $g\circ f$ is diff'ble at a and
$$D(g\circ f)(a) = (Dg)(f(a))\cdot(Df)(a)$$
This generalizes the 1D chain rule!

Pf $$(g\circ f)(a+h) = g(f(a+h)) = g(\underbrace{f(a)}_{a} + \underbrace{Df(a)\cdot h + e_1(h)}_{h})$$
$$= g(f(a)) + (Dg)(f(a))(Df(a)h + e_1(h)) + e_2(Df(a)\cdot h + e_1(h))$$

NTS: 1. $Dg(f(a))\cdot e_1(h) \in o(h)$ easy!

2. $e_2(Df(a)\cdot h + e_1(h)) \in o(h)$

Lemma If $e \in o(h)$ & $|\lambda(h)| \le C\cdot|h|$ for small h, then $e\circ\lambda \in o(h)$

Pf NTS $(e\circ\lambda)(0) = 0$ (easy) and $|e(\lambda(h))| < \epsilon|h|$ for $|h| < \delta$, some δ.

write $|e(\lambda(h))| < \frac{\epsilon}{C}|\lambda(h)| < \epsilon|h|$, provided $h < \delta = \min(\frac{\delta_1}{C}, \delta_2)$

where $|\lambda(h)| < C|h|$ on $B_{\delta_2}(0)$, & $|e(y)| < \frac{\epsilon}{C}|y|$ on $B_{\delta_1}(0)$

Hour 10, October 1 2021: The Chain Rule.
HW2 due at midnight; HW3 on web by midnight. Read Along. Spivak p. 15-19.
Riddle Along. Can you fit 21 3x1 tromino pieces on a chess board with one square (any one) removed?

Def: $f:\mathbb{R}^n\to\mathbb{R}^m$ diff'ble at a: $f(a+h)=f(a)+\underset{f'(a)}{Df(a)}\cdot h+o(h)$

$o(h):=\{e:\mathbb{R}^n\to\mathbb{R}^m : e(0)=0 \ \&\ \lim_{h\to 0}\frac{e(h)}{|h|}=0\}$

Thm The chain rule: If $f:\mathbb{R}^n\to\mathbb{R}^m$ is diff'ble at $a\in\mathbb{R}^n$, and $g:\mathbb{R}^m\to\mathbb{R}^p$ is diff'ble at $f(a)$, then $g\circ f$ is diff'ble at a and

$$D(g\circ f)(a)=(Dg)(f(a))\cdot(Df)(a)$$

This generalizes the 1D chain rule.

Pf $(g\circ f)(a+h)=g(f(a+h))=g(\underbrace{f(a)}_{a}+\underbrace{Df(a)\cdot h+e_1(h)}_{h})$

$=g(f(a))+(Dg)(f(a))(Df(a)h+e_1(h))+e_2(Df(a)\cdot h+e_1(h))$

NTS: 1. $Dg(f(a))\cdot e_1(h)\in o(h)$ (easy!)

2. $e_2(Df(a)\cdot h+e_1(h))\in o(h)$

Lemma If $e\in o(h)$ & $|\lambda(h)|\le C\cdot|h|$ for small h, then $e\circ\lambda\in o(h)$

Pf NTS $(e\circ\lambda)(0)=0$ (easy) and $|e(\lambda(h))|<\epsilon|h|$ for $|h|<\delta$, some δ.

write $|e(\lambda(h))|<\frac{\epsilon}{C}|\lambda(h)|<\epsilon|h|$, Provided $h<\delta=\min(\frac{\delta_1}{C},\delta_2)$

where $|\lambda(h)|<C|h|$ on $B_{\delta_2}(0)$, & $|e(y)|<\frac{\epsilon}{C}|y|$ on $B_{\delta_1}(0)$

Facts on Df/f':
1. f const $\Rightarrow f'=0$
2. f a lin-trans $\Rightarrow Df(a)=f$
3. If $s:\mathbb{R}^2\to\mathbb{R}$ is $s(a,b)=a+b$, then $Ds(a,b)(x,y)=x+y$
$D+=+\qquad (+)'=(1\ 1)$
3' $d(a,b)=a-b$.
4 $f=\begin{pmatrix}f_1\\ \vdots\\ f_m\end{pmatrix}\Rightarrow Df(a)(h)=\begin{pmatrix}Df_1(a)h\\ \vdots\\ Df_m(a)h\end{pmatrix}\qquad f'(a)=\begin{pmatrix}\underline{\quad f_1'(a)\quad}\\ \vdots\\ \underline{\quad f_m'(a)\quad}\end{pmatrix}$

5. If $p(a,b)=ab$ $Dp(a,b)(x,y)=bx+ay$ $\quad p'(a,b)=(b\ \ a)$

Hour 11, October 4 2021: A chain-rule example.
Read Along. Spivak p. 19-34.
Riddle Along. Can you fit 125 4x2x1 boxes in one 10x10x10 cube?

$$F(a+h) = F(a) + DF(a)h + o(h) = f(a) + f'(a)h + o(h)$$

$$(g\circ f)'(a) = g'(f(a))\,f'(a)$$

Facts on DF/f':

1. f const $\Rightarrow$ $f'=0$.
2. F a lin-trans $\Rightarrow$ $DF(a)=F$
3. If $s:\mathbb{R}^2\to\mathbb{R}$ is $s(a,b)=a+b$, then $Ds(a,b)(x,y)=x+y$
   $D+ = +$ $\quad (+)' = (1\ 1)$

on board

Computing the differential of $\frac{x}{y} = e^{\log x - \log y}$:

$$\begin{tikzcd}[column sep=huge]
\mathbb{R}^2_{x>0,\,y>0} \arrow[r, "F_1:(x,y)\mapsto(\log x,\log y)", "D:\begin{pmatrix}1/x & \\ & 1/y\end{pmatrix}"'] \arrow[rrr, bend right, "x,y\mapsto x/y"'] & \mathbb{R}^2_{\xi,\eta} \arrow[r, "F_2:(\xi,\eta)\mapsto(\xi-\eta)", "D:\ (1\ \ -1)"'] & \mathbb{R}_\gamma \arrow[r, "F_3:\gamma\mapsto e^\gamma", "D:\ e^\gamma"'] & \mathbb{R}
\end{tikzcd}$$

$$D:\ e^{\log x-\log y}\cdot(1\ \ -1)\cdot\begin{pmatrix}1/x & \\ & 1/y\end{pmatrix} = \left(y\ \ -\frac{x}{y^2}\right)$$

Fact. $F=\begin{pmatrix}F_1\\ \vdots\\ F_m\end{pmatrix} \Rightarrow DF(a)(h)=\begin{pmatrix}DF_1(a)h\\ \vdots\\ DF_m(a)h\end{pmatrix}$ $\qquad F'(a)=\begin{pmatrix}\overline{\ F_1'(a)\ }\\ \vdots \\ \overline{\ F_m'(a)\ }\end{pmatrix}$

Exercises. 1. Compute F' for $F(x,y)=x\cdot y$

2. compute $(f/g)'$

Partial Derivatives: D_i, $D_{i,j}=D_{j,i}$, min, max.

Thm If $f:\mathbb{R}^n\to\mathbb{R}$ is diff-able at a, then all it's partial derivatives exist at a and

$$Df(a) = (D_1f(a)\ \ldots\ldots\ D_nf)(a)$$

Thm If D_if exist and are cont. near a, then f is diff-able at a.

Hour 12, October 6 2021: Partial derivatives.
Read Along. Spivak p. 19-34.
Riddle Along. Can you fit uncountably many Y-shapes (of any size) in the plane?

correction: $\frac{x}{y} = e^{\log x - \log y}$ only for $x, y > 0$!
[also a minor m.n order issue]

Partial Derivatives: D_i, min, max.

Thm If $f: \mathbb{R}^n \to \mathbb{R}$ is diffable at a, then all it's partial derivatives exist at a and

$$Df(a) = (D_1 f(a) \ldots \ldots D_n f)(a)$$

main Thm For $f: \mathbb{R}^n \to \mathbb{R}$,

use "Axis crawl"

if $D_i f$ exist and are cont. near a,

then f is diffable at a & $f'(a) = (D_1 f(a) \ldots D_n f(a))$

$$= \left(\frac{\partial f}{\partial x_1}(a) \ldots \frac{\partial f}{\partial x_n}(a)\right)$$

Cor. For $f: \mathbb{R}^n \to \mathbb{R}^m$,

if $D_i f_j$ exist and are cont. near a,

then f is diffable at a & $f'(a) = \begin{pmatrix} D_1 f_1(a) & \ldots & D_n f_1(a) \\ & \vdots & \\ D_n f_1(a) & \ldots & D_n f_m(a) \end{pmatrix}$

"The Jacobian matrix of f at a"

Aside/Lemma Let $R \subset \mathbb{R}^n$ be a rectangle, and $f: R \to \mathbb{R}^m$ be ~~cont.~~ diffable. Suppose $|D_i f_j(x)| \leq M$ in $\mathrm{int}(R)$. Then $\forall x, y \in R$,

$$|f(x) - f(y)| \leq \cancel{\text{const}} \cdot M \cdot |x - y|$$

$n \cdot M$

Q. Can you deduce "main thm" from "Aside/Lemma" w/o going through telescopic summation/MVT once again?

Higher partials, partials commute, C^∞.

Hour 13, October 8 2021: Partial derivatives, the IFT.
Read Along. Spivak p. 19-34.
HW3 is due at midnight, HW on web by midnight.
Happy Thanksgiving! No class on Monday.
Riddle Along. Can you find uncountably many subsets of N such that each two of them have a finite intersection?

Thm $f:\mathbb{R}^n\to\mathbb{R}$, $a\in\mathbb{R}^n$, $D_if(a)$ exist & are cont. near a

$\Rightarrow f$ is diff'able at a & $f'(a)=(\partial_1 f(a)\ \dots\ \partial_n f(a))$

Pf $b_i=(a_1+h_1,\dots a_i+h_i, a_{i+1}\dots a_n)$ $b_0=a$ $b_n=a+h$ $\boxed{\begin{matrix} b_i \\ a=b_0\ b_1\end{matrix}}$

$f(a+h)-f(a)-\sum(\partial_i f)h_i=$ board line.

Finish proof!

Aside/Lemma Let $R\subset\mathbb{R}^n$ be a rectangle, and $f:R\to\mathbb{R}^m$ be ~~cont.~~ diff'able. Suppose $|D_if_j(x)|\leq M$ in $\operatorname{int}(R)$. Then $\forall x,y\in R$,

$$|f(x)-f(y)|\leq \underset{n\cdot m}{\cancel{\text{const}}}\cdot M\cdot|x-y|$$

Q. Can you deduce "main thm" from "Aside/Lemma" w/o going through telescopic summation/MVT once again?

Higher partials, partials commute, C^∞.

Thm (IFT) $f:\mathbb{R}^n\to\mathbb{R}^n$ cont. diff'able in an open set containing $a\in\mathbb{R}^n$, $f'(a)$ invertible $\Rightarrow\exists$ open $V\ni a$, open $W\ni b=f(a)$ s.t. $f|_V:V\to W$ is invertible with $f^{-1}:=(f|_V)^{-1}$ is cont., diff'able, and with

$$*\quad (f^{-1})'(y)=\left[f'(f^{-1}(y))\right]^{-1}$$

Well known as hard... My goal: Convince you that it isn't!

1. Dispatch $*$.
2. WLOG, $f'(a)=I$ (also wlog $a=b=0$, but we don't care)
3. Strategy:

Hour 14, October 13 2021: The Inverse Function Theorem.
Read Along. Spivak p. 34-40.
Riddle Along. Can you find uncountably many subsets of N such that for each two of them A and B, either A\subset B or B\subset A?

Thm (The Inverse Function Theorem, IFT)

$f:\mathbb{R}^n\to\mathbb{R}^n$ cont. diff'able in an open set A containing $a\in\mathbb{R}^n$, $f'(a)$ invertible $\Rightarrow \exists$ open $V\ni a$, open $W\ni b=f(a)$ s.t. $f|_V:V\to W$ is invertible with $f^{-1}:=(f|_V)^{-1}$ is cont., diff'able, and with

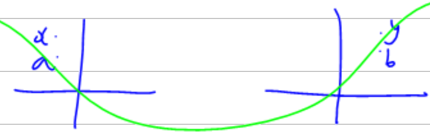
$$* \quad (f^{-1})'(y)=\left[f'(f^{-1}(y))\right]^{-1}$$ (dispatch first!)

WLOG, $f'(a)=I$. Given that, "All Scale Fidelity";

$$|(f(x_1)-f(x_2))-(x_1-x_2)|<\frac{1}{257}|x_1-x_2|$$

where $x_1,x_2\in B_r(a)$

~~Idea of the proof~~



done first

1. Let $W=B_{r/2}(b)$. $\forall y\in W$ $\exists x\in B_r(a)$ s.t. $f(x)=y$.

$x_1=a+(y-b)$

$x_2=x_1+(y-f(x_1))$

$x_3=x_2+(y-f(x_2))$

$|x_n-x_{n-1}|=|(x_{n-1}-x_{n-2})-(f(x_{n-1})-f(x_{n-2}))|$

$\leq\frac{1}{257}|x_{n-1}-x_{n-2}|$

So $x_n\in B_r(a)$, (x_n) is Cauchy, $\lim x_n=x$ exists, & $f(x)=y$.

Now let $V=f^{-1}(W)$; $f|_V:V\to W$ is onto & 1-1!

2. f^{-1} is cont. Indeed,

$$\Big|\underbrace{(x_1-x_2)}_{\alpha}-\underbrace{(f(x_1)-f(x_2))}_{\beta}\Big|\leq\frac{1}{257}\underbrace{|x_1-x_2|}_{\alpha}$$

$$|\alpha-\beta|\leq\frac{1}{257}|\alpha|\leq\frac{1}{257}(|\beta|+|\alpha-\beta|)$$

So $|f^{-1}(y_1)-f^{-1}(y_2)-(y_1-y_2)|\leq\frac{1}{256}|y_1-y_2|$ So $|\alpha-\beta|\leq\frac{1}{256}|\beta|$

So $\left| |f^{-1}(y_1)-f^{-1}(y_2)| - |y_1-y_2| \right| \leq \frac{1}{256}|y_1-y_2|$

So $|f^{-1}(y_1)-f^{-1}(y_2)| \leq \frac{257}{256}|y_1-y_2|$

So f^{-1} is cont.

<u>3</u> f^{-1} is diff'able at b:

$$f^{-1}(\underbrace{b+h}_{y_2}) = f^{-1}(\underbrace{b}_{y_1}) + I \cdot h + e(h)$$

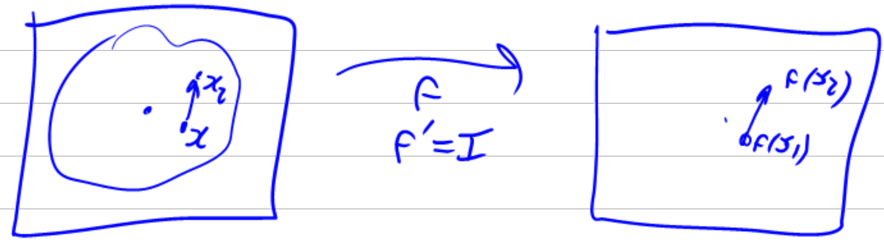
$$e(h) = f^{-1}(y_2) - f^{-1}(y_1) - (y_2-y_1)$$

So $|e(h)| \leq \frac{1}{256}|h|$.

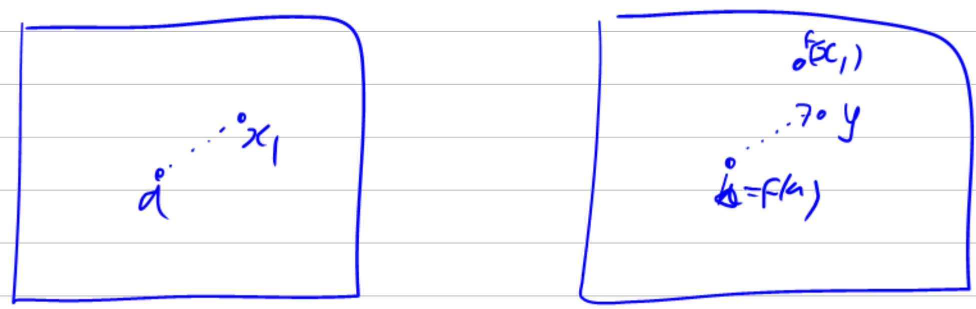
So f^{-1} is diff'able at b, hence everywhere.

$\square$

All Scale Fidelity



$$\left|(x_2 - x_1) - (f(x_2) - f(x_1))\right| \leq \frac{1}{257}\left|x_2 - x_1\right|$$



$$x_1 = a + y - f(a)$$
$$x_2 = a + y - f(x_1)$$

Hour 15, October 15 2021: The Inverse Function Theorem (2).
Read Along. Spivak p. 34-40.
HW4 is due at midnight; HW5 will be online by midnight.
Last day of video recordings!

Thm (IFT) $a \in A \underset{\text{open}}{\subset} \mathbb{R}^n$ $f: A \to \mathbb{R}^n$ cont. diff'ble, $f'(a)$ invertible,

$\Rightarrow \exists$ open $V \ni a$, open $W \ni b = f(a)$ s.t. $f|_V : V \to W$ is invertible with

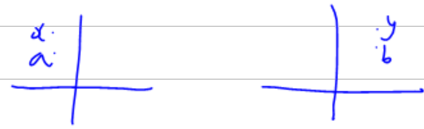
$f^{-1} := (f|_V)^{-1}$ cont. diff'ble with $(f^{-1})'(y) = [f'(f^{-1}(y))]^{-1}$.

Pf WLOG, $f'(a) = I$. Given that, "All Scale Fidelity",

$$|(f(x_1) - f(x_2)) - (x_1 - x_2)| < \frac{1}{257}|x_1 - x_2| \text{ near } a.$$

Board line

Idea of the proof



1. Let $W = B_{r/2}(b)$. $\forall y \in W$ $\exists x \in B_r(a)$ s.t. $f(x) = y$.

$x_1 = a + (y - b)$

$x_2 = x_1 + (y - f(x_1))$

$x_3 = x_2 + (y - f(x_2))$

$|x_n - x_{n-1}| = |(x_{n-1} - x_{n-2}) - (f(x_{n-1}) - f(x_{n-2}))|$

$\leq \frac{1}{257}|x_{n-1} - x_{n-2}|$

So $x_n \in B_r(a)$, (x_n) is Cauchy,
$\lim x_n = x$ exists, & $f(x) = y$.

Now let $V = f^{-1}(W)$; $f|_V : V \to W$ is onto & 1-1 !

2. f^{-1} is cont. Indeed,

$$|\underbrace{(x_1 - x_2)}_{\alpha} - \underbrace{(f(x_1) - f(x_2))}_{\beta}| \leq \frac{1}{257}|\underbrace{x_1 - x_2}_{\alpha}|$$

$$|\alpha - \beta| \leq \frac{1}{257}|\alpha| \leq \frac{1}{257}(|\beta| + |\alpha - \beta|)$$

So $|\alpha - \beta| \leq \frac{1}{256}|\beta|$

done line

So $|f^{-1}(y_1) - f^{-1}(y_2) - (y_1 - y_2)| \leq \frac{1}{256}|y_1 - y_2|$

So $|f^{-1}(y_1) - f^{-1}(y_2)| \leq \frac{257}{256}|y_1 - y_2|$

So f^{-1} is cont.

$\underline{3}$ f^{-1} is diffable at b:

$$f^{-1}(\underbrace{b+h}_{y_2}) = f^{-1}(\underbrace{b}_{y_1}) + I \cdot h + e(h)$$

$$e(h) = f^{-1}(y_2) - f^{-1}(y_1) - (y_2 - y_1)$$

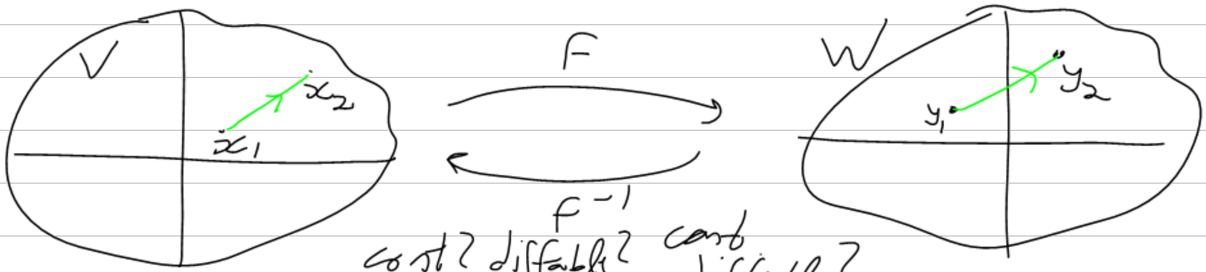
so $|e(h)| \leqslant \frac{1}{2\delta}|h|$.

So f^{-1} is diffable at b, hence everywhere.

Why is it cont. diffable? $\square$

What about higher derivatives?

Hour 16 MAT257 Analysis II on October 18, 2021: The Inverse / Implicit Function Theorems.
Read Along: Spivak 34-45.
Riddle Along: On any pair of potatos, can you draw a pair of 3D congruent curves?



$(F^{-1})'(y) = [F'(F^{-1}(y))]^{-1}$

cont? diffable? cont diffable?

ASF: $|(x_1-x_2)-(y_1-y_2)| \leq \frac{1}{257}|x_1-x_2|$
$\frac{1}{256}|y_1-y_2|$

board line

$\Rightarrow |x_1-x_2| \leq \frac{257}{256}|y_1-y_2| \Rightarrow$ cont. !

F^{-1} is diffable at b:

$F^{-1}(\underbrace{b+h}_{y_2}) = F^{-1}(\underbrace{b}_{y_1}) + Ih + e(h)$

$e(h) = F^{-1}(y_2) - F^{-1}(y_1) - (y_2-y_1)$

So $|e(h)| \leq \frac{1}{256}|h|$.

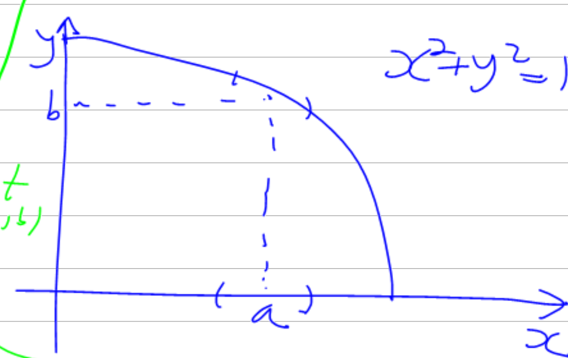
So F^{-1} is diffable at b, hence everywhere. $\square$

But why is F^{-1} cont. diffable?

What about higher derivatives?

done line

The Implicit Function Thm



Thm Given a cont. diffable (at least near (a,b))

$F: \mathbb{R}^n_{x_1 \ldots x_n} \times \mathbb{R}^k_{y_1 \ldots y_k} \longrightarrow \mathbb{R}^k$

and $(a,b) \in \mathbb{R}^n \times \mathbb{R}^k$ s.t. $f(a,b)=0$, such that

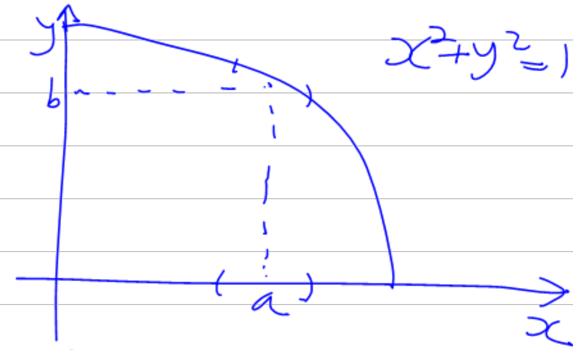
$\underline{\qquad\qquad\qquad\qquad}$, $\exists$ nbd A of a, nbd B of b, & $\exists!\, g: A \to B$

s.t. $g(a)=b$ & $\forall z \in A$ $f(z, g(z))=0$. Furthermore,

g is cont. diffable & $g' = \underline{\qquad\qquad}$.

Hour 17 MAT257 Analysis II on October 20, 2020: The Implicit Function Theorem.
Read Along: Spivak 40-45. Read Ahead: Spivak 46-52, "Integration".
Riddle Along: Save Omar!

The Implicit Function Thm



Thm Given $f:\mathbb{R}^n_{x_1\dots x_n}\times\mathbb{R}^k_{y_1\dots y_k}\longrightarrow\mathbb{R}^k$ cont. diff'able near $(a,b)\in\mathbb{R}^n\times\mathbb{R}^k$ and s.t. $f(a,b)=0$ and $\underbrace{\partial f/\partial y \text{ invertible}}$, $\exists$ nbd A of a, nbd B of b, & $\exists!\, g:A\to B$ s.t. $g(a)=b$ & $\forall z\in A$ $f(z,g(z))=0$. Furthermore, g is cont. diff'able & $g' = \underbrace{-\left(\frac{\partial f}{\partial y}\right)^{-1}\frac{\partial f}{\partial x}}$.

Pf First compute g'. Then:

$$f(z,y)=0 \iff \begin{cases} x=z\\ f(x,y)=0\end{cases} \quad \text{so w/ } H(x,y)=\begin{pmatrix}x\\ f(x,y)\end{pmatrix}$$

this is $H\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}z\\0\end{pmatrix}$ where $H\begin{pmatrix}a\\b\end{pmatrix}=\begin{pmatrix}a\\0\end{pmatrix}$. If $H'\begin{pmatrix}a\\b\end{pmatrix}$ is non-singular, H^{-1} exists near $\begin{pmatrix}a\\b\end{pmatrix}$, so for z near a $\exists!\,(x,y)$ s.t. $H\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}z\\0\end{pmatrix}$; so set

$$g(z)=\pi_2\circ H^{-1}\begin{pmatrix}z\\0\end{pmatrix}.$$

* When is H' invertible?

Hour 18 MAT257 Analysis II on October 22, 2021: The definition of the integral.
Read Along: Spivak 46-56. HW5 due, HW6 assigned.
~~TT1 discussion on Wednesday next week. Until then, just keep on top of things!~~
Riddle Along: A spherical loaf of bread is put into a bread-cutting machine. Which slice gets the most crust?

$$\int_M dW = \int_{\partial M} W$$

$R = \prod [a_i, b_i]$ $\quad P: (P_i)$ where P_i partitions $[a_i, b_i]$

$$P_i = (a_i = t_{i0} < \dots < t_{iN_i} = b_i)$$

R is divided into a union of nearly-disjoint subrectangles $\prod [t_{i,j_i-1}, t_{ij_i}]$ $\quad \prod N_i$ of them.

$V(R) := \prod (b_i - a_i)$ $\quad$ Claim $\quad V(R) = \sum_{S \in P} V(S)$

$f: R \to \mathbb{R}$ bdd. $\quad$ E.g. $f_1 \equiv c$, $\quad f_2(x) = \begin{cases} 1 & \forall i\ x_i \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$

$m_S(f) \quad M_S(f) \quad L(f,P), U(f,P) \quad L(f,P) \le U(f,P)$

def!

done line

Lemma If P' refines P, $L(f,P) \le L(f,P')$
$U(f,P) \ge U(f,P')$

Corollary If P & P' are any two partitions.

$$L(f,P) \le U(f,P')$$

def
$$U(f) := \inf_P U(f,P) \qquad L(f) = \sup_P L(f,P)$$

def integrable

Thm f is integrable $\iff \forall \epsilon > 0\ \exists P$ s.t.

$$U(f,P) - L(f,P) < \epsilon.$$

Goal: f is integrable iff f is cont. except on a tiny set.

Def Measure 0 [using either open or closed rectangles]

Hour 19 MAT257 Analysis II on October 25, 2021: The definition of the integral (2).
Read Along: Spivak 46-56.
TT1 discussion on Wednesday. Until then, just keep on top of things!

$$\int_M dw = \int_{\partial M} w$$

$R = \prod [a_i, b_i]$ $P: \left(P_i = (a_i = t_{i0} \leq \dots \leq t_{iN_i} = b_i)\right)_{i=1}^n$

$R = \bigcup_j \prod [t_{i,j_i-1}, t_{i,j_i}]$ * nearly disjoint, * $\prod N_i$ of them.

$v(R) := \prod (b_i - a_i)$ Given $f: R \to \mathbb{R}$ bdd,

$m_S(f) := \inf_{x \in S} f(x)$ $M_S(f) := \sup_{x \in S} f(x)$

$L(f,P) = \sum_S v(S) m_S(f) \leq U(f,P) := \sum_S v(S) M_S(f)$

def! board line

Lemma If P' refines P, $L(f,P) \leq L(f,P')$
$U(f,P) \geq U(f,P')$

Corollary If P & P' are any two partitions.
$$L(f,P) \leq U(f,P')$$

def
$$U(f) := \inf_P U(f,P) \qquad L(f) = \sup_P L(f,P)$$

def integrable

Thm f is integrable $\iff \forall \epsilon > 0\ \exists P$ s.t.
$$U(f,P) - L(f,P) < \epsilon$$

Thm cont. functions are integrable. done line

Pf Cover R w/ open rects w/ small $\bigcirc$ on closure, use to get a partition.

Goal: f is integrable iff f is cont. except on a tiny set.

Def Measure 0 [using either open or closed rectangles]

Hour 20 MAT257 Analysis II on Wednesday October 27, 2021: Integrability and continuity, countability.
Read Along: Spivak 46-56.
Riddle Along: Your turn!

On TT1:
* Tuesday November 2, 5-7PM (Toronto time), at EX 320.
* Material: Everything up to and including today.
* No accessories, closed notes, closed book.
* How to study: go over some older exams. Yet more important: make sure that you understand every single bit of class material so far!
* An email with links and info about further office hours will be out later today.

Claim Always, $L(f,P) \leq U(f,P')$

Def $U(f) := \inf_P U(f,P)$ $\quad L(f) = \sup_P L(f,P)$

Def integrable: $U(f) = L(f)$

Thm f is integrable $\iff \forall \epsilon > 0\ \exists P$ s.t. $U(f,P) - L(f,P) < \epsilon$

Thm. Cont. fnctns are integrable.

board line

Def $o(f,A)$ $O(f,a)$

Pf Cover R w/ open sets w/ small O on closure, use to get a partition.

Goal: f is integrable iff f is cont. except on a tiny set.

Def measure 0 [using either open or closed rectangles]

Finite sets, countable sets

done line

The Cantor set.

Subsets.

A countable union of meas-0.

Aside on countable sets.
1. Def.
2. finite sets.
3. Subsets of cntble
4. Finite unions; countable unions
5. Products
6. $\mathbb{Q}$
7. But not $\mathbb{R}$!

Hour 21 MAT257 Analysis II on October 29, 2021: Countability, measure 0 and content 0, integrability and continuity.
Read Along: Spivak 46-56.

Goal: ($f: \mathbb{R} \to \mathbb{R}$ integrable) $\Leftrightarrow$ (f is cont. except on a set of measure 0)

Def A is meas-0 means $\forall \epsilon > 0$ $\exists$ open ~~closed~~ rectangles (R_i) s.t.

1. $A \subset \bigcup R_i$ 2. $\sum v(R_i) < \epsilon$

Examples Finite & countable sets, a line in $\mathbb{R}^2$, are all meas-0

Aside X countable: $\exists f: \mathbb{N} \twoheadrightarrow X$.

1. Finite sets 2. Subsets of cntble.

board line.

3. Finite & cntble unions of cntble

4. Products.

5. $\mathbb{Q}$ 6. But not $\mathbb{R}$!

Back to meas-0: The Cantor set.

Subsets.

A countable union of meas-0.

Def content 0

done line

Thm Compact + measure 0 $\Rightarrow$ content 0.

Thm $[a,b]$ does not have content 0,
(hence not measure 0, hence not countable)

Skipped Thm Same for $\prod [a_i, b_i]$.

Hour 22 MAT257 Analysis II on November 1, 2021: measure 0 and content 0, integrability and continuity.
Term test 1 is tomorrow at 5pm at EX320. Good luck!
Format: n Questions, 4n+2 pages on 2n+1 sheets: Cover, 2n solution pages, and 2n+1 scratch pages. Adjust your style!
Plenty office hours today and tomorrow! Check Q/Announcements.
Read Along: Spivak 46-56.

Goal: ($f: R \to \mathbb{R}$ integrable) $\Longleftrightarrow$ (f is cont. except on a set of measure 0)

Def $A \subset \mathbb{R}^n$ has measure 0 / content 0 if for every $\epsilon > 0$ it can be covered w/ countably / finitely many rectangles whose total volume is $< \epsilon$.

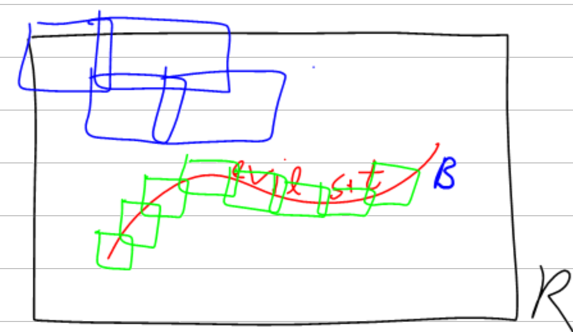
board line

Thm Compact + measure 0 $\Rightarrow$ content 0

Thm $R = \prod [a_i, b_i]$ does not have content 0, (hence not measure 0, hence not countable)

Reminder: $o(f,a)$ Thm f is cont. at a iff $o(f,a) = 0$.

Pf of Goal, $\Leftarrow$.



On to $\Rightarrow$:

$B_n := \{ b \in R : o(f,b) > \frac{1}{n} \}$

$B = \bigcup B_n$

Show that B_n has content 0.

likely skip

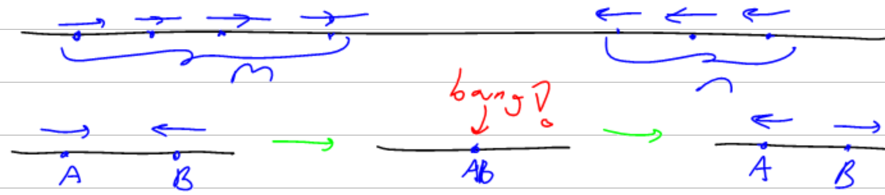
Aside If A is closed and $\epsilon > 0$, $\{ a \in A : o(f,a) \geq \epsilon \}$ is closed.

Aside So disc(f) is F_σ and cont(f) is G_δ

Riddle Is every F_σ set disc(f) for some f?

Is every G_δ set cont(f) for some f?

Hour 23 MAT257 Analysis II on November 3, 2021: integrability and continuity, Fubini.
TT1 results: Likely after reading week.
Read Along: Spivak 46-56, 56-62.
Riddle Along: How many bangs?



Thm. ($f: R \to \mathbb{R}$ integrable) $\iff$ (f is cont. except on a set of measure 0)

$\Leftarrow$: Done

board line

$\Rightarrow$: $E_n := \{x \in R : o(f,x) > \frac{1}{n}\}$ $\quad E = \bigcup E_n$

Show that B_n has content 0.

Common misconception: "measure-0 sets can be ignored".

Yet true: "content-0 sets can be ignored."

$\chi_C(x) = 1_C(x)$ For $C \subset R$, $\mathrm{Vol}(C)$ Aka "content", area, length.

Claim χ_C integrable $\iff$ bd C has meas 0.

Def $\int_C f := \int f\chi_C$ [may not make sense even if C is open!]

done line

$L\int = \underline{\int} = \underline{\int} := \sup L(f,P)$

$U\int = \overline{\int} = \overline{\int} := \inf U(f,P)$

Given $f: R = A \times B \to \mathbb{R}$, set $\underline{F}(x) = \underline{\int_B} f(x,y)\,dy$ $\quad \overline{F}(x) = \overline{\int_B} f(x,y)\,dy$

Thm (not really Fubini)

If f is integrable, then $\int_{A \times B} f = \int_A \underline{F} = \int_A \overline{F}$

Comments 1. f cont. $\Rightarrow$ . . . $\quad$ 2. $\int_{[0,1]^2} x \cdot y \,dx\,dy = \frac{1}{4}$

3. $f(x,-)$ integrable except on a finite set.

4. $f(x,y) = \begin{cases} 1 + \frac{1}{q} & x, y \in \mathbb{Q},\ x = p/q \\ 1 & \text{otherwise} \end{cases} =$

Hour 24 MAT257 Analysis II on November 5, 2021: Fubini's Theorem.
Read Along: Spivak 56-62.
Reading week! No classes, tutorials, office hours until Nov 15!
HW7 will be assigned within days and due Nov 19.
Riddle Along: n prisoners. Each wears a tower of infinitely-many randomly-chosen b/w hats. Simultaneously each needs to point at a black hat on their head. How can they maximize the chance that they will all get it right? Can they do better than 1/2^n?

Thm $f:(R=A\times B\subset\mathbb{R}^n\times\mathbb{R}^m)\to\mathbb{R}$ integrable, $\underline{g}(x):=\underline{\int_B} f(x,y)\,dy$

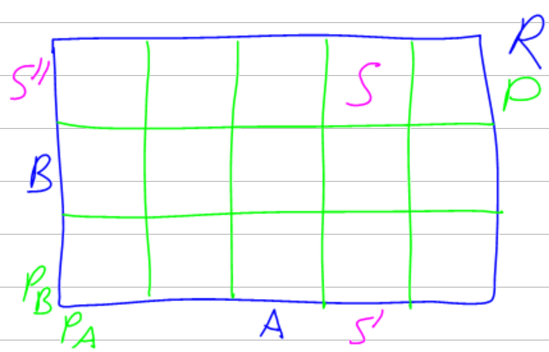
$$\Rightarrow \int_R f\,dx\,dy=\int_A \underline{g}(x)\,dx=\int_A \overline{g}(x)\,dx \qquad \overline{g}(x):=\overline{\int_B} f(x,y)\,dy$$

Comments 1. f cont. $\Rightarrow\ldots$ 2. $\int_{[0,1]^2} x\cdot y\,dx\,dy=\frac14$

3. $f(x,-)$ integrable except on a finite set.

4. $f(x,y)=\begin{cases}1+\frac1q & x,y\in\mathbb{Q},\ x=p/q\\ 1 & \text{otherwise}\end{cases}$ = done line

Proof Given $P=P_A\times P_B$, write each $S\in P$ as $S=S'\times S''$, $S'\in P_A$, $S''\in P_B$.



$$L(f,P)=\sum_{S'\in P_A} v(S')\sum_{S''\in P_B} v(S'')\inf_{S'\times S''} f$$

$$=\sum_{S'\in P_A} v(S')\sum_{S''\in P_B}\inf_{x\in S'} v(S'')\inf_{y\in S''} f(x,y)$$

$$\leq\sum_{S'\in P_A} v(S')\inf_{x\in S'}\underbrace{\sum_{S''\in P_B} v(S'')\inf_{y\in S''} f(x,y)}_{L(f(x,-),P_B)\leq \underline{g}(x)}$$

$$\leq\sum_{S'\in P_A} v(S')\inf_{x\in S'}\underline{g}(x)=L(\underline{g},P_A)$$

Aside $\inf_x h_k(x)\leq h_k(x)$
so $\sum\inf_x h_k(x)\leq\sum h_k(x)$
so $\sum\inf h_k(x)\leq\inf\sum h_k(x)$

Similarly $U(f,P)\geq U(\overline{g},P_A)$, so

$$L(f,P)\leq L(\underline{g},P_A)\begin{matrix}\leq L(\overline{g},P_A)\leq\\ \leq U(\underline{g},P_A)\leq\end{matrix} U(\overline{g},P_A)\leq U(f,P)$$

$\square$

Hour 25 MAT257 Analysis II on November 15, 2021: Fubini's Theorem (2).
Read Along: Spivak 56-62.
TT1: Not yet; sorry.
Riddle Along: n prisoners. Each wears a tower of infinitely-many randomly-chosen b/w hats. Simultaneously each needs to point at a black hat on their head. How can they maximize the chance that they will all get it right? Can they do better than 1/2^n?

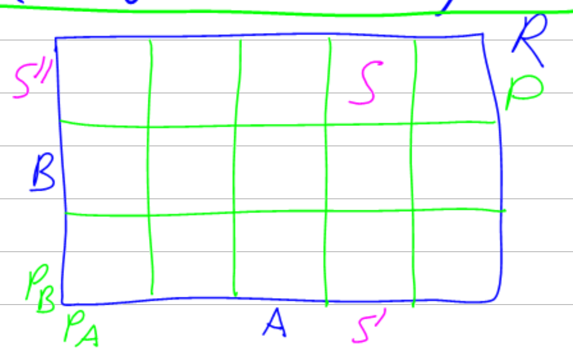
<u>Thm</u> $f:(R=A\times B\subset\mathbb{R}^n\times\mathbb{R}^m)\to\mathbb{R}$ integrable, $\underline{g}(x):=\underline{\int_B} f(x,y)\,dy$

$$\Rightarrow \int_R f\,dx\,dy=\int_A \underline{g}(x)\,dx=\int_A \overline{g}(x)\,dx \qquad \overline{g}(x):=\overline{\int_B} f(x,y)\,dy$$

board line

(cont? No worries)

<u>Proof</u> Given $P=P_A\times P_B$, write each



$S\in P$ as $S=S'\times S''$, $S'\in P_A$, $S''\in P_B$.

$$L(f,P)=\sum_{S'\in P_A} v(S')\sum_{S''\in P_B} v(S'')\inf_{S'\times S''} f$$

$$=\sum_{S'\in P_A} v(S')\sum_{S''\in P_B}\inf_{x\in S'} v(S'')\inf_{y\in S''} f(x,y)$$

Aside
$\inf_x h_k(x)\le h_k(x)$
so $\sum\inf_x h_k(x)\le\sum h_k(x)$
so $\sum\inf h_k(x)\le\inf\sum h_k(x)$

$$\le\sum_{S'\in P_A} v(S')\inf_{x\in S'}\underbrace{\sum_{S''\in P_B} v(S'')\inf_{y\in S''} f(x,y)}_{L(f(x,-),P_B)\le\underline{g}(x)}$$

$$\le\sum_{S'\in P_A} v(S')\inf_{x\in S'}\underline{g}(x)=L(\underline{g},P_A)$$

Similarly $U(f,P)\ge U(\overline{g},P_A)$, so

$$L(f,P)\le L(\underline{g},P_A)\begin{array}{l}\le L(\overline{g},P_A)\le\\ \le U(\underline{g},P_A)\le\end{array} U(\overline{g},P_A)\le U(f,P)$$

done line ... $\square$

<u>Thm (POI)</u> $\mathcal{U}=\{U\}$ an open cover of $A\subset\mathbb{R}^n\Rightarrow$

$\exists$ countable $\Phi=\{\varphi_i:W\to[0,1]\}$ C^∞ on an open set $W\supset A$ s.t.

1. Φ is "locally finite" ... as in 2020, yet not done...

Hour 26 MAT257 Analysis II on November 17, 2021: Partitions of unity.
Read Along: Spivak 63-66. TT1 Results: later today.
Riddle recap: n prisoners. Each wears a tower of infinitely-many randomly-chosen b/w hats. Simultaneously each needs to point at a black hat on their head. How can they maximize the chance that they will all get it right? Can they do better than 1/2^n?

Philosophy on POI, then....

Thm (POI) Given $A \subset \mathbb{R}^n$ & $\mathcal{U}$ an open cover thereof, we can find a countable collection $\Phi = \{\varphi_i : W \to [0,1]\}$ of C^∞ functions defined on some open $W \supset A$, s.t.

1. Φ is locally finite: Each $x \in W$ has some open neighborhood $V \ni x$, s.t. "loc. fin."
$$|\{i : \operatorname{supp}(\varphi_i) \cap V \neq \emptyset\}| < \infty$$
2. $\forall x \in A \quad \sum \varphi_i(x) = 1$. "Sum=1"
3. $\forall \varphi_i \in \Phi \quad \exists U \in \mathcal{U}$ s.t. $\operatorname{supp} \varphi_i \subset U$. "subordinate"

Prel. 1 $\exists$ smooth flat-top mountains:

Mt. Conner, Aus

If $\underset{\text{compact}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists f \in C^\infty(\mathbb{R}^n)$ s.t.
$$f|_C \equiv 1, \quad \operatorname{supp} f \subset U$$

Steps 1. $\exists$ smooth 1D seashores: $\sigma(x) = 0 \quad x \leq 0$, $\sigma(x) > 0 \quad x > 0$; $\sigma(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$

2. $\exists$ smooth 1D bumps: $\beta_\epsilon(x) \geq 0$, $\beta_\epsilon(0) > 0$, $\operatorname{supp} \beta_\epsilon \subset [-\epsilon, \epsilon]$; $\beta_\epsilon(x) = \sigma(\epsilon + x) \cdot \sigma(\epsilon - x)$

3. $\exists$ Smooth nD bumps $\beta(a) > 0$, $\beta(x) = 0 \quad |x-a| > \epsilon$; $\beta_{a,\epsilon} = \beta_{\epsilon^2}\left(\sum (x_i - a_i)^2\right)$

4. $\exists$ smooth 1D steps $\theta(x) = 0 \quad x \leq 0$, $\theta(x) = 1 \quad x \geq 1$; $\theta(x) = \frac{1}{Z} \int_0^x \beta_{\frac{1}{2}, \frac{1}{2}}(t)\,dt$

5. Finish the proof.

Prel 2 If $C \subset U \subset \mathbb{R}^n$ (C cmpct, U open), $\exists$ cmpct D s.t.

$$C \subset \operatorname{int} D \subset D \subset U$$

Back to PO1: Given $A, \mathcal{U}$ $\exists \varphi_i$ loc fin, sum $=1$ subordinate.

Case I A is cmpct.

Pf WLOG $\mathcal{U} = \{U_i\}_1^n$ is finite. Shrink U_i to a cmpct $C_i \subset U_i$ s.t. $\{\operatorname{int} C_i\}$ covers A.

Find ψ_i on $\cup U_i$ w/ $\psi_i|_{C_i} \equiv 1$, $\operatorname{supp} \psi_i \subset U_i$ &

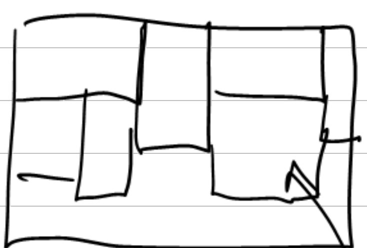
f s.t. $f|_A \equiv 1$, $\operatorname{supp} f \subset \bigcup \operatorname{int} C_i$

& set

$$\varphi_i(x) = \begin{cases} f(x) \dfrac{\psi_i(x)}{\sum \psi_i(x)} & x \in \bigcup \operatorname{int} C_i \\ 0 & \text{otherwise} \end{cases}$$

TA meeting:

$\mathbb{Q} \not\ni$



$a \notin \mathbb{Z}$

each is semi-integral - at least one side is rational.

$$\int_{R} e^{2\pi i(x+y)} = \int_{[a,b]} e^{2\pi i x} \int_{[c,d]} e^{2\pi i y}$$

Hour 27 MAT257 Analysis II on November 19, 2021: Partitions of unity (2).
Read Along: Spivak 63-66.
HW8 on web by midnight.

Thm (PO1) $\underset{\text{any}}{A} \subset \mathbb{R}^n$, $\mathcal{U}$ an open cover, $\Rightarrow \exists \underset{\text{open}}{W} \supset A$ &

a countable collection $\Phi = \{\varphi_i : W \to [0,1]\} \subset C^\infty$ s.t.

1. $\forall x \in W$ $\exists$ nbd $V \ni x$, s.t.

   $|\{i : \operatorname{supp}(\varphi_i) \cap V \neq \emptyset\}| < \infty$ "loc. fin."

2. $\forall x \in A$ $\sum \varphi_i(x) = 1$. "sum=1"

3. $\forall \varphi_i \in \Phi$ $\exists U \in \mathcal{U}$ s.t. $\operatorname{supp} \varphi_i \subset U$. "subordinate"

Prel. 1 Given $\underset{\text{compact}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists F \in C^\infty(\mathbb{R}^n)$ s.t.

$$F|_C \equiv 1, \quad \operatorname{supp} F \subset U$$

Steps 1. $\exists$ smooth 1D staircases: $\sigma(x) = 0$ $x \le 0$, $\sigma(x) > 0$ $x > 0$

board line

TTT Discussion, then.....

2. $\exists$ smooth 1D bumps: $\beta_\epsilon(x) \geq 0$ $\beta_\epsilon(0) > 0$, $\operatorname{supp} \beta_\epsilon \subset [-\epsilon, \epsilon]$ $\quad \beta_\epsilon(x) = \sigma(\epsilon + x) \cdot \sigma(\epsilon - x)$

3. $\exists$ smooth nD bumps $\beta(a) > 0$, $\beta(x) = 0$ $|x - a| > \epsilon$ $\quad \beta_{a,\epsilon} = \beta_{\epsilon^2}(\sum (x_i - a_i)^2)$

4. $\exists$ smooth 1D steps $\theta(x) = 0$ $x \le 0$, $\theta(x) = 1$ $x \geq 1$ $\quad \theta(x) = \frac{1}{Z} \int_0^x \beta_{\frac{1}{2}, \frac{1}{2}}(t)\,dt$

5. Finish the proof.

Show SmoothLego@.nb

done line

Prel 2 If $\underset{\text{cmpct}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists$ cmpct D s.t.

$$C \subset \operatorname{int} D \subset D \subset U$$

Back to PO1: Given $A, \mathcal{U}$ $\exists \varphi_i$ loc fin, sum=1, subordinate.

Hour 28 MAT257 Analysis II on November 22, 2021: Partitions of unity (3).
Read Along: Spivak 63-66.

Thm (PO1) $\underset{\text{any}}{A} \subset \mathbb{R}^n$, $\mathcal{U}$ an open cover, $\Rightarrow \exists \underset{\text{open}}{W} \supset A$ & cntble $\Phi = \{\varphi_i : W \to [0,1]\} \subset C^\infty$ s.t.

1. "loc. fin" 2. "Sum=1" 3. "subordinate"

Prel. 1 Given $\underset{\text{compact}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists f \in C^\infty(\mathbb{R}^n)$ s.t.

$$f|_C \equiv 1, \quad \operatorname{supp} f \subset U$$

board line

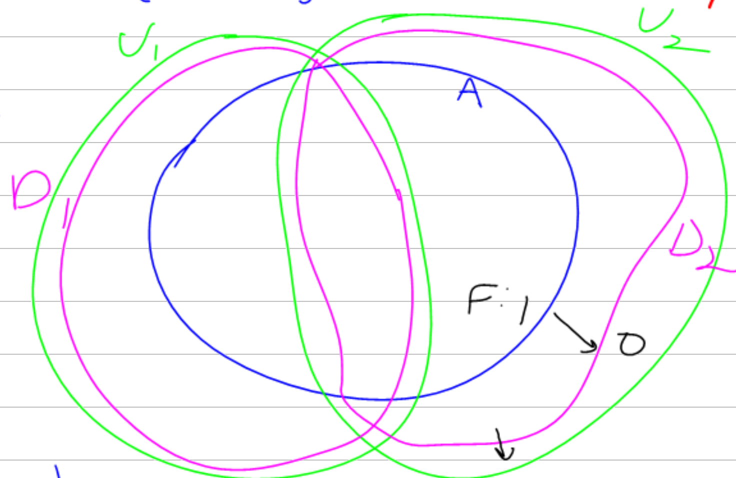
Prel 2 If $\underset{\text{cmpct}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists$ cmpct D s.t.

$$C \subset \operatorname{int} D \subset D \subset U$$

Case I A is cmpct.

Pf WLOG $\mathcal{U} = \{U_i\}_1^n$ is finite. Shrink U_i to a cmpct $D_i \subset U_i$ s.t. $\{\operatorname{int} D_i\}$ covers A } blundered here; needs repeat.

By prev. work, Find $\psi_i : \mathbb{R}^n \to [0,1]$ smooth, $\psi_i|_{D_i} \equiv 1$, $\operatorname{supp} \psi_i \subset U_i$



Define

$$\varphi_i(x) = \begin{cases} F(x) \cdot \dfrac{\psi_i(x)}{\sum \psi_i(x)} & \\ 0 & x \notin \bigcup_{i} \operatorname{int} D_i \end{cases}$$

where F is a "flat top" $F|_A \equiv 1$, $\operatorname{supp} F \subset \bigcup \operatorname{int} D_i$

done line

Read Along: Spivak 63-66.
Riddle Along: Cars A,B,C,D drive in the Sahara Desert on generic straight lines and at constant speed; it is known that A meets B (they arrive at the same place at the same time), A meets C, A meets D, B meets C, and B meets D. Does C necessarily meet D?

Thm (PO1) $\underset{\text{any}}{A} \subset \mathbb{R}^n$, $\mathcal{U}$ an open cover, $\Rightarrow \exists \underset{\text{open}}{W} \supset A$ & cntble $\Phi = \{\varphi_i : W \to [0,1]\} \subset C^\infty$ s.t.

1. "loc. fin." 2. "Sum=1" 3. "subordinate"

Prel. 1 Given $\underset{\text{compact}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists f \in C^\infty(\mathbb{R}^n)$ s.t.

$$f|_C \equiv 1, \quad \operatorname{supp} f \subset U$$

Prel 2 If $\underset{\text{cmpct}}{C} \subset \underset{\text{open}}{U} \subset \mathbb{R}^n$, $\exists$ cmpct D s.t. $C \subset \operatorname{int} D \subset D \subset U$.

Blunder: $\underset{\text{cmpct}}{A} \subset \bigcup_{i=1}^{p} U_i$, U_i open $\Rightarrow \exists$ cmpct $D_i \subset U_i$ w/ $A \subset \bigcup_{i=1}^{p} \operatorname{int} D_i$

board line

Pf $E_1 = A \setminus \bigcup_{i=2}^{p} U_i \subset U_1$, find D_1 w/ $E_1 \subset \operatorname{int} D_1 \subset D_1 \subset U_1$

then $\operatorname{int} D_1 \cup \bigcup_{i>1} U_i \supset A$

Suppose $D_1 \ldots D_{q-1}$ have been found s.t.

$$D_i \subset U_i \quad \& \quad \bigcup_{i<q} \operatorname{int} D_i \cup \bigcup_{i \geq q} U_i \supset A$$

Set $E_q = A \setminus \left(\bigcup_{i<q} \operatorname{int} D_i\right) \setminus \left(\bigcup_{i>q} U_i\right) \subset U_q$ & find D_q

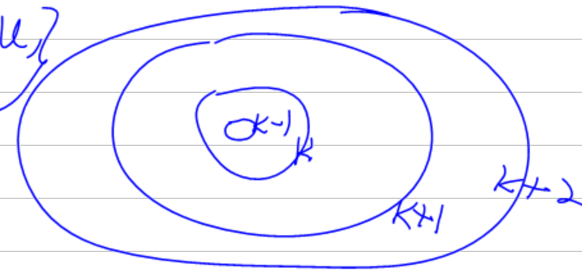
s.t. $E_q \subset \operatorname{int} D_q \subset D_q \subset U_q$

then $\bigcup_{i \leq q} \operatorname{int} D_i \cup \bigcup_{i>q} U_i \supset A$

and we can continue...

<u>Case II</u> $A = \bigcup_{k=0}^{\infty} A_k$, A_k cmpct, $A_k \subset \operatorname{int} A_{k+1}$; $A_0 = \emptyset$

$\mathcal{U}_k = \{ U \cap \operatorname{int} A_{k+2} \cap (A_{k-1})^c : U \in \mathcal{U}, k \geq 1 \}$



Find Φ_k for $A_{k+1} \setminus \operatorname{int} A_k$, let

$\{\bar{\varphi}_i\} = \Phi = \bigcup \Phi_k$ (still countable!)

and set $\varphi_i(x) = \dfrac{\bar{\varphi}_i(x)}{\sum \bar{\varphi}_i(x)}$

P.S. A is open.

<u>Case III</u> A open. Take $A_k = \{ x : |x| \leq k \ \& \ \operatorname{dist}(x, A^c) \geq \frac{1}{k} \}$.

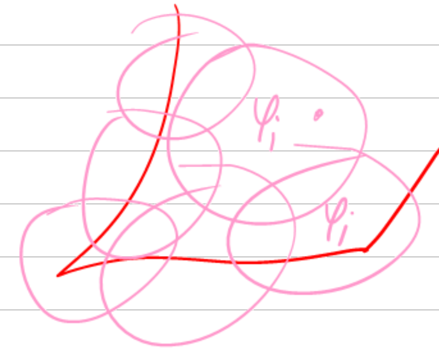
<u>Case IV</u> Any A.

Hour 29 MAT257 Analysis II on November 24, 2021: Partitions of unity (4).
Read Along: Spivak 63-66.
Riddle Along: Cars A,B,C,D drive in the Sahara Desert on generic straight lines and at constant speed; it is known that A meets B (they arrive at the same place at the same time), A meets C, A meets D, B meets C, and B meets D. Does C necessarily meet D?

TA meeting

Thm (PO1) $A \subset \mathbb{R}^n$ (any), $\mathcal{U}$ an open cover of A, $\Rightarrow \exists W \supset A$ (open) & cntble $\Phi = \{\varphi_i : W \to [0,1]\} \subset C^\infty$ s.t.

1. "loc. fin." $\forall x \in W\ \exists$ nbd V s.t. $|\{i : \operatorname{supp}\varphi_i \cap V \neq \emptyset\}| < \infty$
2. "Sum=1" $\forall x \in A$ $\sum \varphi_i(x) = 1$
3. "subordinate" $\forall i\ \exists U \in \mathcal{U}$ $\operatorname{supp}\varphi_i \subset U$.



Prel. 1 Given $C \subset U \subset \mathbb{R}^n$ (C compact, U open), $\exists f \in C^\infty(\mathbb{R}^n)$ s.t. $f|_C = 1$, $\operatorname{supp} f \subset U$

Prel 2 If $C \subset U \subset \mathbb{R}^n$ (C cmpct, U open), $\exists$ compact D s.t. $C \subset \operatorname{int} D \subset D \subset U$.

Blunder: A (cmpct) $\subset \bigcup_{i=1}^{p} U_i$, U_i open $\Rightarrow \exists$ cmpct $D_i \subset U_i$ w/ $A \subset \bigcup_{i=1}^{p} \operatorname{int} D_i$

board line

Pf $E_1 = A \setminus \bigcup_{i=2}^{p} U_i \subset U_1$, find D_1 w/ $E_1 \subset \operatorname{int} D_1 \subset D_1 \subset U_1$

then $\operatorname{int} D_1 \cup \bigcup_{i>1} U_i \supset A$

Suppose $D_1 \dots D_{q-1}$ have been found s.t.

$D_i \subset U_i$ & $\bigcup_{i<q} \operatorname{int} D_i \cup \bigcup_{i \geq q} U_i \supset A$

Set $E_q = A \setminus \left(\bigcup_{i<q} \operatorname{int} D_i\right) \setminus \left(\bigcup_{i>q} U_i\right) \subset U_q$ & find D_q

s.t. $E_q \subset \operatorname{int} D_q \subset D_q \subset U_q$

then $\bigcup_{i \leq q} \operatorname{int} D_i \cup \bigcup_{i>q} U_i \supset A$

and we can continue...

Hour 30 MAT257 Analysis II on November 26, 2021: Integration NT.
Read Along: Spivak 66-74.
Riddle Along: Do you know a device fine enough to tell that that there is a difference of air pressure at your forehead and at your chin?

Goal: $\int_A^{NT} f := \sum_i \int \varphi_i f$

POI: Given $A, \mathcal{U}$ $\exists \varphi_i$ loc fin, sum $=1$, subordinate.

Known: A open. need: A general

border line

do!

Prel 1 (HW7 Q1) $\int_R f+g = \int_R f + \int_R g$ Prel 2 (HW7 Q3) $f \le g \Rightarrow \int_R f \le \int_R g$

Anti-prel 3. $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\dots = \log 2$ $1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots = \frac{\pi}{4}$

Riemann re-arrangement thm: These can be rearranged to get any sum you want! [or nothing at all]

PF using good & evil forces.

True whenever $\sum |a_i| = \infty$, $a_i \to 0$.

157: If a_i is "abs. convergent", $\sum |a_i| < \infty$, rearrangements work!

On to $\int^{NT}$ - - - -

Suppose $\underbrace{f: A}_{\text{not nec. bndd.}} \xrightarrow{\text{open}} \mathbb{R}$ locally bndd but, not necessarily bndd and with disc(f) of meas-0.

Let $\mathcal{U} = \{U\}$ be a cover of A by bndd open sets contained in A and let $\Phi = \{\varphi\}$ be a POI of A sub to $\mathcal{U}$. Then $\forall \varphi \in \Phi$, $\int \varphi |f|$ makes sense. Call f "$(\mathcal{U}, \Phi)$-integrable" if $\sum_i \int \varphi_i |f|$ converges. Then $\sum_i \int \varphi_i f$

where?

is absolutely convergent. Define $\int_A^{(\mathcal{U}, \Phi)} f = \sum_i \int \varphi_i f$

Thm 1. f is $(\mathcal{U},\Phi)$-integ $\iff$ $(\mathcal{U}',\Phi')$-integ

and $\int_A^{(\mathcal{U},\Phi)} f = \int_A^{(\mathcal{U}',\Phi')} f$ (so $\int^{NT}$ makes sense)

2. If A & f are bdd, then f is integ (NT)

3. If also A is Jordan-meas, then $\int_A^{NT} f = \int_A f$.

bdd w/ meas-0 bdry; 1_A integrable

Pf

1. $$\int_A^{(\mathcal{U},\Phi)} g \underset{(1)}{=} \sum_i \int \varphi_i g \underset{(2)}{=} \sum_i \int \left(\sum_j \varphi_j'\right)\varphi_i g \underset{(3)}{=} \sum_i \sum_j \int \varphi_j' \varphi_i g$$

$$\underset{(4)}{=} \sum_j \sum_i \int \varphi_i \varphi_j' g \underset{(3)}{=} \sum_j \int \left(\sum_i \varphi_i\right)\varphi_j' g \underset{(2)}{=} \sum_j \int \varphi_j' g \underset{(1)}{=} \int_A^{(\mathcal{U}',\Phi')} g$$

For $g=|f|$: (1): ignore. (2): sum $=1$! (3): A finite sum! (4): all ≥ 0

For $g=f$: (1) def (2) sum $=1$ (3) a finite sum (4) absolute convergence.

2. If $|f|\leq M$ & $A\subset R$ rect, & if $F\subset\Phi$ is finite,

$$\sum_{\varphi\in F}\int_A \varphi|f| = \int_A \Big(\sum_{\varphi\in F}\varphi\Big)|f| \leq 1\cdot M\cdot \mathrm{vol}(R) \ldots$$

3. If also A is Jordan-measurable, find a compact $C\subset A$ s.t. $\mathrm{vol}(A-C)<\epsilon$. For only finitely many i's, $\operatorname{supp}\varphi_i \cap C \neq 0$; let N be bigger than the biggest of those. Then

$$\left|\int_A f - \sum_{i=1}^N \varphi_i f\right| \leq \int_A \left|f - \sum \varphi_i f\right|$$

$$\leq M\int_A \Big(1-\sum_{i=1}^N \varphi_i\Big) \leq M\int_{A-C} 1 \leq M\epsilon \quad \square$$

Hour 31 MAT257 Analysis II on November 29, 2021: Integration NT (2).
Read Along: Spivak 66-74.
Riddle Along: Back to the cars in the Sahara and to the 99 bands.

$f: \underset{\text{open}}{A} \to \mathbb{R}$ loc. bndd, cont. except meas-0

$\mathcal{U}$: cover of A by bndd open sets contained in A on which f is bndd.

$\Phi = \{\varphi_i\}$: PO1 for A subo. to $\mathcal{U}$

"f is $(\mathcal{U},\Phi)$-integ": $\sum \int \varphi_i |f| < \infty$

In that case, $\int_A^{(\mathcal{U},\Phi)} f := \sum_i \int_A \varphi_i f$

Thm f is $(\mathcal{U},\Phi)$-integ $\Leftrightarrow$ $(\mathcal{U}',\Phi')$-integ in that case, $\int_A^{(\mathcal{U},\Phi)} f = \int_A^{(\mathcal{U}',\Phi')} f$ (so $\int^{NT}$ makes sense)

board line

Pf

1. $$\int_A^{(\mathcal{U},\Phi)} g \underset{(1)}{=} \sum_i \int \varphi_i g \underset{(2)}{=} \sum_i \int \Big(\sum_j \varphi_j'\Big)\varphi_i g \underset{(3)}{=} \sum_i \sum_j \int \varphi_j' \varphi_i g$$
$$\underset{(4)}{=} \sum_j \sum_i \int \varphi_i \varphi_j' g \underset{(3)}{=} \sum_j \int \Big(\sum_i \varphi_i\Big)\varphi_j' g \underset{(2)}{=} \sum_j \int \varphi_j' g \underset{(1)}{=} \int_A^{(\mathcal{U}',\Phi')} g$$

For $g=|f|$: (1): ignore. (2): sum $=1$! (3): A finite sum! (4): all ≥ 0

For $g=f$: (1) def (2) sum $=1$ (3) a finite sum (4) absolute convergence.

Thm 1. If A & f are bndd, then f is integ (NT)

2. If also A is Jordan-meas, then $\int_A^{NT} f = \int_A f$.
bndd w/ meas-0 bdry; 1_A integrable

Pf 1. If $|f| \leq M$ & $A \subset R$ rect, & if $F \subset \Phi$ is finite,

$$\sum_{\varphi \in F} \int_A \varphi |f| = \int_A \Big(\sum_{\varphi \in F} \varphi\Big)|f| \leq 1 \cdot M \cdot \mathrm{vol}(R) \ldots$$

done 11:11

2. If also A is Jordan-measurable, find a compact $C \subset A$ s.t. $\text{Vol}(A-C) < \epsilon$. For only finitely many i's, $\text{supp}\, \varphi_i \cap C \neq 0$; let N be bigger than the biggest of those. Then

$$\left| \int_A f - \sum_{i=1}^{N} \varphi_i f \right| \leq \int_A \left| f - \sum \varphi_i f \right|$$

$$\leq M \int_A \left(1 - \sum_{i=1}^{N} \varphi_i\right) \leq M \int_{A-C} 1 \leq M\epsilon. \quad \square$$

Hour 32 MAT257 Analysis II on December 1, 2021: Integration NT (3).
Read Along: Spivak 66-74.
Riddle Along: Back to the cars in the Sahara and to the 99 bands.

$f: \underset{\text{open}}{A} \to \mathbb{R}$ loc. bndd, cont. except meas-0

$\mathcal{U}$: cover of A by bndd open sets contained in A on which f is bndd.

$\Phi = \{\varphi_i\}$: POI for A subo. to $\mathcal{U}$

"f is NT-integ": $\sum \int \varphi_i |f| < \infty$ & $\int_A^{NT} f := \sum_i \int_A \varphi_i f$

Thm 1. If A & f are bndd, then f is integ (NT) ✓

2. If also A is Jordan-meas, then $\int_A^{NT} f = \int_A f$.

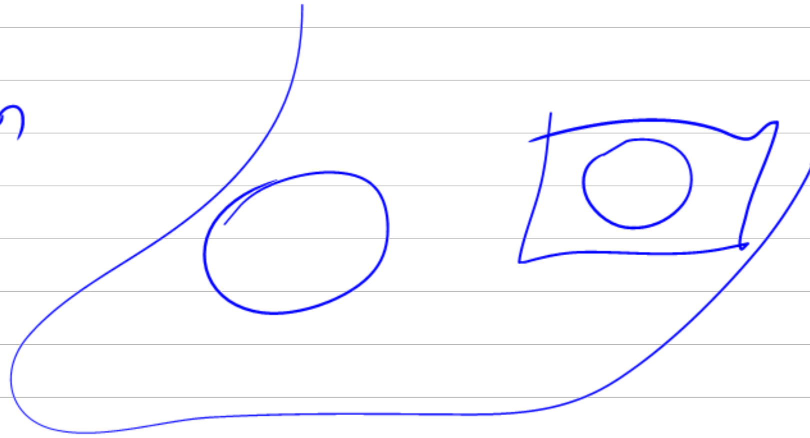
board line

pf
2. If also A is Jordan-measurable, find a compact $C \subset A$ s.t. $\mathrm{Vol}(A-C) < \epsilon$. For only finitely many i's, supp $\varphi_i \cap C \neq \emptyset$; let N be bigger than the biggest of those. Then

$$\left| \int_A f - \sum_{i=1}^{N} \varphi_i f \right| \leq \int_A \left| f - \sum \varphi_i f \right|$$

$$\leq M \int_A \left(1 - \sum_{i=1}^{N} \varphi_i\right) \leq M \int_{A-C} 1 < M\epsilon. \quad \square$$

$A \subset \mathbb{R}^n$
open
$f : A \to \mathbb{R}$
loc bndd
cont. except
meas-0



$$\mathcal{U} = \{ V : V \subset A,\ V \text{ open},\ f \text{ is bndd on } V,\ V \text{ is bndd} \}$$

$\Phi = (\varphi_i)$ a POU for A subo. to $\mathcal{U}$

f is NT-integ : $\sum_i \int \varphi_i |f| < \infty$

$$\int_A^{NT} f = \sum_i \int \varphi_i f$$

Thm The above is indep of $\mathcal{U}, \Phi$

Thm If A is bndd & f is bndd, f is NT-integ

Thm If also A is Jordan-meas, $\text{meas}(\text{bd} A) = 0$
A is bnd.

$$\int_A^{old} f = \int_A^{NT} f.$$

Thm 1. lin of $\int^{NT}$

2. Fubini NT (only for cont. functns)

$A \times B$
$\varphi_i \quad \psi_j$

$\varphi_i(x)\psi_j(y)$

3. $\int_{\mathbb{R}}^{NT} f = \int_{-\infty}^{\infty} f(x)\, dx$

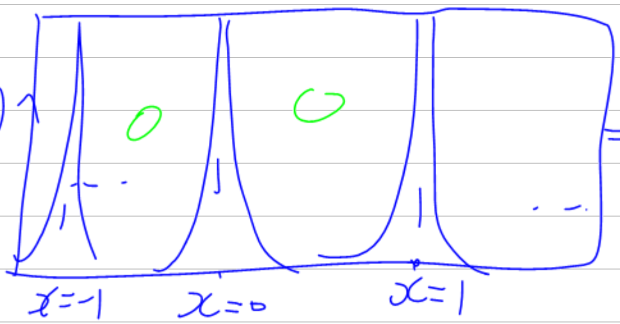
Hour 33 Dec 3 Integration over (4), Changes of Variables (COV)
Spivak 66–74.

$A, F, U, \Phi, \int^{NJ}$

0. Agrees w/ $\int^{old}$ when ~~it~~ can
1. Linear
~~2. Fubini~~

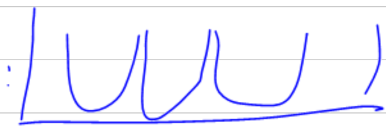
Add: $f \le g \Rightarrow \int f \le \int g$.
Shame on me!

board line.



$=: f_1(x,y)$ Integrable?

$$g_1(x) = \int f_1(x,y)\,dy$$

$g_1(x)$: 

$$F_2(x,y) = \sum_{n=1}^{\infty} \frac{1}{2^n} f_1(nx,y) \qquad g_2(x) = \int f_2(x,y)\,dy$$ will be ∞ on every rational x.

Fact 3. $B \subset A$ (open, open), f integrable on $A \Rightarrow f$ integrable on B.

Pf NTS $\sum_{i=1}^{N} \int_B \varphi_i^B |f|$ is bdd. Indeed $= \int_B \sum_{i=1}^{N} \varphi_i^B |f| = \int_A \sum_1^N \varphi_i^B |f| \le \int_A |f|$ $\square$

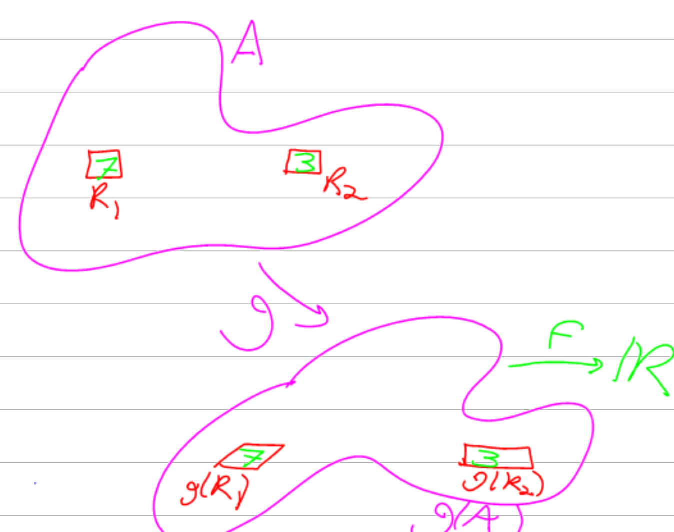
Fact 4. If f is integrable on an open A and $A = \bigcup A_n$, $A_n \subset A_{n+1}$, all open then $\lim_{n\to\infty} \int_{A_n} f = \int_A f$.

Pf will be on HW.

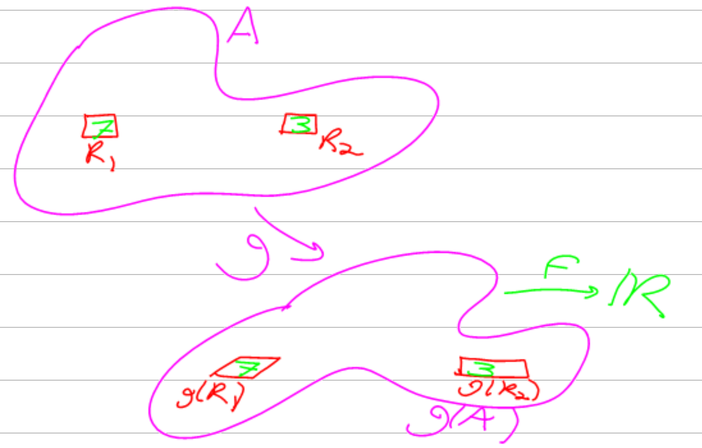
done line

Thm (Change of Variables, "COV")
Let $A \subset \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ cont. diffable, 1–1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $f: g(A) \to \mathbb{R}$ is integrable, then

$$\int_{g(A)} f = \int_A (f \circ g)\,|\det g'|$$



Hour 34 MAT257 Analysis II on December 6, 2021: Changes of variable, Gaussian integration, volumes of spheres and balls.
Read Along: Spivak 66-74.
Note that HW10 is due on Thursday, not Friday!



Thm (Change of Variables, "COV")
Let $A \subset \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ cont. diff'able, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $F: g(A) \to \mathbb{R}$ is integrable, then

$$\int_{g(A)} F = \int_A (F \circ g)\,|\det g'|$$

Compute $I_1 = \int_{\mathbb{R}} e^{-x^2/2}\,dx$, "the most important integral in math".

$$I_2 = \int_{\mathbb{R}^2} e^{-(x^2+y^2)/2}\,dx\,dy \overset{\text{IOU}}{=} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy\, e^{-x^2/2} e^{-y^2/2} = \int dx\, e^{-x^2/2} \int dy\, e^{-y^2/2} = I_1^2$$

$\Big\|$ $x = r\cos\theta$, $y = r\sin\theta$

$$\int dr\,d\theta\, r e^{-r^2/2} \overset{\text{IOU}}{=} 2\pi \int dr\, r e^{-r^2/2} \overset{\text{last class}}{=} 2\pi \cdot 1 = 2\pi \quad \Rightarrow I_1 = \sqrt{2\pi}$$

Paying the debt: <u>Claim</u> If A is an increasing union of open sets: $A = \bigcup A_k$, $A_k \subset A_{k+1}$, A_k open & if $F: A \to \mathbb{R}$ is non-negative $F \geq 0$, loc. bndd, cont except meas 0, and if $\int_{A_k} F$ all exist & are uniformly bndd by M,

then $\int_A F$ exists & is bndd by M.

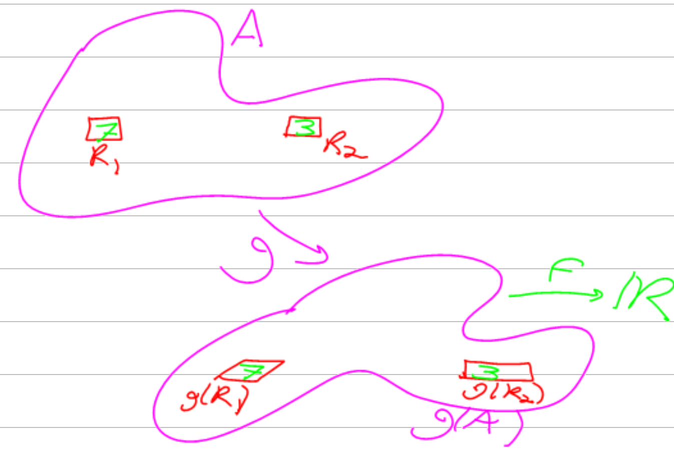
$\Longrightarrow$ I_2 & I_1 both exist, $I_1 = \lim_{R\to\infty} \int_{-R}^{R} e^{-x^2/2}\,dx$

use $\frac{x^2}{2} \geq x - 1$ $\qquad$ $I_2 = \lim_{R\to\infty} \cdots$

Hour 35 MAT257 Analysis II on December 8, 2021: COV(2), IOUs, volumes of spheres and balls.
Read Along: Spivak 66-74.
Note that HW10 is due on Thursday, not Friday!
Riddle Along: Handout 1206.nb

<u>Thm</u> (Change of Variables, "COV") Let $A \subset \mathbb{R}^n$ be open, $g: A \to \mathbb{R}^n$ cont. diffable, 1-1, and s.t. $\forall x \in A$ $g'(x)$ is invertible. If $F: g(A) \to \mathbb{R}$ is integrable, then

$$\int_{g(A)} F = \int_A (F \circ g)|\det g'|$$



In $\int_{\mathbb{R}} e^{-x^2/2}dx$ IOU$_1$: $\displaystyle\int_{\mathbb{R}\times\mathbb{R}} e^{-\frac{x^2+y^2}{2}}dx\,dy \overset{?}{=} \left(\int_{\mathbb{R}} e^{-x^2/2}dx\right)\left(\int_{\mathbb{R}} e^{-y^2/2}dy\right)$

board line

$\uparrow N \to \infty$ $\quad$ $\uparrow N \to \infty$ $\quad$ $\uparrow N \to \infty$

b_P increases $b_P = \int\left(\sum_{i=1}^{P}\varphi_i\right)F$ to I_2; a_N increases; NTS, to the same thing

$$a_N := \int_{[-N,N]^2} e^{-\frac{x^2+y^2}{2}}dx\,dy \overset{\text{Fubini}}{=} \left(\int_{[-N,N]}\quad\right)\left(\int_{[-N,N]}\quad\right)$$

$\forall N \exists P$ $b_P \geq a_N$ $\quad$ $\forall P \exists N$ $a_N > b_P$ ... $\square$.

also on board

$A = [0, \cancel{\infty}R) \times [0, 2\pi] \Longleftrightarrow A' = (0, \cancel{\infty}R) \times (0, 2\pi)$

You can always ignore closed sets of meas. 0.

Let's compute like physicists!

$\sigma_n : \mathrm{Vol}(S^n) \quad S^n = \{z \in \mathbb{R}^{n+1} : |z| = 1\}$

$$(2\pi)^{\frac{n+1}{2}} = I_{n+1} = \int_{\mathbb{R}^{n+1}} e^{-|z|^2/2} dz = \sigma_n \int_0^\infty r^n e^{-r^2/2} dr = \sigma_n \tau_n$$

$$\tau_{n-2} = \int_0^\infty r^{n-2} e^{-r^2/2} = \frac{1}{n-1} \int_0^\infty r^n e^{-r^2/2} = \frac{1}{n-1} \tau_n \quad \text{so}$$

$$\sigma_n = \frac{(2\pi)^{(n+1)/2}}{\tau_n} = 2\pi \frac{(2\pi)^{(n-1)/2}}{(n-1)\tau_{n-2}} = \frac{2\pi}{n-1}\sigma_{n-2}$$

$\sigma_0 = 2 \quad \beta_0 = \emptyset$
$\sigma_1 = 2\pi \quad \beta_1 = 2$
$\sigma_2 = 4\pi \quad \beta_2 = \pi$
$\sigma_3 = 2\pi^2 \quad \beta_3 = 4\pi/3$
$\vdots$

And $\beta_n = \mathrm{Vol}(B_n) = \dfrac{\sigma_{n-1}}{n}$

Proof of the COV formula:

And globally? Duh? so what?

"Locally, g is mostly a composition of

layer-preserving transformations"

what?
why?
so what?

Hour 36 MAT257 Analysis II on December 9, 2021: My Favourite First-Year Analysis Theorem
(Following http://www.math.toronto.edu/~drorbn/Talks/MAASeaway-1810/)