Pensieve header: The WG Algebra: testing, knots, optimization.

Loading Knot Data

In[∘]:=
```
Once[<< KnotTheory`]
```

```
Loading KnotTheory` version of February 2, 2020, 10:53:45.2097.
Read more at http://katlas.org/wiki/KnotTheory.
```

Defining a Group

In[∘]:=
```
DeclareGroup[S_k_] := Module[{α, β, e, γs},
  Clear[G, n, g, ι, m, inv];
  G = PermutationCycles /@ (Permutations@Range@k);
  n = Length[G];
  Do[g[α] = e = G[[α]]; ι[e] = α, {α, n} ];
  m[] = ι[Cycles[{}]];
  Do[m[α, β] = ι[g[α] ~PermutationProduct~ g[β]], {α, n}, {β, n} ];
  m[α_] := α; m[α_, β_, γs__] := m[m[α, β], γs];
  Do[inv[α] = ι[InversePermutation[g[α]]], {α, n}]
 ]
```

In[∘]:=
```
DeclareGroup[S_3];
Table[m[i, j], {i, n}, {j, n}] // MatrixForm
```

Out[∘]//MatrixForm=

1	2	3	4	5	6
2	1	4	3	6	5
3	5	1	6	2	4
4	6	2	5	1	3
5	3	6	1	4	2
6	4	5	2	3	1

Defining WG

In[∘]:=
```
Basis[] = {1};
Basis[i_, is___] := Flatten@Table[W_i[α, β] Basis[is], {α, n}, {β, n}]
```

In[∘]:=
```
Basis[1, 2]
```

Out[∘]=
```
{W1[1, 1] W2[1, 1], W1[1, 1] W2[1, 2], W1[1, 1] W2[1, 3], W1[1, 1] W2[1, 4], W1[1, 1] W2[1, 5],
 W1[1, 1] W2[1, 6], W1[1, 1] W2[2, 1], W1[1, 1] W2[2, 2], W1[1, 1] W2[2, 3], … 1278 …,
 W1[6, 6] W2[5, 4], W1[6, 6] W2[5, 5], W1[6, 6] W2[5, 6], W1[6, 6] W2[6, 1], W1[6, 6] W2[6, 2],
 W1[6, 6] W2[6, 3], W1[6, 6] W2[6, 4], W1[6, 6] W2[6, 5], W1[6, 6] W2[6, 6]}
```

large output | show less | show more | show all | set size limit...

In[∘]:=
```
m_{i_,j_→k_}[ℰ_] :=
  Expand[ℰ] /. W_i[α_, β_] W_j[γ_, δ_] ⧴ If[m[α, β] == m[β, γ], W_k[α, m[β, δ]], 0];
η_{i_}[ℰ_] := Expand[ℰ Sum[W_i[α, m[]], {α, n}]];
```

In[∘]:=
```
Δ_{i_→j_,k_}[ℰ_] := Expand[ℰ /. W_i[α_, β_] ⧴ Sum[W_j[γ, β] W_k[m[α, inv[γ]], β], {γ, n}]];
ε_{i_}[ℰ_] := Expand[ℰ /. W_i[α_, β_] ⧴ If[α == m[], 1, 0]];
```

In[∘]:=
```
S_{i_}[ℰ_] := Expand[ℰ /. W_i[α_, β_] ⧴ W_i[m[inv[β], inv[α], β], inv[β]]];
```

In[∘]:=
```
R_{i_,j_} := Sum[W_i[α, m[]] W_j[β, α], {α, n}, {β, n}];
R̄_{i_,j_} := Sum[W_i[α, m[]] W_j[β, inv@α], {α, n}, {β, n}];
```

Testing the Axioms of an IHOP+R

m is associative:

In[∘]:=
```
b = Basis[1, 2, 3]; (b // m_{1,2→1} // m_{1,3→1}) == (b // m_{2,3→2} // m_{1,2→1})
```

Out[∘]= True

η is a unit:

In[∘]:=
```
b = Basis[1]; (b // η_2 // m_{1,2→1}) == b == (b // η_2 // m_{1,2→1})
```

Out[∘]= True

Δ is co-associative:

In[∘]:=
```
b = Basis[1]; (b // Δ_{1→1,2} // Δ_{2→2,3}) == (b // Δ_{1→1,3} // Δ_{1→1,2})
```

Out[∘]= True

ε is a co-unit:

In[∘]:=
```
b = Basis[1]; (b // Δ_{1→1,2} // ε_2) == b == (b // Δ_{1→2,1} // ε_2)
```

Out[∘]= True

ε is an algebra morphism:

```
In[◦]:= b = Basis[1, 2]; (b // ε₁ // ε₂) == (b // m_{1,2→1} // ε₁)
Out[◦]= True
```

m is an algebra morphism:

```
In[◦]:= b = Basis[1, 3]; (b // Δ_{1→1,2} // Δ_{3→3,4} // m_{1,3→1} // m_{2,4→2}) == (b // m_{1,3→1} // Δ_{1→1,2})
Out[◦]= True
```

S is an algebra anti-morphism:

```
In[◦]:= b = Basis[1, 2]; (b // m_{1,2→1} // S₁) == (b // S₁ // S₂ // m_{2,1→1})
Out[◦]= True
```

S is a co-algebra anti-morphism:

```
In[◦]:= b = Basis[1]; (b // S₁ // Δ_{1→1,2}) == (b // Δ_{1→2,1} // S₁ // S₂)
Out[◦]= True
```

S is a convolution inverse of the identity:

```
In[◦]:= b = Basis[1]; (b // Δ_{1→1,2} // S₂ // m_{1,2→1}) == (b // ε₁ // η₁) == (b // Δ_{1→1,2} // S₁ // m_{1,2→1})
Out[◦]= True
```

S is involutive:

```
In[◦]:= b = Basis[1]; (b // S₁ // S₁) == b
Out[◦]= True
```

Reidemeister 2:

```
In[◦]:= (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2}) == (1 // η₁ // η₂) == (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{4,2→2})
Out[◦]= True
```

Reidemeister 3:

```
In[◦]:= (R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) == (R_{2,3} R_{1,4} R_{5,6} // m_{1,5→1} // m_{2,6→2} // m_{3,4→3})
Out[◦]= True
```

Compatibility of *R* and (Δ, *m*):

```
In[◦]:= {(R_{1,3} // Δ_{1→1,2}) == (R_{2,3} R_{1,4} // m_{3,4→3}), (R_{1,2} // Δ_{2→2,3}) == (R_{0,2} R_{1,3} // m_{0,1→1})}
Out[◦]= {True, True}
```

Compatibility of *R* and *ε*:

```
In[◦]:= {(R_{1,2} // ε₁) == (1 // η₂), (R_{1,2} // ε₂) == (1 // η₁)}
Out[◦]= {True, True}
```

Compatibility of *R* and *S*:

```
In[•]:= (R_{1,2} // S_1) == R̄_{1,2} == (R_{1,2} // S_2)
Out[•]= True
```

Does R1 hold?

```
In[•]:= {R_{1,2} // m_{1,2→1}, 1 // η_1}
Out[•]= {W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 4] + W_1[5, 5] + W_1[6, 6],
  W_1[1, 1] + W_1[2, 1] + W_1[3, 1] + W_1[4, 1] + W_1[5, 1] + W_1[6, 1]}
```

Computing the Knot Invariant for S_3

```
In[•]:= Z1[K_] := Z1[PD[K]];
Z1[pd_PD] := Module[{z},
  z = Expand[Times @@ pd /. x : X[i_, j_, k_, l_] :→ If[PositiveQ@x, R_{l,i}, R̄_{j,i}]];
  Do[z = z // m_{1,k→1}, {k, 2 Length@pd}];
  z]
```

```
In[•]:= tab1 = Table[K → Echo[Timing[Z1[K]]], {K, AllKnots[{3, 5}]}]
» {1.03125, W_1[1, 1] + 3 W_1[2, 2] + 3 W_1[3, 3] + W_1[4, 1] + W_1[5, 1] + 3 W_1[6, 6]}
Out[•]= $Aborted
```

Edge-vertex convention: an oriented edge carries the same label as the vertex ending it.

```
In[•]:= Z2[K_] := Z2[PD@K];
Z2[pd_PD] := Module[{z, done, st, c, mn, k},
  z = 1; done = {}; st = Range[2 Length@pd];
  Do[
   z *= c /. X[i_, j_, _, l_] :→ If[PositiveQ@c, mn = {i, l}; R_{l,i}, mn = {i, j};
        R̄_{j,i}];
   Do[
    If[MemberQ[done, k + 1], z = z // m_{k,k+1→k}; st = st /. k + 1 → k];
    If[MemberQ[done, k - 1], z = z // m_{st[[k-1]],k→st[[k-1]]}; st = st /. k → st[[k - 1]]],
    {k, mn}];
   done = done ⋃ mn,
   {c, List @@ pd}];
  z ]
```

```
tab2 = Table[K → Echo[Timing[Z2[K]]], {K, AllKnots[{3, 6}]}]
```

```
» {0.125, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 1] + W1[5, 1] + 3 W1[6, 6]}
» {0.15625, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]}
» {6.35938, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]}
» {1.34375, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]}
» {1.875, W1[1, 1] + 3 W1[2, 1] + 3 W1[3, 1] + W1[4, 4] + W1[5, 5] + 3 W1[6, 1]}
» {1.3125, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 4] + W1[5, 5] + W1[6, 1]}
» {2.45313, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]}
```

```
Out[]= {Knot[3, 1] → {0.125, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 1] + W1[5, 1] + 3 W1[6, 6]},
  Knot[4, 1] → {0.15625, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]},
  Knot[5, 1] → {6.35938, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]},
  Knot[5, 2] → {1.34375, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]},
  Knot[6, 1] → {1.875, W1[1, 1] + 3 W1[2, 1] + 3 W1[3, 1] + W1[4, 4] + W1[5, 5] + 3 W1[6, 1]},
  Knot[6, 2] → {1.3125, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 4] + W1[5, 5] + W1[6, 1]},
  Knot[6, 3] → {2.45313, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]}}
```

```
In[]:= ThinPosition[K_] := Module[{todo, done, pd, c},
   todo = List @@ PD@K; done = {}; pd = PD[];
   While[todo =!= {},
    AppendTo[pd, c = RandomChoice@MaximalBy[todo, Length[done ⋂ List @@ #] &]];
    todo = DeleteCases[todo, c];
    done = done ⋃ List @@ c];
   pd
  ]
```

```
In[]:= Z3[K_] := Z2@ThinPosition@K;
```

```
In[]:= tab3 = Table[K → Echo[Timing[Z3[K]]], {K, AllKnots[{3, 7}]}]
```

```
» {0.125, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 1] + W1[5, 1] + 3 W1[6, 6]}
» {0.15625, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]}
» {0.0625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]}
» {0.203125, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]}
» {0.203125, W1[1, 1] + 3 W1[2, 1] + 3 W1[3, 1] + W1[4, 4] + W1[5, 5] + 3 W1[6, 1]}
» {0.0625, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 4] + W1[5, 5] + W1[6, 1]}
» {0.171875, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]}
» {0.140625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]}
» {0.203125, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]}
» {0.0625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]}
» {0.296875, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 4] + W1[5, 5] + 3 W1[6, 6]}
» {0.296875, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]}
» {0.203125, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 1] + W1[5, 1] + W1[6, 6]}
» {0.21875, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 4] + W1[5, 5] + 3 W1[6, 6]}

Out[]= {Knot[3, 1] → {0.125, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 1] + W1[5, 1] + 3 W1[6, 6]},
  Knot[4, 1] → {0.15625, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]},
  Knot[5, 1] → {0.0625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]},
  Knot[5, 2] → {0.203125, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]},
  Knot[6, 1] → {0.203125, W1[1, 1] + 3 W1[2, 1] + 3 W1[3, 1] + W1[4, 4] + W1[5, 5] + 3 W1[6, 1]},
  Knot[6, 2] → {0.0625, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 4] + W1[5, 5] + W1[6, 1]},
  Knot[6, 3] → {0.171875, W1[1, 1] + W1[2, 1] + W1[3, 1] + W1[4, 1] + W1[5, 1] + W1[6, 1]},
  Knot[7, 1] → {0.140625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]},
  Knot[7, 2] → {0.203125, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]},
  Knot[7, 3] → {0.0625, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 4] + W1[5, 5] + W1[6, 6]},
  Knot[7, 4] → {0.296875, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 4] + W1[5, 5] + 3 W1[6, 6]},
  Knot[7, 5] → {0.296875, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 5] + W1[5, 4] + W1[6, 6]},
  Knot[7, 6] → {0.203125, W1[1, 1] + W1[2, 2] + W1[3, 3] + W1[4, 1] + W1[5, 1] + W1[6, 6]},
  Knot[7, 7] → {0.21875, W1[1, 1] + 3 W1[2, 2] + 3 W1[3, 3] + W1[4, 4] + W1[5, 5] + 3 W1[6, 6]}}
```

Computing the Knot Invariant for S_4

In[◦]:=
```
DeclareGroup[S4];
Table[m[i, j], {i, n}, {j, n}] // MatrixForm
```

Out[◦]//MatrixForm=

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24
2	1	4	3	6	5	8	7	10	9	12	11	14	13	16	15	18	17	20	19	22	21	24	23
3	5	1	6	2	4	9	11	7	12	8	10	15	17	13	18	14	16	21	23	19	24	20	22
4	6	2	5	1	3	10	12	8	11	7	9	16	18	14	17	13	15	22	24	20	23	19	21
5	3	6	1	4	2	11	9	12	7	10	8	17	15	18	13	16	14	23	21	24	19	22	20
6	4	5	2	3	1	12	10	11	8	9	7	18	16	17	14	15	13	24	22	23	20	21	19
7	8	13	14	19	20	1	2	15	16	21	22	3	4	9	10	23	24	5	6	11	12	17	18
8	7	14	13	20	19	2	1	16	15	22	21	4	3	10	9	24	23	6	5	12	11	18	17
9	11	15	17	21	23	3	5	13	18	19	24	1	6	7	12	20	22	2	4	8	10	14	16
10	12	16	18	22	24	4	6	14	17	20	23	2	5	8	11	19	21	1	3	7	9	13	15
11	9	17	15	23	21	5	3	18	13	24	19	6	1	12	7	22	20	4	2	10	8	16	14
12	10	18	16	24	22	6	4	17	14	23	20	5	2	11	8	21	19	3	1	9	7	15	13
13	19	7	20	8	14	15	21	1	22	2	16	9	23	3	24	4	10	11	17	5	18	6	12
14	20	8	19	7	13	16	22	2	21	1	15	10	24	4	23	3	9	12	18	6	17	5	11
15	21	9	23	11	17	13	19	3	24	5	18	7	20	1	22	6	12	8	14	2	16	4	10
16	22	10	24	12	18	14	20	4	23	6	17	8	19	2	21	5	11	7	13	1	15	3	9
17	23	11	21	9	15	18	24	5	19	3	13	12	22	6	20	1	7	10	16	4	14	2	8
18	24	12	22	10	16	17	23	6	20	4	14	11	21	5	19	2	8	9	15	3	13	1	7
19	13	20	7	14	8	21	15	22	1	16	2	23	9	24	3	10	4	17	11	18	5	12	6
20	14	19	8	13	7	22	16	21	2	15	1	24	10	23	4	9	3	18	12	17	6	11	5
21	15	23	9	17	11	19	13	24	3	18	5	20	7	22	1	12	6	14	8	16	2	10	4
22	16	24	10	18	12	20	14	23	4	17	6	19	8	21	2	11	5	13	7	15	1	9	3
23	17	21	11	15	9	24	18	19	5	13	3	22	12	20	6	7	1	16	10	14	4	8	2
24	18	22	12	16	10	23	17	20	6	14	4	21	11	19	5	8	2	15	9	13	3	7	1

In[◦]:=
```
Table[K → Echo[Timing[Z3[K]]], {K, AllKnots[{3, 7}]}]
```

» $\{4.57813, W_1[1,1]+5W_1[2,2]+5W_1[3,3]+4W_1[4,1]+4W_1[5,1]+5W_1[6,6]+5W_1[7,7]+W_1[8,8]+4W_1[9,1]+W_1[10,10]+4W_1[10,19]+W_1[11,11]+4W_1[11,14]+4W_1[12,1]+4W_1[13,1]+4W_1[14,11]+W_1[14,14]+5W_1[15,15]+4W_1[16,1]+W_1[17,17]+W_1[18,18]+4W_1[18,23]+4W_1[19,10]+W_1[19,19]+4W_1[20,1]+4W_1[21,1]+5W_1[22,22]+4W_1[23,18]+W_1[23,23]+W_1[24,24]\}$

» $\{7.1875, W_1[1,1]+W_1[2,1]+W_1[3,1]+4W_1[4,1]+4W_1[5,1]+W_1[6,1]+W_1[7,1]+W_1[8,1]+4W_1[9,1]+W_1[10,1]+W_1[11,1]+4W_1[12,1]+4W_1[13,1]+W_1[14,1]+W_1[15,1]+4W_1[16,1]+W_1[17,1]+W_1[18,1]+W_1[19,1]+4W_1[20,1]+4W_1[21,1]+W_1[22,1]+W_1[23,1]+W_1[24,1]\}$

» $\{11.2031, W_1[1,1]+W_1[2,2]+W_1[3,3]+W_1[4,4]+W_1[5,5]+W_1[6,6]+W_1[7,7]+W_1[8,8]+W_1[9,9]+W_1[10,19]+W_1[11,14]+W_1[12,12]+W_1[13,13]+W_1[14,11]+W_1[15,15]+W_1[16,16]+W_1[17,17]+W_1[18,23]+W_1[19,10]+W_1[20,20]+W_1[21,21]+W_1[22,22]+W_1[23,18]+W_1[24,24]\}$

Out[◦]= $Aborted

Computing the knot invariant for S_4 by summing over conjugacy classes

In[◦]:=
```
DeclareGroup[S4];
```

pdf

```
In[∘]:= ConjugacyClasses := Module[{sea = Range@n, ccs = {}, cc},
          While[Length@sea > 0,
           cc = Union[Table[m[inv[α], First@sea, α], {α, n}]];
           AppendTo[ccs, cc];
           sea = Complement[sea, cc];
          ]; ccs]
```

```
In[∘]:= CC = ConjugacyClasses
Out[∘]= {{1}, {2, 3, 6, 7, 15, 22}, {4, 5, 9, 12, 13, 16, 20, 21}, {8, 17, 24}, {10, 11, 14, 18, 19, 23}}
```

pdf

```
In[∘]:= (* $CIS is a Conjugation Invariant Set *)
        R_{i_,j_} := Sum[W_i[α, 1] W_j[β, α], {α, $CIS}, {β, $CIS}];
        R̄_{i_,j_} := Sum[W_i[α, 1] W_j[β, inv@α], {α, $CIS}, {β, $CIS}];
        η_{i_}[ℰ_] := Expand@Sum[ℰ W_i[α, 1], {α, $CIS}];
```

Reidemeister 2:

```
In[∘]:= Table[
         (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{2,4→2}) == (1 // η_1 // η_2) == (R_{1,2} R̄_{3,4} // m_{1,3→1} // m_{4,2→2}),
         {$CIS, CC} ]
Out[∘]= {True, True, True, True, True}
```

Reidemeister 3:

```
In[∘]:= Table[
         (R_{1,2} R_{4,3} R_{5,6} // m_{1,4→1} // m_{2,5→2} // m_{3,6→3}) == (R_{2,3} R_{1,4} R_{5,6} // m_{1,5→1} // m_{2,6→2} // m_{3,4→3}),
         {$CIS, CC} ]
Out[∘]= {True, True, True, True, True}
```

In[∘]:=
```
Z4[K_] := Z4[PD@K];
Z4[pd_PD] := Sum[
   Module[{z, done, st, c, mn, k},
    z = 1; done = {}; st = Range[2 Length@pd];
    Do[
     z *= c /. X[i_, j_, _, l_] :> If[PositiveQ@c, mn = {i, l}; R_{l,i}, mn = {i, j};
         R̄_{j,i}];
     Do[
      If[MemberQ[done, k + 1], z = z // m_{k,k+1→k}; st = st /. k + 1 → k];
      If[MemberQ[done, k - 1], z = z // m_{st⟦k-1⟧,k→st⟦k-1⟧}; st = st /. k → st⟦k - 1⟧]],
      {k, mn}];
     done = done ⋃ mn,
     {c, List @@ pd}];
    z ],
   {$CIS, CC} ];
Z5[K_] := Z4@ThinPosition@K;
```

In[∘]:=
```
Table[K → Echo[Timing[Z5[K]]], {K, AllKnots[{3, 7}]}]
```

KnotTheory: Loading precomputed data in PD4Knots`.

» {0.109375, $W_1[1,1] + 5W_1[2,2] + 5W_1[3,3] + 4W_1[4,1] + 4W_1[5,1] + 5W_1[6,6] + 5W_1[7,7] + W_1[8,8] + 4W_1[9,1] + W_1[10,10] + 4W_1[10,19] + W_1[11,11] + 4W_1[11,14] + 4W_1[12,1] + 4W_1[13,1] + 4W_1[14,11] + W_1[14,14] + 5W_1[15,15] + 4W_1[16,1] + W_1[17,17] + W_1[18,18] + 4W_1[18,23] + 4W_1[19,10] + W_1[19,19] + 4W_1[20,1] + 4W_1[21,1] + 5W_1[22,22] + 4W_1[23,18] + W_1[23,23] + W_1[24,24]$}

» {0.171875, $W_1[1,1] + W_1[2,1] + W_1[3,1] + 4W_1[4,1] + 4W_1[5,1] + W_1[6,1] + W_1[7,1] + W_1[8,1] + 4W_1[9,1] + W_1[10,1] + W_1[11,1] + 4W_1[12,1] + 4W_1[13,1] + W_1[14,1] + W_1[15,1] + 4W_1[16,1] + W_1[17,1] + W_1[18,1] + W_1[19,1] + 4W_1[20,1] + 4W_1[21,1] + W_1[22,1] + W_1[23,1] + W_1[24,1]$}

» {0.734375, $W_1[1,1] + W_1[2,2] + W_1[3,3] + W_1[4,4] + W_1[5,5] + W_1[6,6] + W_1[7,7] + W_1[8,8] + W_1[9,9] + W_1[10,19] + W_1[11,14] + W_1[12,12] + W_1[13,13] + W_1[14,11] + W_1[15,15] + W_1[16,16] + W_1[17,17] + W_1[18,23] + W_1[19,10] + W_1[20,20] + W_1[21,21] + W_1[22,22] + W_1[23,18] + W_1[24,24]$}

» {1.65625, $W_1[1,1] + W_1[2,2] + W_1[3,3] + W_1[4,4] + W_1[5,5] + W_1[6,6] + W_1[7,7] + W_1[8,8] + W_1[9,9] + W_1[10,19] + W_1[11,14] + W_1[12,12] + W_1[13,13] + W_1[14,11] + W_1[15,15] + W_1[16,16] + W_1[17,17] + W_1[18,23] + W_1[19,10] + W_1[20,20] + W_1[21,21] + W_1[22,22] + W_1[23,18] + W_1[24,24]$}

» {0.28125, $W_1[1,1] + 5W_1[2,1] + 5W_1[3,1] + W_1[4,4] + W_1[5,5] + 5W_1[6,1] + 5W_1[7,1] + W_1[8,1] + W_1[9,9] + 4W_1[10,1] + W_1[10,17] + 4W_1[11,1] + W_1[11,24] + W_1[12,12] + W_1[13,13] + 4W_1[14,1] + W_1[14,24] + 5W_1[15,1] + W_1[16,16] + W_1[17,1] + 4W_1[18,1] + W_1[18,8] + 4W_1[19,1] + W_1[19,17] + W_1[20,20] + W_1[21,21] + 5W_1[22,1] + 4W_1[23,1] + W_1[23,8] + W_1[24,1]$}

» {1.10938, $W_1[1,1] + W_1[2,1] + W_1[3,1] + W_1[4,4] + W_1[5,5] + W_1[6,1] + W_1[7,1] + W_1[8,1] + W_1[9,9] + W_1[10,17] + W_1[11,24] + W_1[12,12] + W_1[13,13] + W_1[14,24] + W_1[15,1] + W_1[16,16] + W_1[17,1] + W_1[18,8] + W_1[19,17] + W_1[20,20] + W_1[21,21] + W_1[22,1] + W_1[23,8] + W_1[24,1]$}

» {1.125, $W_1[1,1] + W_1[2,1] + W_1[3,1] + W_1[4,1] + W_1[5,1] + W_1[6,1] + W_1[7,1] + W_1[8,1] + W_1[9,1] + W_1[10,1] + W_1[11,1] + W_1[12,1] + W_1[13,1] + W_1[14,1] + W_1[15,1] + W_1[16,1] + W_1[17,1] + W_1[18,1] + W_1[19,1] + W_1[20,1] + W_1[21,1] + W_1[22,1] + W_1[23,1] + W_1[24,1]$}

» $\{0.328125, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 5] + W_1[5, 4] + W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 13] +$
$W_1[10, 10] + W_1[11, 11] + W_1[12, 20] + W_1[13, 9] + W_1[14, 14] + W_1[15, 15] + W_1[16, 21] + W_1[17, 17] +$
$W_1[18, 18] + W_1[19, 19] + W_1[20, 12] + W_1[21, 16] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]\}$

» $\{0.328125, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + 4 W_1[4, 5] + 4 W_1[5, 4] +$
$W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + 4 W_1[9, 13] + W_1[10, 10] + W_1[11, 11] + 4 W_1[12, 20] +$
$4 W_1[13, 9] + W_1[14, 14] + W_1[15, 15] + 4 W_1[16, 21] + W_1[17, 17] + W_1[18, 18] +$
$W_1[19, 19] + 4 W_1[20, 12] + 4 W_1[21, 16] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]\}$

» $\{1.98438, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + 4 W_1[4, 4] + 4 W_1[5, 5] +$
$W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + 4 W_1[9, 9] + W_1[10, 19] + W_1[11, 14] + 4 W_1[12, 12] +$
$4 W_1[13, 13] + W_1[14, 11] + W_1[15, 15] + 4 W_1[16, 16] + W_1[17, 17] + W_1[18, 23] +$
$W_1[19, 10] + 4 W_1[20, 20] + 4 W_1[21, 21] + W_1[22, 22] + W_1[23, 18] + W_1[24, 24]\}$

» $\{2.45313, W_1[1, 1] + 5 W_1[2, 2] + 5 W_1[3, 3] + W_1[4, 4] + W_1[5, 5] + 5 W_1[6, 6] + 5 W_1[7, 7] + W_1[8, 8] +$
$W_1[9, 9] + 4 W_1[10, 10] + W_1[10, 19] + 4 W_1[11, 11] + W_1[11, 14] + W_1[12, 12] + W_1[13, 13] + W_1[14, 11] +$
$4 W_1[14, 14] + 5 W_1[15, 15] + W_1[16, 16] + W_1[17, 17] + 4 W_1[18, 18] + W_1[18, 23] + W_1[19, 10] +$
$4 W_1[19, 19] + W_1[20, 20] + W_1[21, 21] + 5 W_1[22, 22] + W_1[23, 18] + 4 W_1[23, 23] + W_1[24, 24]\}$

» $\{2.125, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 5] + W_1[5, 4] + W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 13] +$
$W_1[10, 10] + W_1[11, 11] + W_1[12, 20] + W_1[13, 9] + W_1[14, 14] + W_1[15, 15] + W_1[16, 21] + W_1[17, 17] +$
$W_1[18, 18] + W_1[19, 19] + W_1[20, 12] + W_1[21, 16] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]\}$

» $\{9.96875, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 1] + W_1[5, 1] + W_1[6, 6] + W_1[7, 7] + W_1[8, 8] +$
$W_1[9, 1] + W_1[10, 10] + W_1[11, 11] + W_1[12, 1] + W_1[13, 1] + W_1[14, 14] + W_1[15, 15] + W_1[16, 1] +$
$W_1[17, 17] + W_1[18, 18] + W_1[19, 19] + W_1[20, 1] + W_1[21, 1] + W_1[22, 22] + W_1[23, 23] + W_1[24, 24]\}$

» $\{2.59375, W_1[1, 1] + 5 W_1[2, 2] + 5 W_1[3, 3] + W_1[4, 4] + W_1[5, 5] + 5 W_1[6, 6] + 5 W_1[7, 7] + W_1[8, 8] +$
$W_1[9, 9] + W_1[10, 10] + 4 W_1[10, 19] + W_1[11, 11] + 4 W_1[11, 14] + W_1[12, 12] + W_1[13, 13] + 4 W_1[14, 11] +$
$W_1[14, 14] + 5 W_1[15, 15] + W_1[16, 16] + W_1[17, 17] + W_1[18, 18] + 4 W_1[18, 23] + 4 W_1[19, 10] +$
$W_1[19, 19] + W_1[20, 20] + W_1[21, 21] + 5 W_1[22, 22] + 4 W_1[23, 18] + W_1[23, 23] + W_1[24, 24]\}$

Out[∘]= $\{\text{Knot}[3, 1] \to$
$\{0.109375, W_1[1, 1] + 5 W_1[2, 2] + 5 W_1[3, 3] + 4 W_1[4, 1] + 4 W_1[5, 1] + 5 W_1[6, 6] + 5 W_1[7, 7] +$
$W_1[8, 8] + 4 W_1[9, 1] + W_1[10, 10] + 4 W_1[10, 19] + W_1[11, 11] + 4 W_1[11, 14] + 4 W_1[12, 1] +$
$4 W_1[13, 1] + 4 W_1[14, 11] + W_1[14, 14] + 5 W_1[15, 15] + 4 W_1[16, 1] +$
$W_1[17, 17] + W_1[18, 18] + 4 W_1[18, 23] + 4 W_1[19, 10] + W_1[19, 19] + 4 W_1[20, 1] +$
$4 W_1[21, 1] + 5 W_1[22, 22] + 4 W_1[23, 18] + W_1[23, 23] + W_1[24, 24]\},$
$\text{Knot}[4, 1] \to \{0.171875, W_1[1, 1] + W_1[2, 1] + W_1[3, 1] + 4 W_1[4, 1] + 4 W_1[5, 1] +$
$W_1[6, 1] + W_1[7, 1] + W_1[8, 1] + 4 W_1[9, 1] + W_1[10, 1] + W_1[11, 1] + 4 W_1[12, 1] +$
$4 W_1[13, 1] + W_1[14, 1] + W_1[15, 1] + 4 W_1[16, 1] + W_1[17, 1] + W_1[18, 1] +$
$W_1[19, 1] + 4 W_1[20, 1] + 4 W_1[21, 1] + W_1[22, 1] + W_1[23, 1] + W_1[24, 1]\},$
$\text{Knot}[5, 1] \to \{0.734375, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 4] + W_1[5, 5] +$
$W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 9] + W_1[10, 19] + W_1[11, 14] + W_1[12, 12] +$
$W_1[13, 13] + W_1[14, 11] + W_1[15, 15] + W_1[16, 16] + W_1[17, 17] + W_1[18, 23] +$
$W_1[19, 10] + W_1[20, 20] + W_1[21, 21] + W_1[22, 22] + W_1[23, 18] + W_1[24, 24]\},$
$\text{Knot}[5, 2] \to \{1.65625, W_1[1, 1] + W_1[2, 2] + W_1[3, 3] + W_1[4, 4] + W_1[5, 5] +$
$W_1[6, 6] + W_1[7, 7] + W_1[8, 8] + W_1[9, 9] + W_1[10, 19] + W_1[11, 14] + W_1[12, 12] +$
$W_1[13, 13] + W_1[14, 11] + W_1[15, 15] + W_1[16, 16] + W_1[17, 17] + W_1[18, 23] +$
$W_1[19, 10] + W_1[20, 20] + W_1[21, 21] + W_1[22, 22] + W_1[23, 18] + W_1[24, 24]\},$
$\text{Knot}[6, 1] \to \{0.28125, W_1[1, 1] + 5 W_1[2, 1] + 5 W_1[3, 1] + W_1[4, 4] + W_1[5, 5] +$
$5 W_1[6, 1] + 5 W_1[7, 1] + W_1[8, 1] + W_1[9, 9] + 4 W_1[10, 1] + W_1[10, 17] + 4 W_1[11, 1] +$
$W_1[11, 24] + W_1[12, 12] + W_1[13, 13] + 4 W_1[14, 1] + W_1[14, 24] + 5 W_1[15, 1] +$
$W_1[16, 16] + W_1[17, 1] + 4 W_1[18, 1] + W_1[18, 8] + 4 W_1[19, 1] + W_1[19, 17] +$

```
    W₁[20, 20] + W₁[21, 21] + 5 W₁[22, 1] + 4 W₁[23, 1] + W₁[23, 8] + W₁[24, 1]},
 Knot[6, 2] → {1.10938, W₁[1, 1] + W₁[2, 1] + W₁[3, 1] + W₁[4, 4] + W₁[5, 5] + W₁[6, 1] +
    W₁[7, 1] + W₁[8, 1] + W₁[9, 9] + W₁[10, 17] + W₁[11, 24] + W₁[12, 12] + W₁[13, 13] +
    W₁[14, 24] + W₁[15, 1] + W₁[16, 16] + W₁[17, 1] + W₁[18, 8] + W₁[19, 17] +
    W₁[20, 20] + W₁[21, 21] + W₁[22, 1] + W₁[23, 8] + W₁[24, 1]}, Knot[6, 3] →
  {1.125, W₁[1, 1] + W₁[2, 1] + W₁[3, 1] + W₁[4, 1] + W₁[5, 1] + W₁[6, 1] + W₁[7, 1] + W₁[8, 1] +
    W₁[9, 1] + W₁[10, 1] + W₁[11, 1] + W₁[12, 1] + W₁[13, 1] + W₁[14, 1] + W₁[15, 1] + W₁[16, 1] +
    W₁[17, 1] + W₁[18, 1] + W₁[19, 1] + W₁[20, 1] + W₁[21, 1] + W₁[22, 1] + W₁[23, 1] + W₁[24, 1]},
 Knot[7, 1] → {0.328125, W₁[1, 1] + W₁[2, 2] + W₁[3, 3] + W₁[4, 5] + W₁[5, 4] +
    W₁[6, 6] + W₁[7, 7] + W₁[8, 8] + W₁[9, 13] + W₁[10, 10] + W₁[11, 11] + W₁[12, 20] +
    W₁[13, 9] + W₁[14, 14] + W₁[15, 15] + W₁[16, 21] + W₁[17, 17] + W₁[18, 18] +
    W₁[19, 19] + W₁[20, 12] + W₁[21, 16] + W₁[22, 22] + W₁[23, 23] + W₁[24, 24]},
 Knot[7, 2] → {0.328125, W₁[1, 1] + W₁[2, 2] + W₁[3, 3] + 4 W₁[4, 5] + 4 W₁[5, 4] +
    W₁[6, 6] + W₁[7, 7] + W₁[8, 8] + 4 W₁[9, 13] + W₁[10, 10] + W₁[11, 11] + 4 W₁[12, 20] +
    4 W₁[13, 9] + W₁[14, 14] + W₁[15, 15] + 4 W₁[16, 21] + W₁[17, 17] + W₁[18, 18] +
    W₁[19, 19] + 4 W₁[20, 12] + 4 W₁[21, 16] + W₁[22, 22] + W₁[23, 23] + W₁[24, 24]},
 Knot[7, 3] → {1.98438, W₁[1, 1] + W₁[2, 2] + W₁[3, 3] + 4 W₁[4, 4] + 4 W₁[5, 5] +
    W₁[6, 6] + W₁[7, 7] + W₁[8, 8] + 4 W₁[9, 9] + W₁[10, 19] + W₁[11, 14] + 4 W₁[12, 12] +
    4 W₁[13, 13] + W₁[14, 11] + W₁[15, 15] + 4 W₁[16, 16] + W₁[17, 17] + W₁[18, 23] +
    W₁[19, 10] + 4 W₁[20, 20] + 4 W₁[21, 21] + W₁[22, 22] + W₁[23, 18] + W₁[24, 24]},
 Knot[7, 4] → {2.45313, W₁[1, 1] + 5 W₁[2, 2] + 5 W₁[3, 3] + W₁[4, 4] + W₁[5, 5] +
    5 W₁[6, 6] + 5 W₁[7, 7] + W₁[8, 8] + W₁[9, 9] + 4 W₁[10, 10] + W₁[10, 19] + 4 W₁[11, 11] +
    W₁[11, 14] + W₁[12, 12] + W₁[13, 13] + W₁[14, 11] + 4 W₁[14, 14] + 5 W₁[15, 15] +
    W₁[16, 16] + W₁[17, 17] + 4 W₁[18, 18] + W₁[18, 23] + W₁[19, 10] + 4 W₁[19, 19] +
    W₁[20, 20] + W₁[21, 21] + 5 W₁[22, 22] + W₁[23, 18] + 4 W₁[23, 23] + W₁[24, 24]},
 Knot[7, 5] → {2.125, W₁[1, 1] + W₁[2, 2] + W₁[3, 3] + W₁[4, 5] + W₁[5, 4] + W₁[6, 6] +
    W₁[7, 7] + W₁[8, 8] + W₁[9, 13] + W₁[10, 10] + W₁[11, 11] + W₁[12, 20] +
    W₁[13, 9] + W₁[14, 14] + W₁[15, 15] + W₁[16, 21] + W₁[17, 17] + W₁[18, 18] +
    W₁[19, 19] + W₁[20, 12] + W₁[21, 16] + W₁[22, 22] + W₁[23, 23] + W₁[24, 24]},
 Knot[7, 6] → {9.96875, W₁[1, 1] + W₁[2, 2] + W₁[3, 3] + W₁[4, 1] + W₁[5, 1] +
    W₁[6, 6] + W₁[7, 7] + W₁[8, 8] + W₁[9, 1] + W₁[10, 10] + W₁[11, 11] + W₁[12, 1] +
    W₁[13, 1] + W₁[14, 14] + W₁[15, 15] + W₁[16, 1] + W₁[17, 17] + W₁[18, 18] +
    W₁[19, 19] + W₁[20, 1] + W₁[21, 1] + W₁[22, 22] + W₁[23, 23] + W₁[24, 24]},
 Knot[7, 7] → {2.59375, W₁[1, 1] + 5 W₁[2, 2] + 5 W₁[3, 3] + W₁[4, 4] + W₁[5, 5] +
    5 W₁[6, 6] + 5 W₁[7, 7] + W₁[8, 8] + W₁[9, 9] + W₁[10, 10] + 4 W₁[10, 19] + W₁[11, 11] +
    4 W₁[11, 14] + W₁[12, 12] + W₁[13, 13] + 4 W₁[14, 11] + W₁[14, 14] + 5 W₁[15, 15] +
    W₁[16, 16] + W₁[17, 17] + W₁[18, 18] + 4 W₁[18, 23] + 4 W₁[19, 10] + W₁[19, 19] +
    W₁[20, 20] + W₁[21, 21] + 5 W₁[22, 22] + 4 W₁[23, 18] + W₁[23, 23] + W₁[24, 24]}}
```

Computing the knot invariant for S_5 by summing over conjugacy classes

In[◦]:=
```
DeclareGroup[S₅];
CC = ConjugacyClasses
```

Out[•]= {{1}, {2, 3, 6, 7, 15, 22, 25, 55, 81, 106}, {4, 5, 9, 12, 13, 16, 20, 21, 31, 39, 46, 49, 57, 60, 75, 79, 82, 100, 104, 105}, {8, 17, 24, 26, 27, 30, 56, 61, 68, 83, 87, 95, 108, 112, 120}, {10, 11, 14, 18, 19, 23, 33, 36, 37, 40, 44, 45, 51, 54, 58, 59, 63, 70, 73, 76, 80, 84, 85, 96, 98, 99, 103, 107, 110, 119}, {28, 29, 32, 41, 48, 50, 62, 66, 67, 71, 77, 88, 89, 92, 93, 102, 111, 114, 115, 118}, {34, 35, 38, 42, 43, 47, 52, 53, 64, 65, 69, 72, 74, 78, 86, 90, 91, 94, 97, 101, 109, 113, 116, 117}}

```
In[•]:= Table[K → Echo[Timing[Z5[K]]], {K, AllKnots[{3, 7}]}]
```

» {23.375, $W_1[1, 1] + 7 W_1[2, 2] + 7 W_1[3, 3] + 7 W_1[4, 1] + 7 W_1[5, 1] + 7 W_1[6, 6] + 7 W_1[7, 7] + 5 W_1[8, 8] + 7 W_1[9, 1] + W_1[10, 10] + 4 W_1[10, 19] + W_1[11, 11] + 4 W_1[11, 14] + 7 W_1[12, 1] + 7 W_1[13, 1] + 4 W_1[14, 11] + W_1[14, 14] + 7 W_1[15, 15] + 7 W_1[16, 1] + 5 W_1[17, 17] + W_1[18, 18] + 4 W_1[18, 23] + 4 W_1[19, 10] + W_1[19, 19] + 7 W_1[20, 1] + 7 W_1[21, 1] + 7 W_1[22, 22] + 4 W_1[23, 18] + W_1[23, 23] + 5 W_1[24, 24] + 7 W_1[25, 25] + 5 W_1[26, 26] + 5 W_1[27, 27] + W_1[28, 25] + W_1[29, 25] + 5 W_1[30, 30] + 7 W_1[31, 1] + W_1[32, 2] + W_1[33, 33] + 4 W_1[33, 73] + W_1[34, 65] + 5 W_1[34, 91] + W_1[35, 72] + 5 W_1[35, 116] + W_1[36, 36] + 4 W_1[36, 98] + W_1[37, 37] + 4 W_1[37, 51] + W_1[38, 94] + 5 W_1[38, 113] + 7 W_1[39, 1] + W_1[40, 40] + 4 W_1[40, 99] + W_1[41, 6] + 5 W_1[42, 69] + W_1[42, 86] + 5 W_1[43, 90] + W_1[43, 117] + W_1[44, 44] + 4 W_1[44, 54] + W_1[45, 45] + 4 W_1[45, 76] + 7 W_1[46, 1] + 5 W_1[47, 64] + W_1[47, 109] + W_1[48, 3] + 7 W_1[49, 1] + W_1[50, 2] + 4 W_1[51, 37] + W_1[51, 51] + W_1[52, 90] + 5 W_1[52, 117] + 5 W_1[53, 94] + W_1[53, 113] + 4 W_1[54, 44] + W_1[54, 54] + 7 W_1[55, 55] + 5 W_1[56, 56] + 7 W_1[57, 1] + W_1[58, 58] + 4 W_1[58, 103] + W_1[59, 59] + 4 W_1[59, 80] + 7 W_1[60, 1] + 5 W_1[61, 61] + W_1[62, 55] + W_1[63, 63] + 4 W_1[63, 85] + W_1[64, 47] + 5 W_1[64, 78] + 5 W_1[65, 34] + W_1[65, 97] + W_1[66, 15] + W_1[67, 55] + 5 W_1[68, 68] + W_1[69, 42] + 5 W_1[69, 101] + W_1[70, 70] + 4 W_1[70, 110] + W_1[71, 22] + 5 W_1[72, 35] + W_1[72, 74] + 4 W_1[73, 33] + W_1[73, 73] + 5 W_1[74, 72] + W_1[74, 116] + 7 W_1[75, 1] + 4 W_1[76, 45] + W_1[76, 76] + W_1[77, 6] + W_1[78, 64] + 5 W_1[78, 109] + 7 W_1[79, 1] + 4 W_1[80, 59] + W_1[80, 80] + 7 W_1[81, 81] + 7 W_1[82, 1] + 5 W_1[83, 83] + W_1[84, 84] + 4 W_1[84, 107] + 4 W_1[85, 63] + W_1[85, 85] + 5 W_1[86, 42] + W_1[86, 101] + 5 W_1[87, 87] + W_1[88, 7] + W_1[89, 81] + W_1[90, 43] + 5 W_1[90, 52] + W_1[91, 34] + 5 W_1[91, 97] + W_1[92, 22] + W_1[93, 81] + 5 W_1[94, 38] + W_1[94, 53] + 5 W_1[95, 95] + W_1[96, 96] + 4 W_1[96, 119] + 5 W_1[97, 65] + W_1[97, 91] + 4 W_1[98, 36] + W_1[98, 98] + 4 W_1[99, 40] + W_1[99, 99] + 7 W_1[100, 1] + W_1[101, 69] + 5 W_1[101, 86] + W_1[102, 3] + 4 W_1[103, 58] + W_1[103, 103] + 7 W_1[104, 1] + 7 W_1[105, 1] + 7 W_1[106, 106] + 4 W_1[107, 84] + W_1[107, 107] + 5 W_1[108, 108] + 5 W_1[109, 47] + W_1[109, 78] + 4 W_1[110, 70] + W_1[110, 110] + W_1[111, 7] + 5 W_1[112, 112] + W_1[113, 38] + 5 W_1[113, 53] + W_1[114, 106] + W_1[115, 15] + W_1[116, 35] + 5 W_1[116, 74] + 5 W_1[117, 43] + W_1[117, 52] + W_1[118, 106] + 4 W_1[119, 96] + W_1[119, 119] + 5 W_1[120, 120]$}

Out[•]= $Aborted

Conclusion

A very elegant theory which we implemented cleanly and were able to improve dramatically.

Overall, a great