Pensieve header: The WG Algebra with testing.

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In[•]:=
```
DeclareGroup[S_k_] := Module[{α, β, e, γs},
  Clear[G, n, g, ι, m, inv];
  G = PermutationCycles /@ (Permutations@Range@k);
  n = Length[G];
  Do[g[α] = e = G[[α]]; ι[e] = α, {α, n}];
  m[] = ι[Cycles[{}]];
  Do[m[α, β] = ι[g[α]~PermutationProduct~g[β]], {α, n}, {β, n}];
  m[α_] := α; m[α_, β_, γs__] := m[m[α, β], γs];
  Do[inv[α] = ι[InversePermutation[g[α]]], {α, n}]
 ]
```

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In[•]:=
```
DeclareGroup[S4];
Table[m[i, j], {i, n}, {j, n}] // MatrixForm
```

Out[•]//MatrixForm=
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$$\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 \\
2 & 1 & 4 & 3 & 6 & 5 & 8 & 7 & 10 & 9 & 12 & 11 & 14 & 13 & 16 & 15 & 18 & 17 & 20 & 19 & 22 & 21 & 24 & 23 \\
3 & 5 & 1 & 6 & 2 & 4 & 9 & 11 & 7 & 12 & 8 & 10 & 15 & 17 & 13 & 18 & 14 & 16 & 21 & 23 & 19 & 24 & 20 & 22 \\
4 & 6 & 2 & 5 & 1 & 3 & 10 & 12 & 8 & 11 & 7 & 9 & 16 & 18 & 14 & 17 & 13 & 15 & 22 & 24 & 20 & 23 & 19 & 21 \\
5 & 3 & 6 & 1 & 4 & 2 & 11 & 9 & 12 & 7 & 10 & 8 & 17 & 15 & 18 & 13 & 16 & 14 & 23 & 21 & 24 & 19 & 22 & 20 \\
6 & 4 & 5 & 2 & 3 & 1 & 12 & 10 & 11 & 8 & 9 & 7 & 18 & 16 & 17 & 14 & 15 & 13 & 24 & 22 & 23 & 20 & 21 & 19 \\
7 & 8 & 13 & 14 & 19 & 20 & 1 & 2 & 15 & 16 & 21 & 22 & 3 & 4 & 9 & 10 & 23 & 24 & 5 & 6 & 11 & 12 & 17 & 18 \\
8 & 7 & 14 & 13 & 20 & 19 & 2 & 1 & 16 & 15 & 22 & 21 & 4 & 3 & 10 & 9 & 24 & 23 & 6 & 5 & 12 & 11 & 18 & 17 \\
9 & 11 & 15 & 17 & 21 & 23 & 3 & 5 & 13 & 18 & 19 & 24 & 1 & 6 & 7 & 12 & 20 & 22 & 2 & 4 & 8 & 10 & 14 & 16 \\
10 & 12 & 16 & 18 & 22 & 24 & 4 & 6 & 14 & 17 & 20 & 23 & 2 & 5 & 8 & 11 & 19 & 21 & 1 & 3 & 7 & 9 & 13 & 15 \\
11 & 9 & 17 & 15 & 23 & 21 & 5 & 3 & 18 & 13 & 24 & 19 & 6 & 1 & 12 & 7 & 22 & 20 & 4 & 2 & 10 & 8 & 16 & 14 \\
12 & 10 & 18 & 16 & 24 & 22 & 6 & 4 & 17 & 14 & 23 & 20 & 5 & 2 & 11 & 8 & 21 & 19 & 3 & 1 & 9 & 7 & 15 & 13 \\
13 & 19 & 7 & 20 & 8 & 14 & 15 & 21 & 1 & 22 & 2 & 16 & 9 & 23 & 3 & 24 & 4 & 10 & 11 & 17 & 5 & 18 & 6 & 12 \\
14 & 20 & 8 & 19 & 7 & 13 & 16 & 22 & 2 & 21 & 1 & 15 & 10 & 24 & 4 & 23 & 3 & 9 & 12 & 18 & 6 & 17 & 5 & 11 \\
15 & 21 & 9 & 23 & 11 & 17 & 13 & 19 & 3 & 24 & 5 & 18 & 7 & 20 & 1 & 22 & 6 & 12 & 8 & 14 & 2 & 16 & 4 & 10 \\
16 & 22 & 10 & 24 & 12 & 18 & 14 & 20 & 4 & 23 & 6 & 17 & 8 & 19 & 2 & 21 & 5 & 11 & 7 & 13 & 1 & 15 & 3 & 9 \\
17 & 23 & 11 & 21 & 9 & 15 & 18 & 24 & 5 & 19 & 3 & 13 & 12 & 22 & 6 & 20 & 1 & 7 & 10 & 16 & 4 & 14 & 2 & 8 \\
18 & 24 & 12 & 22 & 10 & 16 & 17 & 23 & 6 & 20 & 4 & 14 & 11 & 21 & 5 & 19 & 2 & 8 & 9 & 15 & 3 & 13 & 1 & 7 \\
19 & 13 & 20 & 7 & 14 & 8 & 21 & 15 & 22 & 1 & 16 & 2 & 23 & 9 & 24 & 3 & 10 & 4 & 17 & 11 & 18 & 5 & 12 & 6 \\
20 & 14 & 19 & 8 & 13 & 7 & 22 & 16 & 21 & 2 & 15 & 1 & 24 & 10 & 23 & 4 & 9 & 3 & 18 & 12 & 17 & 6 & 11 & 5 \\
21 & 15 & 23 & 9 & 17 & 11 & 19 & 13 & 24 & 3 & 18 & 5 & 20 & 7 & 22 & 1 & 12 & 6 & 14 & 8 & 16 & 2 & 10 & 4 \\
22 & 16 & 24 & 10 & 18 & 12 & 20 & 14 & 23 & 4 & 17 & 6 & 19 & 8 & 21 & 2 & 11 & 5 & 13 & 7 & 15 & 1 & 9 & 3 \\
23 & 17 & 21 & 11 & 15 & 9 & 24 & 18 & 19 & 5 & 13 & 3 & 22 & 12 & 20 & 6 & 7 & 1 & 16 & 10 & 14 & 4 & 8 & 2 \\
24 & 18 & 22 & 12 & 16 & 10 & 23 & 17 & 20 & 6 & 14 & 4 & 21 & 11 & 19 & 5 & 8 & 2 & 15 & 9 & 13 & 3 & 7 & 1
\end{pmatrix}$$

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In[•]:=
```
Basis[] = {1};
Basis[i_, is___] := Flatten@Table[W_i[α, β] Basis[is], {α, n}, {β, n}]
```

In[•]:=
```
Basis[left]
```

Out[•]=
```
{W_left[1, 1], W_left[1, 2], W_left[1, 3], W_left[1, 4], W_left[1, 5], W_left[1, 6],
 W_left[2, 1], W_left[2, 2], W_left[2, 3], W_left[2, 4], W_left[2, 5], W_left[2, 6],
 W_left[3, 1], W_left[3, 2], W_left[3, 3], W_left[3, 4], W_left[3, 5], W_left[3, 6],
 W_left[4, 1], W_left[4, 2], W_left[4, 3], W_left[4, 4], W_left[4, 5], W_left[4, 6],
 W_left[5, 1], W_left[5, 2], W_left[5, 3], W_left[5, 4], W_left[5, 5], W_left[5, 6],
 W_left[6, 1], W_left[6, 2], W_left[6, 3], W_left[6, 4], W_left[6, 5], W_left[6, 6]}
```

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In[•]:=
```
Basis[mid, rgt]
```

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Out[•]=
```
{W_mid[1, 1] W_rgt[1, 1], W_mid[1, 1] W_rgt[1, 2], W_mid[1, 1] W_rgt[1, 3],
 ··· 1290 ···, W_mid[6, 6] W_rgt[6, 4], W_mid[6, 6] W_rgt[6, 5], W_mid[6, 6] W_rgt[6, 6]}
```

large output | show less | show more | show all | set size limit...

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In[•]:=
```
m_i_,j_→k_[ℰ_] :=
  Expand[ℰ] /. W_i[α_, β_] W_j[γ_, δ_] :> If[m[α, β] == m[β, γ], W_k[α, m[β, δ]], 0];
η_i_[ℰ_] := Expand[ℰ Sum[W_i[α, m[]], {α, n}]];
```

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In[•]:=
```
Δ_i_→j_,k_[ℰ_] := Expand[ℰ /. W_i[α_, β_] :> Sum[W_j[γ, β] W_k[m[α, inv[γ]], β], {γ, n}]];
ε_i_[ℰ_] := Expand[ℰ /. W_i[α_, β_] :> If[α == m[], 1, 0]];
```

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In[•]:=
```
S_i_[ℰ_] := Expand[ℰ /. W_i[α_, β_] :> W_i[m[inv[β], inv[α], β], inv[β]]];
```

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In[•]:=
```
R_i_,j_ := Sum[W_i[α, m[]] W_j[β, α], {α, n}, {β, n}];
R̄_i_,j_ := Sum[W_i[α, m[]] W_j[β, inv@α], {α, n}, {β, n}];
```

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In[•]:=
```
b = Basis[1, 2, 3]; (b // m_1,2→1 // m_1,3→1) == (b // m_2,3→2 // m_1,2→1)
```

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Out[•]=
```
True
```

In[•]:=
```
b = Basis[1, 2, 3]; (b // m_2,1→1 // m_1,3→1) == (b // m_2,3→2 // m_1,2→1)
```

Out[•]=
```
{W_1[1, 1], W_1[1, 2], W_1[1, 3], W_1[1, 4], W_1[1, 5],
  W_1[1, 6], 0, 0, 0, 0, 0, 0, 0, 0, ··· 46628 ···, 0, 0, 0, 0, 0, 0, 0, 0,
  W_1[6, 1], W_1[6, 2], W_1[6, 3], W_1[6, 4], W_1[6, 5], W_1[6, 6]} == ··· 1 ···
```

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```
In[•]:= b = Basis[1]; (b // η2 // m1,2→1) == b == (b // η2 // m1,2→1)
```

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```
In[•]:= b = Basis[1]; (b // Δ1→1,2 // Δ2→2,3) == (b // Δ1→1,3 // Δ1→1,2)
```

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```
In[•]:= b = Basis[1]; (b // Δ1→1,2 // ε2) == b == (b // Δ1→2,1 // ε2)
```

pdf

```
In[•]:= b = Basis[1, 2]; (b // ε1 // ε2) == (b // m1,2→1 // ε1)
```

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```
In[•]:= b = Basis[1, 3]; (b // Δ1→1,2 // Δ3→3,4 // m1,3→1 // m2,4→2) == (b // m1,3→1 // Δ1→1,2)
```

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```
In[•]:= b = Basis[1]; (b // S1 // S1) == b
```

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```
Out[•]= True
```

pdf

```
In[•]:= b = Basis[1]; (b // Δ1→1,2 // S2 // m1,2→1) == (b // ε1 // η1) == (b // Δ1→1,2 // S1 // m1,2→1)
```

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```
In[•]:= (R1,2 R̄3,4 // m1,3→1 // m2,4→2) == (1 // η1 // η2) == (R1,2 R̄3,4 // m1,3→1 // m4,2→2)
```

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```
Out[•]= True
```

pdf

```
In[•]:= (R1,2 R4,3 R5,6 // m1,4→1 // m2,5→2 // m3,6→3) == (R2,3 R1,4 R5,6 // m1,5→1 // m2,6→2 // m3,4→3)
```

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```
Out[•]= True
```

pdf

```
In[•]:= {(R1,3 // Δ1→1,2) == (R2,3 R1,4 // m3,4→3), (R1,2 // Δ2→2,3) == (R0,2 R1,3 // m0,1→1)}
```

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```
In[•]:= {(R1,2 // ε1) == (1 // η2), (R1,2 // ε2) == (1 // η1)}
```

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```
In[•]:= (R1,2 // S1) == R̄1,2 == (R1,2 // S2)
```

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Does R1 hold?

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```
In[•]:= {R1,2 // m1,2→1, 1 // η1}
```

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```
In[•]:= Ks = {PD[X[1, 4, 2, 5], X[3, 6, 4, 1], X[5, 2, 6, 3]],
    PD[X[4, 2, 5, 1], X[8, 6, 1, 5], X[6, 3, 7, 4], X[2, 7, 3, 8]],
    PD[X[1, 6, 2, 7], X[3, 8, 4, 9], X[5, 10, 6, 1], X[7, 2, 8, 3], X[9, 4, 10, 5]],
    PD[X[1, 4, 2, 5], X[3, 8, 4, 9], X[5, 10, 6, 1], X[9, 6, 10, 7], X[7, 2, 8, 3]]};
```

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```
In[•]:= Z[pd_PD] := Module[{z},
   z = Expand[Times @@ pd /. x : X[i_, j_, k_, l_] :> If[PositiveQ@x, Rl,i, R̄j,i]];
   Do[z = z // mi,k→1, {k, 2 Length@pd}];
   z]
```

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In[•]:=
```
Table[K → Echo[Timing[Z[K]]], {K, Ks}]
```